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On a conjecture of Lemke and Kleitman

by

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1. Introduction. Throughout this paper, let G be an additively written finite cyclic group of order $|G| = n$. By a *sequence* over G we mean a finite sequence of terms from G which is unordered and repetition of terms is allowed. We view sequences over G as elements of the free abelian monoid $\mathcal{F}(G)$ and use multiplication notation. Thus a sequence S of length $|S| = l$ is written in the form $S = (n_1 g) \cdots (n_l g)$ where $n_1, \ldots, n_l \in \mathbb{N}$ and $g \in G$. We call S a *zero-sum sequence* if the sum of S is zero, i.e. $\sigma(S) = \sum_{i=1}^{l} n_i g = 0$. If S is a zero-sum sequence, but no proper nontrivial subsequence of S has sum zero, then S is called a *minimal zero-sum sequence*. Recall that the index of a sequence S over G is defined as follows.

DEFINITION 1.1. For a sequence over G,

$$S = (n_1 g) \cdots (n_l g) \quad \text{where } 1 \le n_1, \ldots, n_l \le n,$$

the *index* of S is defined by $\operatorname{ind}(S) = \min\{\|S\|_g \mid g \in G \text{ with } G = \langle g \rangle\}$ where

$$\|S\|_g = \frac{n_1 + \cdots + n_l}{\operatorname{ord}(g)}.$$

Clearly, S has sum zero if and only if $\operatorname{ind}(S)$ is an integer. There are also slightly different definitions of the index in the literature, but they are all equivalent (see [8, Lemma 5.1.2]).

The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. The notion of the index of a sequence was introduced by Chapman, Freeze and Smith [1]. It was first addressed by Lemke and Kleitman (in a conjecture [10, p. 344]), used as a key tool by Geroldinger [7, p. 736], and

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later investigated by Gao [3] in a systematical way. Since then it has received a great deal of attention (see for example [2, 5, 8, 9, 11–18]).

A conjecture of Lemke and Kleitman [10, p. 344] states (in the language of index) that if G is a cyclic group of order n and S is a sequence over G of length $|S| = n$, then there exists a subsequence T of S such that $\mathrm{ind}(T) = 1$. A counterexample was given in [6] for the special case when $n = 2 + 4k$ with $k > 5$. In this paper, we investigate the index of sequences regarding the above mentioned conjecture. In Section 2, we show that the conjecture holds under an additional assumption on the highest multiplicity of an element occurring in the sequence (namely $\mathsf{h}(G) \geq n/3$). Section 3 provides general counterexamples to the conjecture. In the last section, we explore the possible maximal index of minimal zero-sum sequences, and suggest a conjecture for the upper bound of the maximum index over C_n when n is a composite number.

In what follows, we recall some frequently used notation and terminology. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and for real numbers a, b, let $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. For $x \in \mathbb{Z}$ and $n \in \mathbb{N}$, $|x|_n$ denotes the least positive residue of x modulo n. Let S be a sequence over G written in the form

$$S = g_1 \cdots g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)}, \qquad \text{with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

Then $\mathsf{v}_g(S) \in \mathbb{N}_0$ is called the *multiplicity* of g in S. Denote by $\mathrm{supp}(S)$, $\mathsf{h}(S)$ and $\sigma(S)$, the *support*, the *height* and the *sum* of S, respectively, i.e. $\mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\}$, $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G\}$, which is the maximum of the multiplicities of g in S, and $\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$.

A sequence T is called a *subsequence* of S and denoted by $T \mid S$ if $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for all $g \in G$. If $T \mid S$, let ST^{-1} denote the subsequence obtained from S by deleting T. Let

$$\Sigma(S) = \{\sigma(T) \mid T \text{ is a subsequence of } S \text{ with } 1 \leq |T| \leq |S|\}.$$

The sequence S is called *zero-sum*, *zero-sum free*, and *minimal zero-sum* if $\sigma(S) = 0 \in G$, $0 \notin \Sigma(S)$, and $\sigma(S) = 0$ but $\sigma(T) \neq 0$ for every $T \mid S$ with $1 \leq |T| < |S|$, respectively. Other notation and terminology follow those in [4] and [8].

2. A positive result. In this section, we show that the above mentioned conjecture holds under the additional assumption that n is odd and $\mathsf{h}(S) \geq n/3$.

THEOREM 2.1. *Let G be a cyclic group of odd order $n \geq 5$ and S a sequence of length $\geq n$ over G. If $\mathsf{h}(S) \geq n/3$, then S has a subsequence T with* $\mathrm{ind}(T) = 1$.

Proof. Assume to the contrary that S does not have any index 1 subsequences. Clearly, S does not contain 0 and $\mathsf{h}(S) < n$ (for otherwise, let $g \in \mathrm{supp}(S)$ with $\mathsf{v}_g(S) = \mathsf{h}(S)$; then $g^{\mathrm{ord}(g)}$ is a subsequence with index 1, a contradiction). Without loss of generality, we may assume that S is such a sequence with $\mathsf{h}(S)$ being maximal. Hence any sequence S' with $|S'| \ge n$ and $\mathsf{h}(S') > \mathsf{h}(S)$ contains an index 1 subsequence. Let $g \in \mathrm{supp}(S)$ with $\mathsf{v}_g(S) = \mathsf{h}(S)$. Then $\langle g\rangle = G$ (for otherwise, $\mathrm{ord}(g) \le n/3$ and thus $g^{\mathrm{ord}(g)}$ is a subsequence with index 1, a contradiction). Write $S = g^{\mathsf{h}(S)}(t_1 g)\cdots(t_m g)$, where $m \ge n - \mathsf{h}(S) \ge 1$, $t_i \in [2, n-1]$ for all $i \in [1, m]$ and $t_1 \le \cdots \le t_m$. We first prove the following claims.

Claim 1. *$\mathsf{h}(S) < n/2$ and $n/3 < t_1 \le t_m < 2n/3$.*

We first observe that $\mathsf{h}(S) < n/2$ and hence $m > n/2 > 2$. Indeed, if $\mathsf{h}(S) \ge n/2$, then there exists l such that $\mathsf{h}(S) + \sum_{i=1}^{l} t_i < n$ and $\mathsf{h}(S) + \sum_{i=1}^{l+1} t_i > n$. If $t_{l+1} \ge \mathsf{h}(S)$, then $n - t_{l+1} < \mathsf{h}(S)$, so S has a subsequence $g^{n-t_{l+1}}(t_{l+1}g)$ with index 1, a contradiction. If $t_{l+1} < \mathsf{h}(S)$, then S has a subsequence of index 1, again a contradiction.

Next we show that $t_1 \ge \mathsf{h}(S) + 2 > n/3$. Assume to the contrary that $t_1 \le \mathsf{h}(S) + 1$. Consider the sequence $S' = S(t_1 g)^{-1} g^{t_1}$. Since $|S'| > |S|$ and $\mathsf{h}(S') = \mathsf{h}(S) + t_1 > \mathsf{h}(S)$, S' contains an index 1 subsequence, say T'. If $\mathsf{v}_g(T') \le \mathsf{h}(S)$, then T' is a subsequence of S, a contradiction. If $\mathsf{v}_g(T') > \mathsf{h}(S)$, then $T = T' g^{-t_1}(t_1 g)$ is a zero-sum subsequence of S and $\mathrm{ind}(T) \le \mathrm{ind}(T')$, so $\mathrm{ind}(T) = 1$, a contradiction. Finally, we observe that $t_m \le n - \mathsf{h}(S) - 1 < 2n/3$, for if $t_m > n - \mathsf{h}(S) - 1$, then $g^{n-t_m}(t_m g)$ is an index 1 subsequence of S, a contradiction. This completes the proof of Claim 1.

Claim 2. *$t_i \ne (n+1)/2$ for every $i \in [1, m]$ and $t_2 \ge (n+3)/2$.*

Assume to the contrary that there exists a $j \in [1, m]$ with $t_j = (n+1)/2$. Let $h = t_j g$. Then $g = 2h$. Consider the sequence $S' = S(g^{\mathsf{h}(S)}h)^{-1} h^{2\mathsf{h}(S)+1}$. Note that $|S'| = |S| + \mathsf{h}(S) > |S|$ and $\mathsf{h}(S') \ge \mathsf{v}_g(h) = 2\mathsf{h}(S) + 1 > \mathsf{h}(S)$, so S' contains an index 1 subsequence T'. Let $k = \mathsf{v}_h(T')$ and $l = \lfloor k/2 \rfloor$. Then $T = T' h^{-2l} g^{l}$ is a zero-sum subsequence of S with $\mathrm{ind}(T) \le \mathrm{ind}(T')$, so $\mathrm{ind}(T) = 1$, a contradiction. Thus $t_i \ne (n+1)/2$ for every $i \in [1, m]$.

We now show that $t_2 \ge (n+3)/2$. If $t_2 < n/2$, then $2n/3 < t_1 + t_2 < n$, and thus $g^{n-t_1-t_2}(t_1 g)(t_2 g)$ is an index 1 subsequence of S, a contradiction. Thus $t_2 \ge n/2$, which together with $t_2 \ne (n+1)/2$ and n being odd implies that $t_2 \ge (n+3)/2$. Claim 2 is proved.

We are now in a position to complete the proof of the theorem. Consider the new sequence $S' = 2S = (2g)^{\mathsf{h}(S)}(|2t_1|_n g)\cdots(|2t_m|_n g)$. Then S' does not contain any index 1 subsequence (as S does not contain any index 1 subsequence). By Claims 1 and 2, we derive that

$$(*) \qquad |2t_i|_n = 2t_i - n \geq 3$$

for each $i \in [2, m]$. Thus

$$2\mathsf{h}(S) + \sum_{i=2}^{2m'+2} |2t_i|_n \geq 2\mathsf{h}(S) + 3(2m'+1) \geq 2\mathsf{h}(S) + 3(m-2) \geq \mathsf{h}(S) + m \geq n$$

where $m' = \lfloor (m-2)/2 \rfloor$, and $2\mathsf{h}(S) < n$. So we may choose the minimal nonnegative integer s such that $2\mathsf{h}(S) + \sum_{i=2}^{2s+2} |2t_i|_n \geq n$ and $s \leq m'$. If $s = 0$, then $2\mathsf{h}(S) + |2t_2|_n \geq n$, so $(n - |2t_2|_n)/2 \leq \mathsf{h}(S)$. Since $|2t_2|_n = 2t_2 - n$ is odd, we see that $(|2t_2|_n g)(2g)^{(n-|2t_2|_n)/2}$ is an index 1 subsequence of S', a contradiction. If $s \geq 1$, then by the definition of s we have $2\mathsf{h}(S) + \sum_{i=2}^{2s} |2t_i|_n < n$. By $(*)$, we get $|2t_{2s+1}|_n + |2t_{2s+2}|_n = 2t_{2s+1} + 2t_{2s+2} - 2n < 2n/3 \leq 2\mathsf{h}(S)$. Thus $n - 2\mathsf{h}(S) \leq \sum_{i=2}^{2s+2} |2t_i|_n < 2\mathsf{h}(S) + \sum_{i=2}^{2s} |2t_i|_n < n$. Let $\alpha = \sum_{i=2}^{2s+2} |2t_i|_n$. Note that α is odd (as $|2t_i|_n = 2t_i - n$ is odd for all $i \in [2, m]$) and $(n - \alpha)/2 \leq \mathsf{h}(S)$. We conclude that

$$(|2t_2|_n g) \cdots (|2t_{2s+2}|_n g) \dot{(}2g)^{(n-\alpha)/2}$$

is an index 1 subsequence of S', a contradiction.

In all cases, we have found contradictions. This completes the proof of Theorem 2.1. ∎

3. Counterexamples. In this section, we provide general counterexamples to the conjecture of Lemke and Kleitman.

THEOREM 3.1. *Let $G = C_n = \langle g \rangle$ be a cyclic group of order n such that $2 \leq d \,|\, n$ and $n > d^2(d^3 - d^2 + d + 1)$. Then the sequence*

$$S = \left(\frac{n}{d} g\right)^{d-1} \left(\left(\frac{n}{d} + d\right) g\right)^{\lfloor n/d^2 \rfloor - d} \prod_{i=0}^{d-1} \left(\left(1 + \frac{in}{d}\right) g\right)^{l},$$

where $l = n/d - d(d-1) - 1$, has no subsequence T with $\mathrm{ind}(T) = 1$.

Proof. Assume to the contrary that S has a subsequence T with $\mathrm{ind}(T) = 1$. Then there exists an element $h \in G$ with $\mathrm{ord}(h) = n$ such that $\|T\|_h = 1$. We set

$$g = jh \quad \text{and} \quad T = \left(\frac{n}{d} g\right)^{u} \left(\left(\frac{n}{d} + d\right) g\right)^{v} \prod_{i=0}^{d-1} \left(\left(1 + \frac{in}{d}\right) g\right)^{x_i},$$

where $j \in [1, n-1]$ with $\gcd(j, n) = 1$, $u \in [0, d-1]$, $v \in [0, n/d^2 - d]$ and $x_i \in [0, l]$ for $i \in [0, d-1]$. Then

$$(3.1) \qquad n\|T\|_g = x_0 + \cdots + x_{d-1} + dv + \frac{n}{d}\big(u + v + x_1 + 2x_2 + \cdots + (d-1)x_{d-1}\big) \equiv 0 \pmod{n}.$$

We note that if $x_0 + \cdots + x_{d-1} = 0$, then $x_0 = \cdots = x_{d-1} = 0$, so

$$dv + \frac{n}{d}(u+v) \equiv 0 \pmod{n}. \tag{$*$}$$

Thus $d^2v \equiv 0 \pmod{n}$, implying $v = 0$ (as $d^2v \in [0, n-d^3]$). By $(*)$, we have $u = 0$, and so T is empty, yielding a contradiction. Thus we must have $x_0 + \cdots + x_{d-1} \geq 1$.

Let $j = nq/d + j_0$, $1 \leq j_0 \leq n/d - 1$. Then

$$j\left(1 + \frac{un}{d}\right) = j_0 + \frac{n}{d}(ju+q).$$

Note that $\gcd(j,d) \mid \gcd(j,n) = 1$, so

$$\left\{ j_0 + \frac{i}{d}n \;\middle|\; 0 \leq i \leq d-1 \right\} = \left\{ \left| j\left(1+\frac{un}{d}\right)\right|_n \;\middle|\; 0 \leq u \leq d-1 \right\}. \tag{3.2}$$

Since $\|T\|_h = 1$, we have

$$\sum_{i=0}^{d-1} x_i \left| j\left(1 + \frac{in}{d}\right)\right|_n \leq n.$$

By (3.2), we derive that

$$j_0 \sum_{i=0}^{d-1} x_i + \frac{n}{d}\sum_{i=0}^{d-1} x_{i'} i \leq n,$$

where i' runs through $[0, d-1]$ as i runs through $[0, d-1]$. Since $x_i \leq l$, we obtain

$$\sum_{i=0}^{d-1} x_i \leq l + \sum_{i=0}^{d-1} x_{i'} i < l + d,$$

implying that

$$x_0 + \cdots + x_{d-1} = \sum_{i=0}^{d-1} x_i \leq l + (d-1).$$

By (3.1), we have

$$x_0 + \cdots + x_{d-1} + dv \equiv 0 \pmod{n/d}. \tag{$**$}$$

Hence

$$v \geq \frac{1}{d}\left(\frac{n}{d} - l - (d-1)\right) = \frac{1}{d}\big(d(d-1) + 1 - (d-1)\big) > d-2,$$

and so

$$v \geq d-1.$$

If $|j(n/d+d)|_n \ge n/d$, then $v \le d-1$ as $v|j(n/d+d)|_n < n\|T\|_h = n$, implying $v = d-1$. Observe that

$$x_0 + \cdots + x_{d-1} - l \ge \frac{n}{d} - dv - l = \frac{n}{d} - d(d-1) - \left(\frac{n}{d} - d(d-1) - 1\right) = 1,$$

so we obtain

$$\begin{aligned}\|T\|_h &= \frac{1}{n}\left(u\left|j\frac{n}{d}\right|_n + v\left|j\left(\frac{n}{d}+d\right)\right|_n + \sum_{i=0}^{d-1} x_i\left|j\left(\frac{in}{d}+1\right)\right|_n\right)\\ &> \frac{1}{n}\left((d-1)\frac{n}{d} + \left(\sum_{i=0}^{d-1} x_i - l\right)\frac{n}{d}\right) \ge 1,\end{aligned}$$

yielding a contradiction.

Next we assume that $|j(n/d+d)|_n < n/d$, and write

$$j = \frac{n}{d^2} j_1 + j_2, \qquad 0 \le j_2 < \frac{n}{d^2},$$

where $n/d^2, j_2$ are positive numbers (not necessarily integers) and j_1, dj_2 are integers. Then

$$\frac{n}{d} > \left|j\left(\frac{n}{d}+d\right)\right|_n = \left|\frac{n}{d}(j+j_1) + dj_2\right|_n > 0,$$

so we derive that $j + j_1 \equiv 0 \pmod{d}$ and $j_2 > 0$. This shows that $j_1 \not\equiv 0 \pmod{d}$ as $\gcd(j,d) = 1$, whence $j > n/d^2$, so $j_0 \ge n/d^2$ (for otherwise, $j_1 \equiv 0 \pmod{d}$). By (3.2), $j_0 \ge n/d^2$ and $\|T\|_h = 1$, so as before we obtain

$$j_0(x_0 + \cdots + x_{d-1}) < n, \quad \text{hence} \quad x_0 + \cdots + x_{d-1} < d^2.$$

Thus

$$x_0 + \cdots + x_{d-1} + dv < d^2 + d\left(\frac{n}{d^2} - d\right) = \frac{n}{d},$$

a contradiction to $(**)$. This completes the proof of Theorem 3.1. ∎

REMARK. (1) Let S be the sequence as given in Theorem 3.1. Since $|S| \ge n = |G|$, S does have a zero-sum subsequence T. It follows from Theorem 3.1 that $\mathrm{ind}(T) \ge 2$. Note that if n is odd, then $\mathsf{h}(S) = l \le n/3 - 7$. If n is even, then $\mathsf{h}(S) = l \le n/2 - 3$.

(2) Let $T = (ng/d)^{d-1}((n/d+d)g)^d g^{n/d-d^2}$. Then T is a subsequence of S with $\mathrm{ind}(T) = 2$ (as $\|T\|_g = 2$).

4. Maximum index of minimal zero-sum sequences. We now investigate the possible maximal index of minimal zero-sum sequences and propose a conjecture for its upper bound. It should be noted that the counterexamples in the previous section were constructed based on the sequences in Theorem 4.5. We first give the following definition.

Definition 4.1. The *maximum index* of minimal zero-sum sequences over C_n is defined as follows:

$$\mathrm{MI}(C_n) = \max_S\{\mathrm{ind}(S)\},$$

where S runs over all minimal zero-sum sequences of elements in C_n.

Gao proposed an upper bound for $\mathrm{MI}(C_n)$:

Conjecture 4.2 ([3, Conjecture 4.2]). *$\mathrm{MI}(C_n) \le c \ln n$ for some absolute constant c.*

It was proved in [16] that the conjecture is not true for an even integer n. Theorem 4.5 below shows that $\mathrm{MI}(C_n)$ could be very large if n has a small divisor. We first recall the following definition.

Definition 4.3. Let S be a minimal zero-sum (resp. zero-sum free) sequence of elements over G. An element g_0 in S is called *splittable* if there exist two elements $x, y \in G$ such that $x + y = g_0$ and $Sg_0^{-1}xy$ is a minimal zero-sum (resp. zero-sum free) sequence as well; otherwise, g_0 is called *unsplittable*. The sequence S is called *splittable* if at least one of elements of S is splittable; otherwise, it is called *unsplittable*.

Lemma 4.4 (Xia, Yuan [16, Lemma 2.14]). *Let S be a minimal zero-sum sequence in a finite abelian group G. Then an element a in S is unsplittable if and only if $\sum(Sa^{-1}) = G \setminus \{0\}$. Thus S is unsplittable if and only if for every element $a \in \mathrm{supp}(S)$ we have $\sum(Sa^{-1}) = G \setminus \{0\}$.*

We now give the main result of this section.

Theorem 4.5. *Let $n, d \ge 2$ be odd positive integers such that $d \mid n$ and $n > d^3$, and let $n/d = d^2t + r$, $0 \le r < d^2$. Then*

$$S = \left(\frac{n}{d}g\right)^{d-1} g^{dt+r} \prod_{i=1}^{d-1}\left(\left(1 + \frac{in}{d}\right)g\right)^{dt}$$

is an unsplittable minimal zero-sum sequence over C_n. Moreover,

$$\mathrm{ind}(S) = \frac{n}{2d} - \frac{dt+r}{2} + 1.$$

Proof. We first show that S is a minimal zero-sum sequence over C_n. Assume that

$$T = \left(\frac{n}{d}g\right)^{u} g^{v} \prod_{i=1}^{d-1}\left(\left(1 + \frac{in}{d}\right)g\right)^{x_i}$$

is a zero-sum subsequence of S. Then $\sigma(T) = 0$, i.e.

$$(v + x_1 + \cdots + x_{d-1}) + \frac{n}{d}\big(u + x_1 + 2x_2 + \cdots + (d-1)x_{d-1}\big) \equiv 0 \pmod{n}, \tag{4.1}$$

where $x_i \in [0, dt]$, $u \in [0, d-1]$, $v \in [0, dt+r]$. If $v = 0$ and $x_i = 0$ for all $i \in [1, d-1]$, by (4.1), we derive that $un/d \equiv 0 \pmod{n}$, which is impossible since $u \le d-1$. Thus $0 < v + x_1 + \cdots + x_{d-1} \le dt + r + dt + \cdots + dt = d^2t + r = n/d$. Since $v + x_1 + \cdots + x_{d-1} \equiv 0 \pmod{n/d}$ by (4.1), we get $v = dt + r$ and $x_i = dt$ for all $i \in [1, d-1]$, so $T = S$. Therefore, S is a minimal zero-sum sequence over C_n.

Next we show that S is an unsplittable minimal zero-sum sequence over C_n. We need only show that ng/d, g and $(1 + in/d)g$, $i \in [0, d-1]$, are unsplittable. We first show that ng/d is unsplittable. By Lemma 4.4, we need to show that $\sum S(ng/d)^{-1} = G \setminus \{0\}$. Since $d(1 + in/d)g = dg$ for $i \in [1, d-1]$, we have

$$\sum\left(S\left(\frac{n}{d}g\right)^{-1}\right) \supset \sum\left(\left(\frac{n}{d}g\right)^{d-2} g^{dt+r}(dg)^{(d-1)t}\right) = \left\{g, \ldots, \frac{(d-1)n}{d}g\right\}.$$

Since $i(1 + in/d)g = (i + jn/d)g$ for each $i \in [1, d-1]$, where $j = |i^2|_d$, and $(d-j-1)ng/d = (d-j-1)ng/d$, we conclude that if $1 \le j \le d-2$, then

$$\begin{aligned}
&\sum\left(S\left(\frac{n}{d}g\right)^{-1}\right) \\
&\quad \supset \sum\left(\left(\frac{(d-j-1)n}{d}g\right)g^{dt+r}(dg)^{t-1}(dg)^{(d-2)t}\left(\left(i + \frac{jn}{d}\right)g\right)\right) \\
&\quad \supset \left\{\left(\frac{(d-1)n}{d} + i\right)g, \ldots, (n-d+i)g\right\}.
\end{aligned}$$

Note that if $i = 1$ or $d-1$, then $j = 1$. Thus

$$\begin{aligned}
\sum\left(S\left(\frac{n}{d}g\right)^{-1}\right) &\supset \left\{\left(\frac{(d-1)n}{d} + 1\right)g, \ldots, (n-d+1)g\right\} \\
&\quad \cup \left\{\left(\frac{(d-1)n}{d} + d - 1\right)g, \ldots, (n-d+d-1)g\right\} \\
&= \left\{\left(\frac{(d-1)n}{d} + 1\right)g, \ldots, (n-1)g\right\} \\
&\qquad \text{(since } (d-1)n/d + d - 1 < n - d + 1\text{)}.
\end{aligned}$$

It follows that $\sum(S(ng/d)^{-1}) = G \setminus \{0\}$. Therefore, ng/d is unsplittable.

Since $d(1 + in/d)g = dg$ for $i \in [1, d-1]$, we have

$$\sum(Sg^{-1}) \supset \sum\left(\left(\frac{n}{d}g\right)^{d-1} g^{dt+r-1}(dg)^{(d-1)t}\right) = G \setminus \{0\}.$$

Hence g is unsplittable.

For $(1 + in/d)g$, $i \in [1, d-1]$, we need only show that $(1 + n/d)g$ is unsplittable (the other proofs are similar and we omit the details). Since

$(1+n/d)g+(1+(d-1)n/d)g=2g$ and $ng/d+(1+(d-1)n/d)g=g$, we have

$$\sum\left(S\left(\left(1+\frac{n}{d}\right)g\right)^{-1}\right)\supset\sum\left(g^{dt+r}(dg)^{(d-1)t-2}(2g)^{d-1}g\right)=\left\{g,\ldots,\left(\frac{n}{d}-1\right)g\right\}$$

and

$$\begin{aligned}\sum\left(S\left(\left(1+\frac{n}{d}\right)g\right)^{-1}\right)&\supset\sum\left(\left(\frac{n}{d}g\right)^{d-2}g^{dt+r}(dg)^{(d-1)t-2}(2g)^{d-1}g\right)\\&=\left\{g,\ldots,\left(\frac{(d-1)n}{d}-1\right)g\right\}.\end{aligned}$$

Since

$$\begin{aligned}&\sum\left(S\left(\left(1+\frac{n}{d}\right)g\right)^{-1}\right)\\&\quad\supset\sum\left(\left(\frac{n}{d}g\right)^{d-1}g^{dt+r}(dg)^{(d-1)t-2}(2g)^{d-1}\left(1+\frac{(d-1)n}{d}\right)g\right)\\&\quad\supset\left\{\frac{(d-1)n}{d}g,\ldots,(n-1)g\right\},\end{aligned}$$

it follows that $\sum(S((1+n/d)g)^{-1})=G\setminus\{0\}$. Therefore, $(1+n/d)g$ is unsplittable.

Finally, we compute the index of the above sequence. Since

$$\|S\|_g=\frac{1}{n}\left(n+dt\sum_{i=1}^{d-1}\frac{in}{d}\right)=\frac{d(d-1)t}{2}+1=\frac{n}{2d}-\frac{dt+r}{2}+1,$$

we have $\operatorname{ind}(S)\le\frac{n}{2d}-\frac{dt+r}{2}+1$. On the other hand, let $g=jh$ with $\gcd(j,n)=1$ and write $j=j_0+ns/d$, $1\le j_0<n/d$. By (3.2), we have

$$\|S\|_h>\frac{d(d-1)t}{2}=\frac{n}{2d}-\frac{dt+r}{2},$$

which implies that $\operatorname{ind}(S)\ge\frac{n}{2d}-\frac{dt+r}{2}+1$, and we are done. ∎

By using a similar argument to that for Theorem 4.5, we can prove the following result.

PROPOSITION 4.6. *Let $n,d\ge 3$ be odd positive integers with $d\,|\,n$ and $n>3d^2$. Let $n/d=3dt+r$, $0\le r<3d$. Then*

$$S=\left(\frac{n}{d}g\right)^{d-1}g^{dt+r}\left(\left(1+\frac{n}{d}\right)g\right)^{dt}\left(\left(1+\frac{(d-1)n}{d}\right)g\right)^{dt}$$

is an unsplittable minimal zero-sum sequence over C_n.

Let S be the same sequence as in Theorem 4.5. Then

$$\operatorname{ind}(S) = \frac{n}{2d} - \frac{dt+r}{2} + 1 = \frac{(d-1)n}{2d^2} - \frac{r(d-1)}{2d} + 1 \le \frac{(d-1)n}{2d^2} + 1.$$

We remark that when n is even, [16, Theorem 3.1] provides an example of an unsplittable minimal zero-sum sequence S such that $\operatorname{ind}(S) \le n/8 + 1 = \frac{(d-1)n}{2d^2} + 1$ with $d = 2$.

We close the paper by making the following conjecture for $\mathrm{MI}(C_n)$ based on the above information.

Conjecture 4.7. *Let n be a composite positive integer and let p be the least prime divisor of n. Then*

$$\mathrm{MI}(C_n) \le \frac{(p-1)n}{2p^2} + 1.$$

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