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Further remarks on Diophantine quintuples

by

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1. Introduction. Let m and n be integers, with m positive. A set of m positive integers is called a *$D(n)$-m-tuple* if the product of any two distinct elements increased by n is a perfect square. In this paper we study exclusively $D(1)$-m-sets, which will be called *Diophantine m-tuples.* When $m = 2$ (3, 4, 5 or 6), we shall speak of *Diophantine pairs* (triples, quadruples, quintuples or sextuples, respectively).

Since Dujella [6] proved that there are no Diophantine sextuples and only finitely many Diophantine quintuples, the major concern is to confirm the folklore conjecture according to which no Diophantine quintuple exists. An essential ingredient of any strategy seems to be a convenient stratification of the set of Diophantine tuples. Quite early the notion of regular Diophantine tuple appeared (see [1]): a Diophantine triple $\{a, b, c\}$ with $a < b < c$ is called *regular* if $c = c_+$, where $c_\pm = a + b \pm 2\sqrt{ab + 1}$, and a Diophantine quadruple $\{a, b, c, d\}$ with $a < b < c < d$ is called *regular* if $d = d_+$, where $d_\pm = a + b + c + 2abc \pm 2\sqrt{(ab + 1)(ac + 1)(bc + 1)}$. A stronger conjecture put forward by [1] and independently by [13] claims that every Diophantine quadruple is regular.

We shall also employ another useful classification and adopt Fujita's point of view, calling $\{a, b, c\}$ a *standard triple* if it satisfies one of the following:

- $c > b^5$ (standard *of the first kind*);
- $b > 4a$ and $c \ge b^2$ (standard *of the second kind*);
- $b > 12a$ and $b^{5/3} < c < b^2$ (standard *of the third kind*).

Each Diophantine quadruple contains at least one standard triple, not necessarily unique, since the properties required in the classification are not mutually exclusive. Large gaps of entries in a Diophantine set facilitate theoretical analysis (e.g., they are necessary in order to employ the

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hypergeometric method), while small gaps are preferable for explicit computations. Therefore, identification of as many as possible standard triples is of interest.

The main outcome of the work reported here is evidence that this classification is not the most appropriate in the study of Diophantine quintuples. Finding a suitable replacement remains a task for future work.

THEOREM 1.1. *The quadruple left after removing the largest entry of a Diophantine quintuple contains no standard triple of the first kind.*

In the process of obtaining such a qualitative information, enhancements of the relative and absolute size of entries of a hypothetical Diophantine quintuple emerge.

THEOREM 1.2. *Let $\{a, b, c, d, e\}$ be a Diophantine quintuple with $a < b < c < d < e$. If $b > 4a$ then $d < \min\{9.5b^4, b^2c, 10^{72.188}\}$ and $b < 10^{35.793} < 6.209 \cdot 10^{35}$. If $b < 4a$ then either $c = a + b + 2\sqrt{ab+1}$ and $b < 10^{17.647} < 4.44 \cdot 10^{17}$, $d < \min\{16b^3, 10^{53.292}\}$, or $c = (4ab+2)(a+b-2\sqrt{ab+1})+2a+2b$ and $d < \min\{c^2, 10^{47.086}\}$, $b < 10^{11.7715} < 5.91 \cdot 10^{11}$.*

This compares favourably with the absolute bound $d < 3.5 \cdot 10^{96}$ just published [8] and even with $d < 10^{74}$ obtained in [20].

The new absolute bounds on the entries of a putative Diophantine quintuple entail improved estimates on the number of such objects. The best published result to date is $6.8 \cdot 10^{32}$, due to Elsholtz, Filipin, and Fujita [8].

THEOREM 1.3. *There are at most 10^{31} Diophantine quintuples.*

The main technical device on which the strength of these theorems relies is a twist of the congruence method, more precisely, of the way to transform a congruence relation into an equality. After the work reported in this paper was completed, a preprint of Wu and He [20] was brought to our attention, where the same idea has been exploited. This approach is employed in the next section, where it is shown that certain Diophantine quadruples $\{a, b, c, d\}$ with $a < b < c < d$ satisfy $c < 9.5b^4$. In particular, this yields Theorem 1.1. The same idea can be adapted for the study of $D(n)$-sets for $n \neq 1$, but we leave this for subsequent work. Theorem 1.2 is proved in Section 3. The proof of Theorem 1.3 requires explicit bounds for relevant arithmetic functions and is presented in the last section of the paper.

2. Standard triples of Diophantine quadruples. The results we shall prove in this section are better than the published ones mainly due to a careful study of standard triples appearing in a Diophantine quadruple.

LEMMA 2.1. *Let (a, b, c, d) be a Diophantine quadruple with $a < b < c < d$ and r the positive integer satisfying $ab + 1 = r^2$.*

(1) *In case the quadruple is regular, the following hold:*

(a) *If* $b > 4a$ *then* $\{a, b, d\}$ *is a standard triple of the second kind. Moreover, when* $c < b^{1.5}$ *then either*

(i) $c = a + b + 2r$ *and* $\{a, c, d\}$ *is a standard triple of the second kind, or*

(ii) $c \geq 4ab + a + b$ *and* $\{a, c, d\}$ *is a standard triple of the third kind.*

(b) *Assume* $b < 4a$. *Then:*

(iii) *If* $c \geq b^3$ *then* $\{a, b, d\}$ *is a standard triple of the first kind. Otherwise, either*

(iv) $c = a + b + 2r$ *and* $\{a, c, d\}$ *is a standard triple of the second kind, or*

(v) $c = 4\,r(r - a)(b - r)$ *and* $\{b, c, d\}$ *is a standard triple of the third kind.*

(2) *In case the quadruple is not regular, the triple* $\{a, c, d\}$ *is standard of the second kind. More precisely,* $d > c^{3.5}$.

Proof. A consequence of [14, Lemma 4] is that either $c = a + b + 2r$ or $c > 4ab$. Using this, one can readily prove that for a regular Diophantine quadruple one has $c(4ab + 1) < d < 4c(ab + 1)$. This and the hypothesis $b > 4a$ imply that $\{a, b, d\}$ is a standard triple of the second kind. When $c = a + b + 2r$, one gets $9a < c < 2.25b$ and therefore $d > 4abc > ac^2$. When $c \geq 4ab+a+b \geq 4ab+4 > 16a^2$ one has, on the one hand, $d < c(4ab+4) \leq c^2$, and, on the other hand, $c > 21a$. Since from the additional condition $c < b^{1.5}$ it follows that $d > 4abc > 4ac^{5/3}$, we conclude that $\{a, c, d\}$ is a standard triple of the third kind.

(b) Part (iii) is obvious from the inequalities $d > 4abc > b^2c \geq b^5$.

For $c = a + b + 2r$ one gets $4a < c < 4b$, so that $d > 4abc > ac^2$ and part (iv) is established. Suppose $c > a + b + 2r$. Then by [14, Theorem 8] there exists an integer $k \geq 2$ such that $c = c_k$ or $c = \bar{c}_k$, where the increasing sequences (c_k) and $(\bar{c}_k)$ are given by the non-homogeneous linear recurrence relation

$$X_{k+2} = (4ab + 2)X_{k+1} - X_k + 2(a + b) \qquad (k \geq 0)$$

and the initial conditions

$$c_0 = 0, \qquad c_1 = a + b + 2r, \qquad \bar{c}_0 = 0, \quad \bar{c}_1 = a + b - 2r.$$

Notice that from the relations

$$c_2 = 4r(a + r)(b + r) > 8abr > 4ab^2 > b^3 > c$$

and

$$\bar{c}_3 = 8r(r - a)(b - r)(2ab + 1) + a + b + 2r > 16abr > 2c$$

it follows that

$$c = \overline{c}_2 = (4ab+2)(a+b-2r) + 2(a+b) \geq 4ab + 2(a+b) + 2$$

(the last inequality follows by noting that, on the one hand, $a+b-2r \geq 0$ with equality if and only if $b = a+2$, and, on the other hand, for $b = a+2$ one has $\overline{c}_2 = c_1$). Hence, $d < 4c(ab+1) < c^2$ and $d > 4abc > b^2c > c^{5/3}$. The conclusion holds once we show that $c > 12b$. This holds for $b \geq 9$ because then

$$c > 4ab + 2a + 2b + 1 \geq b^2 + 3.5b + 1.5 > 12b.$$

It remains to examine the regular Diophantine quadruples with $a+2 < b < 4a$, $b \leq 8$, and $c = \overline{c}_2 < b^3$. It is readily seen that these conditions imply $a \leq 5$ and that there is a unique such Diophantine quadruple, $(a,b,c,d) = (3, 8, 120, 11781)$. Now it is clear that indeed $c = 120 > 96 = 12b$.

Suppose now that the Diophantine quadruple under study is not regular. From [14, Lemma 4] one learns that $c > 4a$, while [6, Proposition 1] ensures $d > c^{3.5}$. Thus, $\{a, c, d\}$ is a standard Diophantine triple of the second kind. ∎

Let $\{a, b, c, d\}$ be a Diophantine quadruple with $a < b < c < d$. Then there exist positive integers r, s, t, x, y, z such that

$$ab+1 = r^2, \quad ac+1 = s^2, \quad bc+1 = t^2,$$
$$ad+1 = x^2, \quad bd+1 = y^2, \quad cd+1 = z^2.$$

Eliminating d from these equations results in a system of generalized Pell equations

$$az^2 - cx^2 = a - c, \tag{1}$$

$$bz^2 - cy^2 = b - c. \tag{2}$$

The z-component of each solution to (1) and respectively to (2) appears in a linear recurrence sequence given by

$$v_0 = z_0, \quad v_1 = sz_0 + cx_0, \quad v_{k+2} = 2sv_{k+1} - v_k,$$

and respectively

$$w_0 = z_1, \quad w_1 = tz_1 + cy_1, \quad w_{k+2} = 2tw_{k+1} - w_k,$$

for certain integers z_0, z_1, x_0, y_1 satisfying (see, e.g., [6] or [11])

$$1 \leq x_0 < \sqrt{\frac{s+1}{2}}, \quad 1 \leq |z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}},$$

$$1 \leq y_1 < \sqrt{\frac{t+1}{2}}, \quad 1 \leq |z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}.$$

The technical part on [9] depends on the following hypothesis. (In order to facilitate comparison, we keep the numbering of the assumptions used there.)

ASSUMPTION 2.2. There exist integers $m \geq 3$, $n \geq 2$, and z_0 such that $|z_0| = 1$, $z = v_{2m} = w_{2n}$, and $c > b^5$.

By work of Dujella [6, Lemma 3] it is known that Assumption 2.2 implies $n \leq m \leq 2n$, $z_0 = z_1$, and $x_0 = y_1 = 1$. Filipin and Fujita [9] derive many other consequences of the same hypothesis. We shall use a more precise comparison of the two indices m and n.

LEMMA 2.2. *On Assumption* 2.2 *one has* $m < 1.2n$.

Proof. Although the statement of [9, Lemma 2.3] reads $m \leq 1.2n$, its proof rules out the equality case. ■

We shall also need the displayed relation (2.9) from [9].

LEMMA 2.3. *Let* $\{a, b, c, d\}$ *be a Diophantine quadruple with* $a<b<c<d$. *Set* $a' = \max\{b - a, a\}$. *If Assumption* 2.2 *holds then*

$$n < \frac{4\log\bigl(4.001a^{1/2}(a')^{1/2}b^2c\bigr)\log\bigl(1.299a^{1/2}b^{1/2}(b-a)^{-1}c\bigr)}{\log(4bc)\log\bigl(0.1053a(a')^{-1}b^{-1}(b-a)^{-2}c\bigr)}.$$

Another ingredient in our argument is the congruence

$$\varepsilon am^2 + sm \equiv \varepsilon bn^2 + tn \pmod{4c}, \quad \text{where} \quad \varepsilon = \pm 1, \tag{3}$$

derived from Assumption 2.2 and [5, Lemma 4]. It will be used in the proof of the crucial lemma below, in which the numerical coefficient is slightly better than that in [20, Lemma 2].

LEMMA 2.4. *Suppose there exist integers* $m \geq n \geq 2$ *and* z_0 *such that* $|z_0| = 1$, $z = v_{2m} = w_{2n}$, $2n \geq m \geq 3$. *Then* $m > 0.5b^{-1/2}c^{1/2}$.

Proof. Assume $m \leq 0.5b^{-1/2}c^{1/2}$. As a first step towards a contradiction, we show that (3) is actually an equality. To this end, we estimate its terms:

$$\max\{am^2, bn^2\} \leq bm^2 \leq 0.25c,$$

$$\max\{sm, tn\} \leq tm \leq 0.5b^{-1/2}c^{1/2}\sqrt{bc+1} < 0.5c + \frac{1}{4b} < 0.6c,$$

whence (3) turns into an equality that can be written as

$$\varepsilon(am^2 - bn^2) = tn - sm.$$

Multiplying both sides by $tn + sm$ and performing minor algebraic manipulations, this is found to be equivalent to

$$(bn^2 - am^2)(c + \varepsilon(tn + sm)) = m^2 - n^2. \tag{4}$$

Notice that $bn^2 - am^2 = 0$ entails $m^2 - n^2 = 0$, which in turn implies $a = b$, absurd. Therefore, since for any Diophantine pair (a, b) one has $a + 2 \leq b$, it follows that

$$\begin{aligned} c &\le tn + sm + m^2 - n^2 \le m\sqrt{bc+1} + m\sqrt{ac+1} + 0.75m^2 \\ &\le 0.5\big(\sqrt{bc+1} + \sqrt{(b-2)c+1} + 0.375b^{-1/2}c^{1/2}\big)b^{-1/2}c^{1/2} \\ &< c, \end{aligned}$$

a contradiction. ∎

Now we can prove the main result of this section.

THEOREM 2.1. *Keep the notation introduced above. Then Assumption* 2.2 *holds for no Diophantine quadruple* $\{a, b, c, d\}$ *with* $a + 2 < b < c < d$.

Proof. Assume $\{a, b, c, d\}$ is a Diophantine quadruple fulfilling Assumption 2.2. We examine three cases separately.

CASE $b \ge 2a$. Before entering into details, we outline the approach. We shall first prove that only at most half a dozen Diophantine pairs (a, b) can be extended to Diophantine triples satisfying Assumption 2.2. For each pair thus determined we obtain sharp upper bounds for the third entry c. Next, computer-aided sieving (using nothing more than the `issquare` command of Pari/GP [18]) outputs no values c in the allowed range for which both $ac+1$ and $bc+1$ are squares. This contradiction shows that Assumption 2.2 does not hold when $b \ge 2a$.

Now we implement the strategy just described.

Note that in the present case one has $a' = b - a$. Clearly, $a^{1/2}(b-a)^{1/2} \le b/2$ by arithmetic-geometric mean inequality, and from $b \ge 2a$ we readily obtain $a^{1/2}b^{1/2}(b-a)^{-1} \le \sqrt{2}$. Even better, for $a = 1$ one has $b^{1/2}(b-1)^{-1} \le 2b^{-1/2}$. Therefore, the inequality from Lemma 2.3 becomes

$$n < \frac{4\log(4.001b^{5/2}c)\log(2.598b^{-1/2}c)}{\log(4bc)\log(0.1053b^{-4}c)} \qquad \text{for } a = 1$$

and

$$n < \frac{4\log(2.001b^{3}c)\log(1.299\sqrt{2}c)}{\log(4bc)\log(0.1053ab^{-4}c)} \qquad \text{for } a \ge 2.$$

As the right sides above decrease with c, from $b^5 < c$ it follows that

$$n < \frac{4\log(4.001b^{15/2})\log(2.598b^{9/2})}{\log(4b^6)\log(0.1053b)} < \frac{45\log(1.204b)\log(1.237b)}{2\log(1.259b)\log(0.1053b)}$$

and

$$n < \frac{4\log(2.001b^{8})\log(1.299\sqrt{2}b^5)}{\log(4b^6)\log(0.1053ab)} < \frac{80\log(1.091b)\log(1.130b)}{3\log(1.259b)\log(0.1053ab)},$$

respectively. Comparison with the lower bound for n provided by the hypothesis $c \ge b^5$ and Lemmas 2.2 and 2.4, namely,

$$n > \tfrac{5}{6}m > \tfrac{5}{12}b^{-1/2}c^{1/2} \ge \tfrac{5}{12}b^2, \tag{5}$$

results in the inequalities

$$b^2 < \frac{54\log(1.204b)\log(1.237b)}{\log(1.259b)\log(0.1053b)} \quad \text{for } a = 1$$

and

$$b^2 < \frac{64\log(1.091b)\log(1.130b)}{\log(1.259b)\log(0.1053ab)} \quad \text{for } a \geq 2.$$

Thus, when $a = 1$ one obtains $b \leq 17$, so that $(a, b) = (1, 8)$ or $(1, 15)$, while for $a \geq 2$ one gets $b \leq 13$, which means that only $(a, b) = (2, 12), (3, 8)$, $(4, 12)$ need further consideration.

For each of these we use inequality (5) and get $c^{1/2}$ bounded by a rational function in $\log c$. For instance, for $(a, b) = (1, 8)$ one has

$$c^{1/2} < 27.153 \frac{\log(677.49c)\log(0.525c)}{\log(32c)\log(390883.2^{-1}c)},$$

whence $c < 89000$. For each of the other four Diophantine pairs the upper bound for c derived with the help of (5) is smaller than the fifth power of the corresponding b. So it only remains to confirm that Assumption 2.2 does not hold when $(a, b) = (1, 8)$. This is obtained by checking that there is no c between $8^5 = 32768$ and 89000 such that both $c + 1$ and $8c + 1$ are perfect squares.

Case $1.45a \leq b < 2a$. For $b < 2a$ one has $a' = a$, so that the inequality from Lemma 2.3 reads

$$n < \frac{4\log(4.001ab^2c)\log\bigl(1.299a^{1/2}b^{1/2}(b-a)^{-1}c\bigr)}{\log(4bc)\log\bigl(0.1053b^{-1}(b-a)^{-2}c\bigr)}.$$

Despite its simplicity, the observation that in this case any Diophantine pair must satisfy either $(a, b) = (8, 15)$ or $a \geq 15$ and $b \geq 24$ is very helpful.

When $(a, b) = (8, 15)$, proceeding as explained in the previous case one gets $c < 10^5 < 15^5$, in contradiction with Assumption 2.2.

Let us now examine the remaining Diophantine pairs. As the function $x \mapsto x/(b - x^2)$ is increasing for $0 < x < \sqrt{b}$, $b - a < b/2$, $a \leq b/1.45$, and $15 \leq a \leq b - 9$ (either equality attained only for $(a, b) = (15, 24)$), one obtains

$$n < \frac{4\log(2.76b^3c)\log(3.4761c)}{\log(4bc)\log(0.4212b^{-3}c)}.$$

The right side decreases when c increases, so that

$$n < \frac{4\log(2.76b^8)\log(3.4761b^5)}{\log(4b^6)\log(0.4212b^2)},$$

that is,

$$n < \frac{40\log(1.1354b)\log(1.283b)}{3\log(1.2599b)\log(0.6489b)}.$$

The key observation is that this inequality yields $n \le 16$. From (5) it then follows that $b \le 6$. Since this inequality contradicts $b \ge 24$, we conclude that Assumption 2.2 does not hold in this case either.

CASE $a+3 \le b < 1.45a$. Note that now $13 \le b-a < 9b/29$, whence

$$n < \frac{4\log(4.001b^3c)\log(0.09993bc)}{\log(4bc)\log(1.0933b^{-3}c)}.$$

Since the right side is a decreasing function in c, from $b^5 < c$ one obtains

$$n < \frac{16\log(1.19b)\log(0.682b)}{\log(1.259b)\log(1.045b)}.$$

This obviously yields $n < 16$, and therefore one can proceed as in the previous case to obtain $b \le 5$. Hence, $a+3 \le 5$, which in turn implies $4 \le a+3 \le b < 2 \cdot 1.45 < 3$, another contradiction.

The proof of Theorem 2.1 is complete. ■

We now use the result just proved to strengthen it in a particular case appearing in later developments. For conciseness, we introduce the following hypothesis on a Diophantine quadruple $\{a,b,c,d\}$ with $a<b<c<d$.

ASSUMPTION 2.1. There exist integers $m \ge n \ge 2$ and z_0 such that $|z_0| = 1$, $z = v_{2m} = w_{2n}$, $m \ge 3$.

LEMMA 2.5. *Suppose Assumption* 2.1 *holds. If* $b \ge 8$ *and* $c > b^4$ *then* $m \le \frac{4}{3}n$.

Proof. As in [6, proof of Lemma 3], the starting point is provided by the inequalities $v_k \ge v_1(2s-1)^{k-1}$ and $w_k \le w_1(2t)^{k-1}$ valid for positive k. In view of the hypothesis $c > b^4 \ge 8^4$, we readily get

$$1.984\sqrt{ac} < 2s-1, \qquad s+1 < 1.016\sqrt{ac}, \qquad t < c^{5/8}.$$

Combining this with the bounds for z_0, z_1, x_0, y_1 recalled above, we obtain

$$\begin{aligned} v_1 = sz_0 + cx_0 &\ge \frac{c^2 - ac - z_0^2}{s|z_0| + cx_0} > \frac{c^2 - ac - z_0^2}{2cx_0} > \frac{c - a - 0.5a^{-0.5}c^{0.5}}{2x_0} \\ &> \frac{0.998c - 0.5a^{-0.5}c^{0.5}}{2x_0} > \frac{0.99c}{2x_0} > 0.694a^{-1/4}c^{3/4} \end{aligned}$$

and

$$w_1 = tz_1 + cy_1 < 2cy_1 < c\sqrt{2(t+1)} < 1.003\sqrt{2}\,c^{21/16} < 1.419c^{21/16}.$$

Hence,

$$0.694a^{-1/4}c^{3/4} \cdot (1.984\sqrt{ac}\,)^{2m-1} < v_{2m} = w_{2n} < 1.419c^{21/16} \cdot (2c^{5/8})^{2n-1},$$

which in turn implies

$$1.984^{2m-1}c^{m+1/4} < 2^{2n+0.033}c^{5n/4+11/16}.$$

It follows that either $2m-1 < 1.0118(2n+0.033)$ or $m+1/4 < 5n/4+11/16$, that is,

$$m < 1.012n + 0.517 \quad \text{or} \quad m < 1.25n + 0.4375.$$

The former inequality is stronger and each implies $n \ge 3$ (because $m \ge 3$). Note that they also imply $m < \frac{4}{3}n$ for $n \ge 6$ and $m \le n+1$ for $n = 3, 4, 5$. The conclusion of our lemma is therefore proven. ∎

LEMMA 2.6. *On Assumption* 2.1, *the inequality*

$$n < \frac{4\log\bigl(4.001a^{1/2}(a')^{1/2}b^2c\bigr)\log\bigl(1.299a^{1/2}b^{1/2}(b-a)^{-1}c\bigr)}{\log(4bc)\log\bigl(0.1053a(a')^{-1}b^{-1}(b-a)^{-2}c\bigr)}$$

(*with* $a' = \max\{b-a, a\}$) *holds whenever* $c \ge 9.5b^4$.

Proof. The proof is the same as the proof of [9, (2.9)]. ∎

THEOREM 2.2. *Suppose Assumption* 2.1 *holds for a Diophantine quadruple* $\{a, b, c, d\}$ *with* $a+2 < b < c < d$. *Then* $c < 9.5b^4$.

Proof. Having in mind Theorem 2.1 and the fact that no Diophantine pair (a, b) with $a+2 < b$ has $b = 10$ or 11, we may supppose $b \ge 12$.

The strategy is the same as in the proof of Theorem 2.1. For contradiction, assume $c \ge 9.5b^4$. The details vary according to the sign of $b - 2a$. Moreover, it is convenient to examine very small values of a separately.

CASE $a = 1$. Now we are facing a difficulty due to the fact that for c close to $9.5b^4$, the term $\log(0.1053b^{-1}(b-1)^{-3}c) \sim \log 1.00035$ becomes pretty small and the quality of the upper bound on n decreases rapidly. A palliative would be to use a lower bound for $b(b-1)^{-1}$ better than the obvious one. Since this expression decreases when b increases, one needs to bound b from above.

Proceeding as in the proof of Theorem 2.1, one replaces c by $9.5b^4$ in Lemma 2.6. Having in mind that $b^{1/2}(b-1)^{-1} \le 1000 \cdot 999^{-1}b^{-1/2}$ when $b \ge 1000$, the reconciliation of the upper bound for n thus obtained with the lower bound derived from

$$n \ge \tfrac{3}{4}m > \tfrac{3}{8}b^{-1/2}c^{1/2} \ge \tfrac{3\sqrt{9.5}}{8}b^{3/2} \tag{6}$$

results in

$$b^{3/2} < 44997.3\frac{\log(1.751b)\log(2.051b)}{\log(2.0699b)},$$

whence $b < 5600$ and $r \le 74$. Taking $\frac{74^2-1}{74^2-2}$ as a lower bound for $b(b-1)^{-1}$ has the favourable consequence of diminishing the value 44997.3 in the above inequality to 17536.2. This in turn implies $r \le 53$, and so $b(b-1)^{-1} \ge \frac{53^2-1}{53^2-2}$, which leads to $r \le 45$. Playing once more the same game one concludes that $b \le 1680$.

At this junction of the proof we start the filtering phase. For each value of r between 4 and 41 we let a computer program search for integers of the form $c = s^2 - 1$ having the properties: $bc + 1 = (r^2 - 1)c + 1$ is a perfect square and c is at least $9.5b^4$ but smaller than b^5. The output list consists of 67 items: for $r = 8$ or 9 a unique c of this kind was found, when $r = 41$ there are three admissible values, and for each of the remaining r there are precisely two corresponding c's. For each of these pairs (b, c) the upper bound for n computed with the help of Lemma 2.6 was much smaller than the lower bound for n given in (6).

Case $b \geq 2a \geq 4$. This time one gets

$$b^{3/2} < 32.301 \frac{\log(1.523b)\log(2.044b)}{\log(2.0699b)},$$

whence $b \leq 21$. For each integer a with $3 \leq a \leq 10$ we write the inequality from Lemma 2.6 for this specific a, and find that b is at most 15 when $a = 3$ and at most 11 when $a \geq 4$. Recall that in this proof one also has $b \geq 12$ to conclude that the only Diophantine pair satisfying all restrictions presently in force is $(a, b) = (2, 12)$. For this pair, inequality (6) and Lemma 2.6 give

$$c^{1/2} < 36.951 \frac{\log(2576.595c)\log(0.6364c)}{\log(48c)\log(c/56981)},$$

whence $c < 195000$, which contradicts the requirement $c > 9.5 \cdot 12^4 > 196000$.

Case $b < 2a$. Note that now one has $a' = a$, and either $(a, b) = (8, 15)$ or $a \geq 15$. Hence, $b - a \geq 7$, so that

$$n < \frac{28\log(1.6816b)\log(1.1201b)}{\log(2.0699b)\log(1.00035b)} < 28.$$

From $b^{3/2} < \frac{27 \cdot 8}{3\sqrt{9.5}}$ it then follows that $b \leq 8$, in contradiction with $b \geq 15$. ∎

3. Bounds for the size of elements in a Diophantine quintuple. Our next goal is to apply all these considerations to a putative Diophantine quintuple $\{a, b, c, d, e\}$ with $a < b < c < d < e$. Recall that the positive integers x, y, z satisfying $ad+1 = x^2$, $bd+1 = y^2$, $cd+1 = z^2$ are solutions to simultaneous Pell equations (1)–(2). Similarly, there exist positive integers α, β, γ, δ such that $ae + 1 = \alpha^2$, $be + 1 = \beta^2$, $ce + 1 = \gamma^2$, $de + 1 = \delta^2$. Elimination of e results in a system of three Pell equations: $a\delta^2 - d\alpha^2 = a - d$, $b\delta^2 - d\beta^2 = b - d$, $c\delta^2 - d\gamma^2 = c - d$. Taken individually, each of these quadratic equations has solutions given by linearly recurrent sequences. In particular, the common unknown can be expressed as $\delta = U_i$, $\delta = V_j$, $\delta = W_k$, where the positive integers i, j, k indicate the rank of appearance in the sequences

$$\begin{aligned} U_0 &= \pm 1, & U_1 &= \pm x + d, & U_{i+2} &= 2xU_{i+1} - U_i, \\ V_0 &= \pm 1, & V_1 &= \pm y + d, & V_{j+2} &= 2yV_{j+1} - V_j, \\ W_0 &= \pm 1, & W_1 &= \pm z + d, & W_{k+2} &= 2zW_{k+1} - W_k. \end{aligned}$$

As explained in [12], all indices are even and satisfy $4 \le k \le j \le i \le 2k$ and $j \ge 6$. Clearly, the sequences (v_m), (w_n) involved in considerations from Section 2 used for the quadruple $\{a, b, d, e\}$ are actually (U_i), (V_j). Therefore, the requirements on the parity and size of indices from Assumptions 2.1 and 2.2 are fulfilled.

The following is a more explicit version of Theorem 1.1.

PROPOSITION 3.1. *Let $\{a, b, c, d, e\}$ be a Diophantine quintuple with $a < b < c < d < e$. If $b > 4a$ then $\{a, b, d\}$ is a standard Diophantine triple of the second kind which satisfies $d < 9.5b^4$. Otherwise, $c < b^3$, $d < 9.5b^4$, and either $c = a + b + 2r$ and $\{a, c, d\}$ is standard of the second kind or $c = (4ab+2)(a+b-2r)+2(a+b)$ and $\{b, c, d\}$ is standard of the third kind.*

Proof. Recall that Fujita [11] has proved that removing the largest element from any Diophantine quintuple results in a regular Diophantine quadruple. The first part of the conclusion then follows directly from Lemma 2.1(1a). In view of the preceding considerations, the inequality $d < 9.5b^4$ is a consequence of Theorem 2.2, which can be applied to (a, b, d, e) because no Diophantine quintuple contains a pair $\{A-1, A+1\}$ with $A \ge 2$ (see [10]). When $b < 4a$ and $c > b^3$ it follows that $d > b^5$, in contradiction with Theorem 2.1 employed for the Diophantine quadruple (a, b, d, e). The last part of the conclusion is given by Lemma 2.1(1b). ∎

The proofs of our next results rely on Matveev's bound [17] for linear forms in logarithms generated by a Diophantine triple (A, B, C) with $A < B < C$. Actually we shall apply its variant mentioned in [12] to the linear form

$$\Lambda = j \log \xi - k \log \eta + \log \mu,$$

where

$$\xi = S + \sqrt{AC}, \qquad \eta = T + \sqrt{BC}, \qquad \mu = \frac{\sqrt{B}(\sqrt{C} \pm \sqrt{A})}{\sqrt{A}(\sqrt{C} \pm \sqrt{B})}.$$

It is known (see [6, (60)]) that

$$0 < \Lambda < \tfrac{8}{3} AC\xi^{-2j}.$$

For a standard Diophantine triple (A, B, C) of the second kind with $C > 10^{28}$ one obtains

$$\begin{aligned} \sqrt{AC+1} &< \sqrt{0.25C^{1.5}+1} < 0.5001C^{0.75}, \\ \sqrt{BC+1} &< \sqrt{C^{1.5}+1} < 1.0001C^{0.75}, \end{aligned}$$

$$\frac{\sqrt{C}\pm\sqrt{A}}{\sqrt{C}\pm\sqrt{B}} < \frac{\sqrt{C}+0.5\sqrt{B}}{\sqrt{C}-\sqrt{B}} < \frac{1+0.5C^{-0.25}}{1-C^{-0.25}} < 1.0001,$$
$$\frac{\sqrt{C}\pm\sqrt{A}}{\sqrt{C}\pm\sqrt{B}} > \frac{\sqrt{C}-0.5\sqrt{B}}{\sqrt{C}+\sqrt{B}} > \frac{1-0.5C^{-0.25}}{1+C^{-0.25}} > 0.9999.$$

Hence, $A_1 = 2\log\xi$ and

$$A_2 = 2\log\eta < 2\log(2\sqrt{BC+1}) < 2\log(2.0002C^{0.75}) < 1.5216\log C.$$

Note that the minimal polynomial of μ is the primitive part of $A^2(C-B)^2X^4 + 4A^2B(C-B)X^3 + 2AB(3AB-AC-BC-C^2)X^2 + 4AB^2(C-A)X + B^2(C-A)^2$, whose leading coefficient a_0 satisfies

$$\frac{1}{4}\left(\frac{C}{B}-1\right) \le a_0 \le A^2(C-B)^2.$$

Therefore, the following estimates are true:

$$\begin{aligned} A_3 &= 4\mathrm{h}(\mu) < \log(1.0001^2AB(C-B)^2) < \log(0.25\cdot 1.0001^2C^3) \\ &< 3\log C, \\ A_3 &> \log\left(0.9999^2\frac{C-B}{4A}\right) > \log\left(0.9999^2\frac{C-B}{B}\right) \\ &> \log(0.9999^2C^{0.4999}) > 0.499\log C, \end{aligned}$$

whence

$$\Omega = A_1A_2A_3 < 9.1296\log\xi(\log C)^2,$$
$$E = \max\left\{\frac{jA_1}{A_3}, \frac{kA_2}{A_3}\right\} < \max\left\{\frac{1.5001j}{0.499}, \frac{1.5216k}{0.499}\right\} < 3.0493j,$$

and

$$W_0 = \log(6eE\log(4e)) < \log(118.678j).$$

Comparison of the lower and upper bounds for Λ results in the inequality

$$\begin{aligned} 2j\log\xi &< 6.4407\cdot 10^8\cdot 29.8847\cdot 16\cdot 9.1296\log\xi\cdot(\log C)^2\log(118.678j) \\ &\quad + \log(8AC) - \log 3, \end{aligned}$$

that is,

$$j < 1.40581\cdot 10^{12}(\log C)^2\log(118.678j). \tag{7}$$

Similar arguments prove the following result.

Lemma 3.1. *Let (a,b,c,d) be a regular Diophantine quadruple with $a < b < c < d$, $a+2 < b < 4a$, and $d > 10^{21}$.*

(a) *If $c = a+b+2\sqrt{ab+1}$ then $j < 1.1169\cdot 10^{12}(\log d)^2\log(82.576j)$.*
(b) *If $c = (4ab+2)(a+b-2\sqrt{ab+1}) + 2(a+b)$ then*

$$j < 1.1169\cdot 10^{12}(\log d)^2\log(69.191j).$$

Now we can prove the main results of this section. Together with Proposition 3.1, they imply Theorem 1.2 from the Introduction. In both proofs we apply results from Section 2 to the Diophantine quadruple (a, b, d, e).

THEOREM 3.1. *Let $\{a, b, c, d, e\}$ be a Diophantine quintuple with $4a < b < c < d < e$. Then $d < 10^{72.188} < 1.542{\cdot}10^{72}$ and $b < 10^{35.793} < 6.209{\cdot}10^{35}$.*

Proof. As $\{a, b, c, d\}$ is a regular Diophantine quadruple, one has $d = d_+ > 4abc > 4b^2$. Hence, Lemma 2.4 yields

$$j > \frac{\sqrt{d}}{\sqrt{b}} > \sqrt{2}\, d^{1/4}.$$

Comparison with the upper bound for j provided by (7) leads to the desired conclusion. ∎

THEOREM 3.2. *Let $\{a, b, c, d, e\}$ be a Diophantine quintuple with $a < b < c < d < e$ and $b < 4a$. If $c = a+b+2\sqrt{ab+1}$ then $d < 10^{53.292} < 1.96 \cdot 10^{53}$ and $b < 10^{17.647} < 4.44 \cdot 10^{17}$, otherwise $c = (4ab+2)(a+b-2\sqrt{ab+1}) + 2a+2b$, $d < 10^{47.086} < 1.22 \cdot 10^{47}$, and $b < 10^{11.7715} < 5.91 \cdot 10^{11}$.*

Proof. By Lemma 2.1 and Proposition 3.1, c has to have one of the specified forms. When $c = a+b+2\sqrt{ab+1}$ one gets $c > 2.25b$, and therefore $d > 2.25b^3$. This together with Lemma 2.4 readily yields $j > (1.5d)^{1/3}$. Similarly, in the complementary case one arrives at $j > d^{3/8}$ by noting that this time $d > (4ab)^2 > b^4$. Combining these lower bounds for j with the upper bounds given in Lemma 3.1, one arrives at the indicated estimates for d and b. ∎

4. An estimate for the number of Diophantine quintuples. We shall follow the strategy of our predecessors: Consider a pair (a, b) that can be extended to a Diophantine quintuple $\{a, b, c, d, e\}$ with $a < b < c < d < e$. First we estimate the number of such pairs (a, b) below a certain threshold and count the possibilities to extend each of them to a Diophantine triple. By Fujita's theorem [11], any Diophantine quadruple which appears in a Diophantine quintuple is uniquely determined by its smallest three elements. According to another result due to Fujita [12, Theorem 1.2], a Diophantine quadruple can be extended to a quintuple in at most four ways. Therefore, we shall only emphasize the new ideas appearing in the proof of estimates for the number of Diophantine triples.

Counting the number of Diophantine pairs formed by the two smallest entries in a Diophantine quintuple is paramount to computing the number of solutions to a congruence derived from the defining condition $ab+1 = r^2$. This is achieved with the help of the next result.

Lemma 4.1. *Let M be a positive integer and $\omega : \mathbb{N} \to \mathbb{N}$ the counting function of distinct prime divisors. The number of solutions of the congruence $X^2 \equiv 1 \pmod{M}$ in the range $1 \le X < M$ is $2^{\omega(M)}$ if M is odd or $M \equiv 4 \pmod{8}$; $2^{\omega(M)-1}$ if $M \equiv 2 \pmod{4}$; $2^{\omega(M)+1}$ if $M \equiv 0 \pmod{8}$. In each case, precisely half of the solutions are in the range $1 \le X < M/2$.*

Proof. The first part is well known—see, for instance, [19]. The map $X \mapsto M - X$ gives a bijection between the solutions of the congruence which are positive and less than $M/2$, and solutions greater than $M/2$ and smaller than M. ∎

There are several papers devoted to the asymptotic behaviour of the number of $D(1)$-m-sets. Dujella [7] has given the main term in the asymptotic formula for the number of Diophantine pairs and triples. Lao [15] has obtained a second term in each of them. Martin and Sitar [16] established the main term in the asymptotic expansion of the number of $D(1)$-quadruples. Since in each of these results the intervening constants and the range of validity remain unspecified, they cannot serve to bound the number of Diophantine quadruples. However, ideas and techniques from their proofs can be employed to this end.

We now give a series of lower and upper bounds for several arithmetic functions needed for the proof of the main result of this section. In all of them, the dominant term agrees with the main term in the corresponding asymptotic formula.

As usual, μ and ω denote the Möbius function and the counting function of distinct prime divisors, respectively.

Lemma 4.2. *For real $x \ge 1$, set $Q(x) := \sum_{n=1}^{x} \mu^2(n)$. Then $Q(x) = \frac{6}{\pi^2}x + P(x)$, with*

(a) $-0.103229\sqrt{x} \le P(x) \le 0.679091\sqrt{x}$ *for* $x \ge 1$,
(b) $|P(x)| \le 0.1333\sqrt{x}$ *for* $x \ge 1664$,
(c) $|P(x)| \le 0.036438\sqrt{x}$ *for* $x \ge 82005$,
(d) $|P(x)| \le 0.02767\sqrt{x}$ *for* $x \ge 438653$.

Proof. (a) is obtained with the help of a script written in gp [18]. The inequality in part (b) is quoted from [3], and the others have been proved in [4]. ∎

These bounds together with familiar techniques like summing by parts and comparison of sums with integrals serve to obtain the next explicit bounds for functions intervening in subsequent proofs.

Lemma 4.3. *For real $x \ge 1$, set*

$$E(x) := \sum_{n=1}^{x} 2^{\omega(n)}, \qquad F(x) := \sum_{n=1}^{x} \frac{2^{\omega(n)}}{n}, \qquad G(x) := \sum_{n=1}^{x} \frac{2^{\omega(2n-1)}}{2n-1}.$$

Then

$$\frac{6}{\pi^2}x\log x - 0.518398x < E(x) < \frac{6}{\pi^2}x\log x + 2.04864x,$$
$$\frac{3}{\pi^2}(\log x)^2 + 0.0895\log x - 0.916 < F(x),$$
$$F(x) < \frac{3}{\pi^2}(\log x)^2 + 2.6565672\log x + 1.123069 + \frac{1.05}{x},$$
$$G(x) < \frac{3}{2\pi^2}(\log x)^2 + 3.1227147\log x + 3.56851 + \frac{0.525}{x}.$$

LEMMA 4.4. *For n, H positive integers, let $d_H(n)$ denote the number of divisors of n in $[1, H]$. Then, for any $N \geq 2$,*

$$\sum_{n=2}^{N} d_H(n^2-1) < N\bigg[\frac{9}{\pi^2}(\log H)^2 + 12.408437\log H - 2.868921 + \frac{24}{\pi^2}\frac{\log H}{H} + \frac{37.98928}{H} + \frac{33}{H^2}\bigg].$$

Proof. Using Lemma 4.1 and the inequalities

$$\sum_{n=2}^{N} d_H(n^2-1) \leq 2\sum_{n=1}^{N}\sum_{\substack{d=1\\ n^2\equiv 1\,(d)}}^{\min(H,n)} 1 \leq 2\sum_{d=1}^{H}\sum_{\substack{n=d\\ n^2\equiv 1\,(d)}}^{N} 1,$$

one finds that the desired sum is bounded from above by

$$2N\sum_{\substack{d=1\\ d\equiv 1\,(2)}}^{H}\frac{2^{\omega(d)}}{d} + 2N\sum_{\substack{d=4\\ d\equiv 4\,(8)}}^{H}\frac{2^{\omega(d)}}{d} + N\sum_{\substack{d=2\\ d\equiv 2\,(4)}}^{H}\frac{2^{\omega(d)}}{d} + 4N\sum_{\substack{d=8\\ d\equiv 0\,(8)}}^{H}\frac{2^{\omega(d)}}{d}.$$

Hence,

$$\sum_{n=2}^{N} d_H(n^2-1) \leq 2NG\Big(\frac{H+1}{2}\Big) + NG\Big(\frac{H+4}{8}\Big) + NG\Big(\frac{H+2}{4}\Big) + NF\Big(\frac{H}{8}\Big).$$

The conclusion follows by applying Lemma 4.3. ∎

As a last preparation before proceeding to the proof of Theorem 1.3 we quote a consequence of [2, Theorem 2.1].

LEMMA 4.5. *For $N \geq 2$,*

$$\sum_{n=1}^{N} 4^{\omega(n)} < \frac{N}{6}(\log N + 2)^3.$$

We shall stratify the set of all Diophantine quintuples according to Lemma 2.1, estimate the number of solutions of each kind, and sum the individual contributions to get an upper bound for the total number of $D(1)$-quintuples.

Below, (a,b,c,d,e) is a Diophantine set with $a<b<c<d<e$. Recall that $d=d_+$, so that $4abc<d<4c(ab+1)$.

Case $b>4a$ and $c=a+b+2r$. Note that $b>4a$ is equivalent to $b>2r$. Hence, $c>4r+a$, and therefore $d>4(r^2-1)(4r+2)>16r^3$. From $d<10^{72.188}$ it follows that $r<10^{23.6613}=:R$. According to Lemma 4.4, the number of Diophantine pairs (a,b) is less than

$$\frac{1}{2}\sum_{r=3}^{R} d_R(r^2-1) < 7.75\cdot 10^{26}.$$

Since c is uniquely determined by a and b, and a Diophantine triple can be extended to a quintuple in at most four ways, the number of Diophantine quintuples of this kind is bounded from above by

$$7.75\cdot 10^{26}\cdot 4 = 3.1\cdot 10^{27}. \tag{8}$$

Case $b>4a$ and $4ab+a+b\le c<b^{1.5}$. Note that $b\ge 8$ whenever $b>4a$, so that $d>4ab(4ab+a+b)>16r^4$ and $r<10^{17.746}=:R_1$. Hence, the number of Diophantine pairs (a,b) is at most

$$\frac{1}{2}\sum_{r=3}^{R_1} d_{R_1}(r^2-1) < 5.646\cdot 10^{20}.$$

Each such pair can be extended to a Diophantine triple by a suitable term of certain binary linearly recurrent sequences. These sequences are outnumbered by the solutions to the congruence $t_0^2\equiv 1 \pmod b$ in the range $[-2^{-1/2}b^{3/4}, 2^{-1/2}b^{3/4}]$ (see [5, Lemma 1]). As $2^{-1/2}b^{3/4}<0.5b$, according to Lemma 4.1, the number of those solutions is bounded from above by $2\cdot 2^{\omega(b)}$. Now we have $b<\sqrt{d/20}<7.709\cdot 10^{35}$, so each positive integer below this bound has at most 24 prime divisors. Thus, the number of recurrent sequences of interest is less than 2^{25}. From [8] we learn that each sequence contains at most three terms eligible as the largest element of a Diophantine triple. Since there are up to four possibilities to extend a triple to a quintuple, we conclude that in this case there are at most

$$5.646\cdot 10^{20}\cdot 2^{25}\cdot 3\cdot 4 < 2.274\cdot 10^{29} \tag{9}$$

Diophantine quintuples.

Case $b>4a$ and $c>b^{1.5}$. As it turns out that quintuples subject to these restrictions have the most important contribution to the grand total, Elsholtz, Filipin, and Fujita [8] introduced the idea of averaging over the factors $2^{\omega(b)}$. This technique is responsible for the impressive strengthening of the previous estimates of the number of Diophantine quintuples. As we shall see below, a slight improvement is still possible.

Let us first examine the subcase where $a \geq 10^{11} =: A_3$. This hypothesis together with the inequalities $10^{72.188} > d > 4ab^{2.5}$ implies that $b < 1.7155 \cdot 10^{24} =: R_2$.

For fixed b, the number of Diophantine pairs (a, b) with $4a < b$ is less than $2^{\omega(b)}$ (see Lemma 4.1). For fixed a and b, the number of sequences containing terms eligible for prolongation is at most $2^{\omega(b)+1}$ and each sequence contributes at most five values for c according to [8]. Having in mind Lemma 4.5, we conclude that the number of Diophantine quintuples with $b < R_2$ is less than

$$4\frac{R_2}{6}(\log R_2 + 2)^3 \cdot 5 \cdot 4 < 4.418 \cdot 10^{30}. \tag{10}$$

In the complementary subcase $a < A_3$ one has $b < (d/4)^{1/2.5} < 4.309 \cdot 10^{28}$ and $r < 1.001\sqrt{A_3 b} < 1.001\sqrt{A_3(d/4)^{1/2.5}} < 6.5709 \cdot 10^{19} =: R_3$. Lemma 4.4 applied in this situation bounds the number of pairs by

$$R_3\left[\frac{9}{\pi^2}(\log A_3)^2 + 12.40844 \log A_3 - 2.868921\right] < 6.0718 \cdot 10^{22}.$$

On noting that the product of the first 22 primes exceeds 10^{30}, we conclude that $\omega(b) \leq 21$. As in each recurrent sequence we can find at most five terms that can serve as c, the number of Diophantine quintuples in this subcase is less than

$$6.0718 \cdot 10^{22} \cdot 2^{22} \cdot 5 \cdot 4 < 5.095 \cdot 10^{30}. \tag{11}$$

Case $b < 4a$ and $c = a + b + 2r$. Now the inequalities $10^{53.292} > d > 4abc > b^2(b/4 + b + b) = 9b^3/4$ yield $b < 10^{17.647} =: R_4$. By Lemmas 4.1 and 4.3, the number of Diophantine pairs is less than

$$2\sum_{b=4}^{R_4} 2^{\omega(b)} < \frac{12}{\pi^2} R_4 \log R_4 + 4.09728 R_4 < 2.38 \cdot 10^{19}.$$

Again c is uniquely determined by a and b, so that the estimate for the number of Diophantine quintuples satisfying the hypotheses in force is

$$2.38 \cdot 10^{19} \cdot 4 = 9.52 \cdot 10^{19}. \tag{12}$$

Case $b < 4a$ and $c = 4r(r - a)(b - r)$. In this case we know that $10^{47.086} > d > 4abc > b^4$, that is, $b < 10^{11.7715} =: R_5$. Since c is unique, the number of Diophantine quintuples is no more than

$$\left(\frac{12}{\pi^2} \log R_5 + 4.09728\right) R_5 \cdot 4 < 8.76 \cdot 10^{13}. \tag{13}$$

Summing up (8)–(13), we conclude that the number of Diophantine quintuples is bounded from above by $9.75 \cdot 10^{30}$.

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