ACTA ARITHMETICA
122.4 (2006)

On the number of solutions of simultaneous Pell equations II

by

Michael A. Bennett (Vancouver), Mihai Cipu (Bucureşti),
Maurice Mignotte (Strasbourg) and Ryotaro Okazaki (Kyoto)

1. Introduction. In the study of Diophantine equations, there arise situations where a given equation is known to have at most finitely many solutions, but where a more quantitative result is not available. For example, if we wish to deduce upper bounds for the number of integral points on a curve defined by $F(x,y)=0$ where $F \in \mathbb{Z}[x,y]$, in many cases of interest, such as those corresponding to parametrized families of elliptic curves, the dependence (or lack thereof) of the bound upon the coefficients of F is unclear (though there are a number of conjectures of related interest).

In some cases, however, rather precise information is available. If a and b are distinct positive integers, then one can show that the number of integral solutions to the simultaneous Diophantine equations

$$x^2 - az^2 = 1, \quad y^2 - bz^2 = 1 \tag{1}$$

is bounded independent of a and b. In fact, a result of the first author (Theorem 1.1 of [4]) implies that there are at most three solutions in positive integers (x,y,z); in special cases, even more precise information is available (cf. e.g. [11] and [12]). We note that equations of this and similar forms arise in a variety of contexts (see e.g. [2], [7] and [8]).

Relatively recently, Yuan [13] strengthened the main result of [4], proving

Theorem 1.1 (Yuan [13]). *If a and b are distinct positive integers with $\max\{a,b\} > 1.4 \cdot 10^{57}$, then the system of equations* (1) *has at most two solutions in positive integers* (x,y,z).

The principal tool in this sharpening is an improved "gap principle" for solutions to (1), arising from a careful study of properties of binary recur-

2000 *Mathematics Subject Classification*: Primary 11D09, 11D25; Secondary 11B39, 11J13, 11J86.

Key words and phrases: simultaneous Diophantine equations, linear forms in logarithms.

Research supported in part by a grant from NSERC.

rence sequences. In this paper, we will apply the arguments of [4] and [13], in conjunction with a new gap principle from unpublished work of the fourth author, to prove

THEOREM 1.2. *If a and b are distinct positive integers, then the system of equations* (1) *has at most two solutions in positive integers* (x, y, z).

As we shall see later, there exist infinitely many pairs (a, b) for which (1) has precisely two such positive solutions, whereby the stated bound is sharp. Further, let us note that this theorem supersedes the main result of [5] (which itself depends upon hitherto unpublished work of Voutier).

The organization of this paper is as follows. We begin by stating a pair of results which enable us to ensure that solutions to (1) are not too close together (in height). In the remainder of the paper, we combine these with lower bounds for linear forms in (three) complex logarithms, an inequality derived from the hypergeometric method of Thue and Siegel, elementary arguments, and a number of reasonably routine computations, to complete the proof of Theorem 1.2.

This paper actually arose from independent work of Bennett and Okazaki, and of Cipu and Mignotte; it is essentially the former. While the details of these differ somewhat, we feel that they are similar enough to warrant joint publication. We will attempt to indicate, as we proceed, where the proof of Cipu and Mignotte differs from that presented here; an expanded version of this other proof can be found in [6].

Arguments similar to those given in this paper may be applied to sharpen various related results (on other families of simultaneous quadratic equations), such as those in Yuan [14]. We will not undertake this here.

2. Gap principles. Let us assume, here and henceforth, that $b > a$ are positive nonsquare integers. We begin by noting that if (x_i, y_i, z_i) is a positive solution to (1), then we may write

$$z_i = \frac{\alpha^{j_i} - \alpha^{-j_i}}{2\sqrt{a}} = \frac{\beta^{k_i} - \beta^{-k_i}}{2\sqrt{b}} \tag{2}$$

where α and β correspond to the fundamental solutions to the equations $x^2 - az^2 = 1$ and $y^2 - bz^2 = 1$ respectively (i.e. the fundamental units in $\mathbb{Q}(\sqrt{a})$ or $\mathbb{Q}(\sqrt{b})$, or small powers thereof) and j_i and k_i are positive integers. As noted in [13], we may assume, without loss of generality, that $a = m^2 - 1$ and $b = n^2 - 1$ with $n > m > 1$ integers (and hence that $\alpha = m + \sqrt{m^2 - 1}$ and $\beta = n + \sqrt{n^2 - 1}$). Let us suppose that (2) holds for $1 \le i \le 3$ where $1 = j_1 < j_2 < j_3$ (since we assume $a = m^2 - 1$, $b = n^2 - 1$, we have $z_1 = 1$).

In this section, we provide a pair of results which ensure that solutions to (1) cannot lie too "close together". The first of these follows immediately from Lemmata 2.2, 2.4 and the proofs of Lemmata 2.6 and 2.7 of [13]. We note that these lemmata, while correct as stated in [13], have proofs which, in a number of cases, are in need of serious repair. For details, we direct the reader to [6].

LEMMA 2.1 (Yuan [13]). *Suppose that a and b are distinct, nonsquare positive integers such that there exist three positive solutions (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) to* (1) *with corresponding j_i satisfying $j_1 < j_2 < j_3$. Then there exists a positive integer q such that either*

(i) $j_3 = qj_2$ *with* $q \geq z_2$, *or*

(ii) $j_3 = 2qj_2 \pm 1$, *with* $q \geq \begin{cases} n/m & \text{if } k_2 = 2, \\ \sqrt{z_2}/n & \text{if } k_2 \text{ is odd}, \\ \sqrt{z_2/n} & \text{if } k_2 \geq 4 \text{ is even.} \end{cases}$

Further, $k_2 \neq 3$.

To apply this result later, it will be helpful to note that

$$z_2 = \frac{\beta^{k_2} - \beta^{-k_2}}{2\sqrt{n^2-1}} \geq 2n\beta^{k_2-2}. \tag{3}$$

For certain "small" values of (j_2, k_2) we will eschew Lemma 2.1 in favor of the following result, which provides a more analytic gap principle, reminiscent of Lemma 2.2 of [4]. Though it usually gives weaker bounds than Lemma 2.1, it yields an improvement precisely in the few cases that represent the majority of our computations.

LEMMA 2.2. *Suppose that a and b are distinct, nonsquare positive integers such that there exist three positive solutions (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) to* (1) *with corresponding α, β, j_i and k_i satisfying $j_1 < j_2 < j_3$ (whence $k_1 < k_2 < k_3$). Then*

$$j_3 - j_2 > 2j_1k_1\log(\beta)(\alpha^{2j_1} - 1).$$

Proof. From our suppositions, we have three points

$$(t_i, u_i) = (j_i\log(\alpha), k_i\log(\beta)) \quad (1 \leq i \leq 3)$$

on the curve

$$\sinh(u) = \sqrt{b/a}\,\sinh(t). \tag{4}$$

Since $b > a$ by assumption, we see that $u > t$ and, taking logarithms,

$$u - t = \log\left(\frac{1 - e^{-2t}}{1 - e^{-2u}}\right) + \frac{1}{2}\log(b/a),$$

whereby, from calculus,

$$\frac{-1}{e^{2t} - 1} < u - t - \frac{1}{2}\log(b/a) < 0. \tag{5}$$

Since

$$u = \sinh^{-1}(\sqrt{b/a}\,\sinh(t))$$

is an analytic solution to (4), we may implicitly differentiate (4) to find that

$$\frac{du}{dt}\cosh(u) = \sqrt{b/a}\,\cosh(t), \tag{6}$$

whereby

$$\frac{du}{dt} = \sqrt{\frac{(b/a)(\sinh^2(t)+1)}{\sinh^2(u)+1}} = \sqrt{\frac{\sinh^2(t)+1}{\sinh^2(t)+a/b}} > 1. \tag{7}$$

Similarly, implicitly differentiating (6) yields

$$\frac{d^2u}{dt^2}\cosh(u) + \left(\frac{du}{dt}\right)^2 \sinh(u) = \sqrt{b/a}\,\sinh(t) = \sinh(u)$$

and so

$$\frac{d^2u}{dt^2} = \left(1 - \left(\frac{du}{dt}\right)^2\right)\tanh(u) < 0. \tag{8}$$

The Mean Value Theorem, together with inequalities (7) and (8), thus implies that

$$0 < \frac{u_2-u_1}{t_2-t_1} - \frac{u_3-u_2}{t_3-t_2} < \frac{u_2-u_1}{t_2-t_1} - 1.$$

On the other hand, from (5) and (7),

$$0 < -u_2 + t_2 + \frac{1}{2}\log(b/a) < -u_1 + t_1 + \frac{1}{2}\log(b/a) < \frac{1}{e^{2t_1}-1},$$

whereby

$$0 < (u_2-u_1) - (t_2-t_1) < \frac{1}{e^{2t_1}-1}$$

and so

$$0 < \frac{u_2-u_1}{t_2-t_1} - \frac{u_3-u_2}{t_3-t_2} < \frac{1}{(t_2-t_1)\,(e^{2t_1}-1)}.$$

It follows that

$$0 < \frac{k_2-k_1}{j_2-j_1} - \frac{k_3-k_2}{j_3-j_2} < \frac{1}{(j_2-j_1)\log(\beta)(\alpha^{2j_1}-1)}.$$

Writing

$$\Delta = \begin{vmatrix} k_2-k_1 & k_3-k_2 \\ j_2-j_1 & j_3-j_2 \end{vmatrix},$$

we thus have

$$j_3 - j_2 > \Delta(\alpha^{2j_1}-1)\log(\beta) > 0.$$

To complete the proof of Lemma 2.2, we need only show that $\Delta \geq 2j_1k_1$. Since the arguments of [4] (see page 193) imply that

$$j_3 \equiv j_2 \equiv 0 \pmod{j_1}, \qquad k_3 \equiv k_2 \equiv 0 \pmod{k_1},$$

it follows, upon expanding

$$x_2 + (z_2/z_1)\sqrt{az_1^2} = (x_1 + \sqrt{az_1^2})^{j_2/j_1}$$

by the binomial theorem and noting that x_1 and az_1^2 have opposite parities, that

$$z_2/z_1 \equiv j_2/j_1 \pmod{2}.$$

Similarly, we may demonstrate the congruence

$$z_2/z_1 \equiv k_2/k_1 \pmod{2}$$

and thus conclude that

$$j_2/j_1 \equiv k_2/k_1 \pmod{2}. \tag{9}$$

Arguing in a like manner gives

$$j_3/j_1 \equiv k_3/k_1 \pmod{2},$$

and hence

$$\frac{\varDelta}{j_1 k_1} = \begin{vmatrix} \frac{k_2}{k_1} - 1 & \frac{k_3}{k_1} - \frac{k_2}{k_1} \\ \frac{j_2}{j_1} - 1 & \frac{j_3}{j_1} - \frac{j_2}{j_1} \end{vmatrix}$$

is a positive even integer. This completes the proof of Lemma 2.2. ∎

3. Linear forms in logarithms. The final ingredient we require to prove Theorem 1.2 is the following lower bound for linear forms in logarithms of algebraic numbers, due to Matveev; here we have specialized Corollary 2.3 of [10] to the case where the algebraic numbers in question lie in a totally real field. In what follows, $h(\gamma)$ denotes the logarithmic Weil height of an algebraic number γ. We note that the approach of the second and third authors differs from that given in this paper, by appealing to recent more refined estimates, specialized to the case of three logarithms. Had we employed these sharpened bounds here, our later computations would have been reduced.

PROPOSITION 3.1 (Matveev [10]). *Suppose that $\mathbb{K}$ is a number field with $\mathbb{K} \subset \mathbb{R}$ and $D = [\mathbb{K} : \mathbb{Q}]$. Let $b_i \in \mathbb{Z}$, $\alpha_i \in \mathbb{K}^*$ for $1 \le i \le n$, and suppose that*

$$[\mathbb{K}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}) : \mathbb{K}] = 2^n.$$

If we define

$$B = \max\{|b_1|, \ldots, |b_n|\}$$

and suppose that

$$A_j \ge \max\{Dh(\alpha_j), |\log(\alpha_j)|\}$$

for $1 \le j \le n$, then if

$$\varLambda = b_1 \log(\alpha_1) + \cdots + b_n \log(\alpha_n) \ne 0,$$

we have

$$\log(|\Lambda|) > -C(n)D^2 A_1 \cdots A_n \log(eD)\log(eB)$$

where

$$C(n) = \min\left\{\frac{e}{2}\, 30^{n+3} n^{4.5}, 2^{6n+20}\right\}.$$

In our situation, we take

$$\Lambda = \log(\sqrt{b/a}) + j_3 \log(\alpha) - k_3 \log(\beta)$$

and hence (see displayed equation (6) of Yuan [13]) have

$$\log(\Lambda) < -2j_3 \log(\alpha) + \log\left(\frac{\alpha^2}{\alpha^2 - 1}\right). \tag{10}$$

Applying Proposition 3.1, we may, via Lemma 2.5 of [13], assume that $D = 4$ and take

$$A_1 = 2\log(n^2 - 1) < 4\log(\beta), \qquad A_2 = 2\log(\beta), \qquad B = j_3.$$

We thus conclude that

$$\log(\Lambda) > -8.5 \cdot 10^{13} \log(\alpha) \log^2(\beta) \log(ej_3).$$

Combining this with (10) thus implies that

$$\frac{j_3}{\log(ej_3)} < 4.25 \cdot 10^{13}\, \log^2(\beta) + 0.5 \log\left(\frac{\alpha^2}{\alpha^2 - 1}\right) \log(\alpha)^{-1} \log(ej_3)^{-1}$$

and hence

$$\frac{j_3}{\log(ej_3)} < 4.26 \cdot 10^{13}\, \log^2(\beta), \tag{11}$$

where the last inequality is a consequence of the fact that

$$\beta > \alpha \geq 2 + \sqrt{3}.$$

4. The case $k_2 = 2$. We are now in a position to begin the proof of Theorem 1.2. Let us first suppose that $k_2 = 2$ (as mentioned previously, we may assume that $j_1 = k_1 = 1$). From (9) we deduce the existence of an integer $l > 1$ such that $j_2 = 2l$. It follows, if $a = m^2 - 1$ and $b = n^2 - 1$, that

$$n = n(l, m) = \frac{\alpha^{2l} - \alpha^{-2l}}{4\sqrt{m^2 - 1}},$$

and hence, in these cases, that equations (1) have in fact two solutions, given by

$$(x_1, y_1, z_1) = (m, n(l, m), 1),$$
$$(x_2, y_2, z_2) = \left(2n(l, m)m - \frac{n(l, m)}{m} - \frac{n(l - 1, m)}{m}, 2n(l, m)^2 - 1, 2n(l, m)\right).$$

We note that m is readily seen to divide $n(l,m)$ for all l, whence this second solution is in fact integral.

If we have $l=2$, then combining Lemma 2.2 with inequality (11) implies that $m < 1.168 \cdot 10^8$. For larger values of l, Lemma 2.1 and (11) together yield the inequalities

(12)

$l=3$	$m \leq 12583$	$l=8$	$m \leq 8$
$l=4$	$m \leq 398$	$l=9$	$m \leq 5$
$l=5$	$m \leq 72$	$l=10$	$m \leq 4$
$l=6$	$m \leq 26$	$l=11$	$m \leq 3$
$l=7$	$m \leq 13$	$12 \leq l \leq 15$	$m=2$

If $l \geq 16$, we derive an immediate contradiction to the fact that $m \geq 2$.

To deal with the remaining pairs (m,n), we note that, as in the argument leading to the displayed equation (12) of [4], we have

$$0 < \frac{j_{i+1}-j_i}{k_{i+1}-k_i} - \frac{\log(\beta)}{\log(\alpha)} < (\log(\alpha)(k_{i+1}-k_i)(\alpha^{2j_i}-1))^{-1}. \tag{13}$$

If $k_2 = 2$, $j_2 = 4$ and $2 \leq m < 1.168 \cdot 10^8$, it follows from (11) that $j_3 < 6.31 \cdot 10^{18}$, whereby, from (13), $k_3 < 2.11 \cdot 10^{18}$. It follows that

$$(\alpha^8 - 1)\log(\alpha) > 2\,(k_3 - 2),$$

at least provided $m \geq 87$, and hence, for such values of m, $\frac{j_3-4}{k_3-2}$ is an (even-indexed) convergent in the infinite simple continued fraction expansion to $\frac{\log(\beta)}{\log(\alpha)}$, say $\frac{j_3-4}{k_3-2} = \frac{p_t}{q_t}$. Combining (13) with the inequality

$$\left|\frac{\log(\beta)}{\log(\alpha)} - \frac{p_t}{q_t}\right| > \frac{1}{(a_{t+1}+2)q_t^2}$$

(see e.g. [9]; here, a_{t+1} denotes the $(t+1)$st partial quotient in the continued fraction expansion to $\frac{\log(\beta)}{\log(\alpha)}$), we deduce the existence of an integer t such that

$$a_{t+1} > \frac{(\alpha^8-1)\log(\alpha)}{2.11 \cdot 10^{18}} - 2,$$

while the corresponding convergent has $q_t < 2.11 \cdot 10^{18}$. A (long) calculation using Pari GP confirms that no such t exists, provided $197 < m < 1.168 \cdot 10^8$. For smaller values of m, occasionally this inequality is satisfied; checking all suitably small convergents to $\frac{\log(\beta)}{\log(\alpha)}$ for $87 \leq m \leq 197$ leads to no further solutions to (1) in these cases. Finally, if $2 \leq m \leq 86$, we may argue as in Baker and Davenport [3] (see also Anglin [1]). For fixed (a,b), it is possible to algorithmically solve equation (1), via a bound like that given by Proposition 3.1, in conjunction with a result from computational Diophantine approximation, due to Baker and Davenport (essentially a precursor to the L^3 lattice basis reduction algorithm). Applying such arguments in a standard way completes the proof of Theorem 1.2, in case $k_2 = 2$ and $j_2 = 4$. The cases with $k_2 = 2$ and $3 \le l \le 15$, for m as in (12), are similar, though the computations required are much easier.

5. The cases $4 \le k_2 \le 8$**.** We will provide details for $k_2 = 4$ and $k_2 = 5$; the other values may be treated in a similar, computationally simpler fashion. Let us suppose first that $k_2 = 4$ and $j_2 = 6$. In this case, Lemma 2.2 implies

$$j_3 \ge 2\log(\beta)(\alpha^2 - 1) + 6. \tag{14}$$

Since we have, in general,

$$\alpha^{j_2 - 1} > \frac{\alpha^2 - 1}{\alpha^2}\,\beta^{k_2 - 1},$$

if we assume that $m > 10^7$ (so that $\alpha > 1.99 \cdot 10^7$), then

$$\alpha > 0.99\beta^{3/5}$$

and so, from $k_2 = 4$, $j_2 = 6$ and (14),

$$j_3 > 1.96\beta^{6/5}\log(\beta).$$

Combining this with inequality (11) leads to the conclusion that $\beta < 5.32 \cdot 10^{13}$ and hence $n < 2.67 \cdot 10^{13}$. Since we have

$$8n^3 - 4n = 32m^5 - 32m^3 + 6m,$$

it follows that m is necessarily even, say $m = 2m_1$, whereby we may write

$$2n^3 - n = 256m_1^5 - 64m_1^3 + 3m_1.$$

If we define

$$\theta = (128m_1^5)^{1/3},$$

we may rewrite this as

$$(\theta - 2^{4/5}3^{-1}\theta^{-1/5})^3 - \left(n - \frac{1}{6n}\right)^3 = \frac{7m_1}{6} - \frac{2}{27m_1} - \frac{1}{12n} + \frac{1}{216n^3}.$$

We thus have $\theta > n$ and hence a careful application of the Mean Value Theorem implies that

$$\{(128m_1^5)^{1/3}\} < \frac{1}{2m_1^{1/3}}, \tag{15}$$

where $\{x\}$ denotes the fractional part of a real number x. Further, the upper bound upon n implies that

$$m_1 < 4.31 \cdot 10^7. \tag{16}$$

To deal with these remaining cases, for each positive integer m_1 for which both (15) and (16) hold (we note that the great majority of values m_1 in the

range (16) fail to satisfy (15)), we use Pari GP to verify that the polynomial
$$p(x) = 2x^3 - x - 256m_1^5 + 64m_1^3 - 3m_1$$
is irreducible over $\mathbb{Q}[x]$, thus ensuring that it has no integral roots. This calculation, while tiresome, is not especially challenging.

Next, suppose that $k_2 = 4$ and $j_2 \geq 8$ is even (as noted earlier, k_2 and j_2 have the same parity). We apply Lemma 2.1 and inequality (3) to conclude that
$$j_3 \geq 2\sqrt{2}\,\beta j_2 - 1.$$
Comparing this with (11) and using the inequality
$$m + \sqrt{m^2 - 1} = \alpha < \beta^{3/(j_2-1)},$$
we thus conclude that $2 \leq m \leq m_0$, where:

j_2	m_0	j_2	m_0	j_2	m_0	j_2	m_0
8	10695892	20	215	32	19	$44 \leq j_2 \leq 46$	6
10	232377	22	119	34	15	$48 \leq j_2 \leq 50$	5
12	20558	24	73	36	12	$52 \leq j_2 \leq 54$	4
14	3862	26	48	38	10	$56 \leq j_2 \leq 64$	3
16	1138	28	34	40	9	$66 \leq j_2 \leq 84$	2
18	448	30	25	42	7		

If $j_2 > 84$, we contradict $m \geq 2$. To complete the case $k_2 = 4$, we argue as for $j_2 = 6$; each choice of j_2 leads to an equation of the form
$$8n^3 - 4n = f_{j_2}(m)$$
where $f_{j_2} \in \mathbb{Z}[x]$ has degree $j_2 - 1$, and $m \leq m_0$. Routine calculations show that no unexpected solutions to (1) accrue.

If $k_2 = 5$, then working modulo 4, we necessarily have $j_2 \equiv 1 \pmod 4$ and so $j_2 \geq 9$. If $j_2 = 9$, then
$$16n^4 - 12n^2 + 1 = 256m^8 - 448m^6 + 240m^4 - 40m^2 + 1$$
and so
$$(128m^4 - 112m^2 + 11)^2 - (32n^2 - 12)^2 = 96m^2 - 23.$$
On the other hand,
$$(128m^4 - 112m^2 + 11)^2 - (128m^4 - 112m^2 + 10)^2 = 256m^4 - 224m^2 + 21.$$
Since
$$0 < 96m^2 - 23 < 256m^4 - 224m^2 + 21$$
for every integer $m > 1$, we thus have
$$128m^4 - 112m^2 + 10 < 32n^2 - 12 < 128m^4 - 112m^2 + 11,$$
a contradiction.

If $k_2 = 5$ and $j_2 \geq 13$, then from Lemma 2.1 we derive the inequality

$$j_3 > 4\sqrt{4n^2 - 3}\, j_2 - 1,$$

whereby, with (11) and $\alpha < \beta^{4/(j_2-1)}$, we find that $m \leq m_0$, for m_0 as follows:

j_2	m_0	j_2	m_0	j_2	m_0
13	187577	37	36	61	6
17	7579	41	23	65	5
21	1105	45	16	$69 \leq j_2 \leq 73$	4
25	306	49	12	$77 \leq j_2 \leq 85$	3
29	122	53	9	$89 \leq j_2 \leq 117$	2
33	61	57	7		

Again, if $j_2 > 117$, we conclude that $m < 2$, a contradiction. Calculations as in the case $k_2 = 4$ complete the proof of Theorem 1.2 when $k_2 = 5$. Similar, computationally less intensive arguments apply for $6 \leq k_2 \leq 8$.

6. The cases $k_2 \geq 9$. To treat the remaining values of k_2, we will appeal to a result of the first author (Corollary 3.3 of [4]). In our situation, this implies the inequality

$$k_3 < \left(\frac{7k_2^2 + 17k_2 - 32}{k_2^2 - 9k_2 + 8}\right) k_2 \leq 86k_2,$$

whereby, since $k_2 \geq 9$,

$$\frac{k_3 - k_2}{k_2 - 1} < 85 + \frac{85}{k_2 - 1} < 97. \tag{17}$$

On the other hand, $k_2 \geq 9$ implies

$$z_2 \geq 256n^8 - 448n^6 + 240n^4 - 40n^2 + 1,$$

whence, from the fact that $n \geq 3$, Lemma 2.1 ensures the inequality

$$j_3 > 780 j_2 - 1$$

and so

$$\frac{j_3 - j_2}{j_2 - 1} > 779. \tag{18}$$

From (13) with $i = 1$ and $i = 2$, we have

$$\left|\frac{j_3 - j_2}{k_3 - k_2} - \frac{j_2 - 1}{k_2 - 1}\right| < (\log(\alpha)(k_2 - 1)(\alpha^2 - 1))^{-1}$$

and so

$$\left|\frac{j_3 - j_2}{j_2 - 1} - \frac{k_3 - k_2}{k_2 - 1}\right| < (\log(\alpha)(j_2 - 1)(\alpha^2 - 1))^{-1}\left(\frac{k_3 - k_2}{k_2 - 1}\right).$$

This inequality, together with (17) and (18), leads to a contradiction, completing the proof of Theorem 1.2.

References

[1] W. S. Anglin, *Simultaneous Pell equations*, Math. Comp. 65 (1996), 355–359.

[2] —, *The Queen of Mathematics: An Introduction to Number Theory*, Kluwer, Dordrecht, 1995.

[3] A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. (2) 20 (1969), 129–137.

[4] M. A. Bennett, *On the number of solutions of simultaneous Pell equations*, J. Reine Angew. Math. 498 (1998), 173–199.

[5] —, *Solving families of simultaneous Pell equations*, J. Number Theory 67 (1997), 246–251.

[6] M. Cipu and M. Mignotte, *On the number of solutions of simultaneous Pell equations*, Université Louis Pasteur, U. F. R. de Mathématiques preprints, 20 pp.

[7] A. Dujella, *There are finitely many Diophantine quintuples*, J. Reine Angew. Math. 566 (2004), 183–214.

[8] —, *On the number of Diophantine m-tuples*, Ramanujan J., to appear.

[9] A. Ya. Khintchine [A. Ya. Khinchin], *Continued Fractions*, 3rd ed., P. Noordhoff, Groningen, 1963.

[10] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers II*, Izv. Ross. Akad. Nauk Ser. Mat. 64 (2000), no. 6, 125–180 (in Russian); English transl.: Izv. Math. 64 (2000), 1217–1269.

[11] P. G. Walsh, *On integer solutions to* $x^2 - dy^2 = 1$, $z^2 - 2dy^2 = 1$, Acta Arith. 82 (1997), 69–76.

[12] —, *Two classes of simultaneous Pell equations with no solutions*, Math. Comp. 68 (1999), 385–388.

[13] P. Z. Yuan, *On the number of solutions of simultaneous Pell equations*, Acta Arith. 101 (2002), 215–221.

[14] —, *Simultaneous Pell equations*, ibid. 115 (2004), 119–131.

Department of Mathematics
University of British Columbia
Vancouver, B.C., V6T 1Z2 Canada
E-mail: bennett@math.ubc.ca

Institute of Mathematics
Romanian Academy
P.O. Box 1-764
RO-014700 Bucureşti, Romania
E-mail: mihai.cipu@imar.ro

U.F.R. de Mathématiques
Université Louis Pasteur
7, Rue René Descartes
67084 Strasbourg Cedex, France
E-mail: mignotte@math.u-strasbg.fr

Department of Mathematics
Doshisha University
Kyoto, Japan
E-mail: rokazaki@mail.doshisha.ac.jp

Received on 18.7.2005
and in revised form on 18.3.2006 (5035)