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On a problem of Sierpiński

by

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1. Introduction. Let $s \geq 2$ be an integer. Denote by μ_s the least integer so that every integer $\ell > \mu_s$ is the sum of exactly s integers > 1 which are pairwise relatively prime. In 1964, Sierpiński [5] posed the problem of determining μ_s. Let $p_1 = 2, p_2 = 3, \ldots$ be the sequence of consecutive primes. In 1965, P. Erdős [3] proved that there exists an absolute constant C such that $\mu_s \leq p_2 + p_3 + \cdots + p_{s+1} + C$. It is easy to see that $p_2 + p_3 + \cdots + p_{s+1} - 2$ is not the sum of exactly s integers > 1 which are pairwise relatively prime. So $\mu_s \geq p_2 + p_3 + \cdots + p_{s+1} - 2$. Let $\mu_s = p_2 + p_3 + \cdots + p_{s+1} + c_s$. Then $-2 \leq c_s \leq C$. It is easy to see that $c_2 = -2$.

Let U be the set of integers of the form

$$p_2^{k_2} + p_3^{k_3} + \cdots + p_{11}^{k_{11}} - p_2 - p_3 - \cdots - p_{11} \leq 1100,$$

where k_i $(2 \leq i \leq 11)$ are positive integers. All elements of U can be listed explicitly by using Mathematica (see Appendix). Let V_s be the set of integers of the form

$$p_{i_1} + \cdots + p_{i_l} - p_{j_1} - \cdots - p_{j_l} \leq 1100,$$

where $2 \leq j_1 < \cdots < j_l \leq s + 1 < i_1 < \cdots < i_l$. It is clear that $0 \in U$ and $0 \in V_s$ $(l = 0)$. Define $U + V_s = \{u + v \mid u \in U, v \in V_s\}$. Then $U + V_s$ is finite.

In this paper the following results are proved. The main results were announced at ICM2010.

THEOREM 1.1. *Let $s \geq 2$ be any given positive integer. Then*

$$c_s = \max\{2n \mid 2n \leq \min\{1100, p_{s+2}\},\, n \in \mathbb{Z},\, 2n \notin U + V_s\}.$$

REMARK 1.2. As examples, by Theorem 1.1 we have $c_{500} = 16$, $c_{900} = 14$, $c_{1000} = 8$, $c_{2000} = 22$ (see the last section).

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Corollary 1.3. *If* $p_{s+2} - p_{s+1} > 1100$, *then*

$$\mu_s = \sum_{i=2}^{s+1} p_i + 1100.$$

In particular, the set of integers $s \geq 2$ *for which this equality is satisfied has asymptotic density* 1.

We pose a problem here:

Problem 1.4. *Find the least positive integer* s *with* $\mu_s = \sum_{i=2}^{s+1} p_i + 1100$.

Basing on the proof of Theorem 1.6 in Section 4, we make the following conjecture.

Conjecture 1.5. *For* $s \geq 3$, *every integer* $l > p_2 + p_3 + \cdots + p_{s+2}$ *is the sum of exactly* s *distinct primes.*

This conjecture would follow from the following statement: "Every odd integer $n \geq p_{s-1} + p_s + p_{s+1} + p_{s+2}$ can be written as the sum of three primes $q_1 < q_2 < q_3$ with $q_1 \geq p_{s-1}$". Since $p_{s-1} < n/4$, by well-known results on the odd Goldbach problem with almost equal primes, this statement is true for all sufficiently large s. Hence, Conjecture 1.5 is true for all sufficiently large s.

Now we sketch the proof of Theorem 1.1. For the details, see Section 4.

(1) We first find a "long" interval $[1102, 3858]$ such that each even number in this interval can be represented as $\sum_{i=2}^{\infty}(p_i^{t_i} - p_i)$. For any even number $2m > 3858$, there exists a prime p_u such that $p_u^2 - p_u \leq 2m - 1102 < p_{u+1}^2 - p_{u+1}$. Then we use the induction hypothesis on $2m - (p_u^2 - p_u)$. By these arguments we know that every even number $n \geq 1102$ can be represented as $\sum_{i=2}^{\infty}(p_i^{t_i} - p_i)$, where t_i are positive integers. One can verify that 1100 cannot be represented in that form.

(2) Denote by μ'_s the least integer, of the same parity as s, so that every integer $\ell > \mu'_s$ of the same parity as s can be expressed as the sum of s distinct integers > 1 which are pairwise relatively prime. Let $\mu'_s = p_2 + \cdots + p_{s+1} + \tau'_s$. Then τ'_s is even.

For $2n > \min\{1100, p_{s+2}\}$, if $\min\{1100, p_{s+2}\} < 2n \leq 1100$, then $s \leq 182$. By calculation we find that $\sum_{i=2}^{s+1} p_i + 2n$ can be expressed as the sum of s distinct odd primes. Now assume that $2n > 1100$. If $2n$ is "large", then we can choose a "large" prime q such that $p_{s+2} + 2n - q > \tau'_s$. By the induction hypothesis, $p_2 + \cdots + p_{s+1} + (p_{s+2} + 2n - q)$ can be expressed as the sum of s distinct integers > 1 which are pairwise relatively prime. Thus $p_2 + \cdots + p_{s+1} + p_{s+2} + 2n$ can be expressed as the sum of $s + 1$ distinct integers > 1 which are pairwise relatively prime. If $2n$ is "small", then by (1)

(we take some $t_i = 1$)

$$2n = \sum_{i=2}^{s+2} (p_i^{t_i} - p_i).$$

Thus

$$p_2 + \cdots + p_{s+1} + p_{s+2} + 2n = \sum_{i=2}^{s+2} p_i^{t_i}.$$

We can easily convert the case $p_2 + \cdots + p_{s+1} + p_{s+2} + 2n + 1$ into $p_1 + p_2 + \cdots + p_{s+1} + (p_{s+2} + 2n - 1)$ and use the induction hypothesis.

Recall that μ'_s is defined in (2) above, and $\tau'_s = \mu'_s - (p_2 + \cdots + p_{s+1})$ is even. The following theorem is a step in the proof of Theorem 1.1, and not an independent result.

THEOREM 1.6.

$$\tau'_s = \max\{2n \mid 2n \le \min\{1100, p_{s+2}\},\ n \in \mathbb{Z},\ 2n \notin U + V_s\}.$$

2. Preliminary lemmas. In this paper, p, q_i are always primes. First we introduce the following lemmas.

LEMMA 2.1 ([2, Lemma 4]). *For every $x > 24$ there exists a prime in $(x, \sqrt{3/2}\,x]$.*

LEMMA 2.2. *Every even number $n \ge 1102$ can be represented as $\sum_{i=2}^{\infty} (p_i^{t_i} - p_i)$, where t_i are positive integers. The integer 1100 cannot be represented in that form.*

Proof. The proof is by induction on even numbers n. For any sets X, Y of integers, define $X + Y = \{x + y : x \in X,\ y \in Y\}$. Let

$$\begin{aligned} U_4 &= \{0, 3^2 - 3, 3^3 - 3, 3^4 - 3, 3^5 - 3, 3^6 - 3, 3^7 - 3\} \\ &\qquad + \{0, 5^2 - 5, 5^3 - 5, 5^4 - 5\} + \{0, 7^2 - 7, 7^3 - 7\}, \\ U_i &= U_{i-1} \cup (U_{i-1} + \{p_i^2 - p_i\}), \qquad i = 5, 6, \ldots. \end{aligned}$$

Using Mathematica, we can list the elements of each U_i and verify that $[1102, 3858] \cap 2\mathbb{Z} \subseteq U_{12}$ and $1100 \notin U_{12}$.

Thus, if n is even with $1102 \le n \le 3858$, then n can be represented as $\sum_{i=2}^{\infty} (p_i^{t_i} - p_i)$, where t_i are positive integers.

Now assume that any even n with $1102 \le n < 2m$ ($2m > 3858$) can be represented as such a sum.

Since $2m - 1102 > 3858 - 1102 = 53^2 - 53$, there exists a prime $p_u \ge 53$ with

$$p_u^2 - p_u \le 2m - 1102 < p_{u+1}^2 - p_{u+1}. \tag{2.1}$$

Then

$$1102 \le 2m - (p_u^2 - p_u) < 2m.$$

By the induction hypothesis, we have

$$2m - (p_u^2 - p_u) = \sum_{i=2}^{\infty}(p_i^{t_i} - p_i),$$

where t_i are positive integers. Hence

$$2m = \sum_{i=2}^{\infty}(p_i^{t_i} - p_i) + (p_u^2 - p_u). \tag{2.2}$$

Now we prove that $t_u = 1$. Indeed, otherwise $t_u \geq 2$ and $2m \geq 2(p_u^2 - p_u)$. By (2.1) we have

$$2(p_u^2 - p_u) - 1102 \leq 2m - 1102 < p_{u+1}^2 - p_{u+1} < p_{u+1}^2 - p_u.$$

Thus

$$2p_u^2 - p_u - 1102 < p_{u+1}^2.$$

Since $p_u \geq 53$, by Lemma 2.1 we have $p_{u+1} \in (p_u, \sqrt{3/2}\, p_u]$. Since

$$\sqrt{3/2}\, p_u \leq \sqrt{2p_u^2 - p_u - 1102},$$

we have

$$p_{u+1}^2 \leq 2p_u^2 - p_u - 1102,$$

a contradiction.

So $t_u = 1$, and by (2.2), $2m$ can be represented in the desired form, completing the proof of the first assertion of the lemma.

Suppose now that 1100 can be expressed as $\sum_{i=2}^{\infty}(p_i^{t_i} - p_i)$, where t_i are positive integers. Then $p_i^{t_i} - p_i \leq 1100$ for all i. If $t_i \geq 2$, then $p_i^2 - p_i \leq 1100$. Thus $p_i < 37$. So $i < 12$. If $t_i \geq 3$, then $p_i^3 - p_i \leq 1100$. Thus $p_i \leq 7 = p_4$. As $p_2^{t_2} - p_2 \leq 1100$ we have $t_2 \leq 6$. As $p_3^{t_3} - p_3 \leq 1100$ we have $t_3 \leq 4$. As $p_4^{t_4} - p_4 \leq 1100$ we have $t_4 \leq 3$. Hence $1100 \in U_{12}$, a contradiction. ∎

LEMMA 2.3. *If $2n < p_{s+2}$ and $\sum_{i=2}^{s+1} p_i + 2n$ is the sum of exactly s integers > 1 which are pairwise relatively prime, then $\sum_{i=2}^{s+1} p_i + 2n$ can be expressed as the sum of powers of s distinct odd primes.*

Proof. Let

$$\sum_{i=2}^{s+1} p_i + 2n = \sum_{i=1}^{s} m_i,$$

where $1 < m_1 < \cdots < m_s$ and $(m_i, m_j) = 1$ for $1 \leq i, j \leq s$, $i \neq j$. By comparing the parities we know that the s integers m_i must all be odd. If one of them has at least two distinct prime factors, then the sum of these s integers is at least $3 \times 5 + p_4 + \cdots + p_{s+2} = p_2 + \cdots + p_{s+1} + p_{s+2} + 7$. This contradicts $2n \leq p_{s+2}$. ∎

3. Proof of Theorem 1.6. For $s \ge 2$, let

$$H(s) = \{p_j - p_i : 2 \le i \le s+1 < j \le 185\} \cup \{p_u + p_v - p_s - p_{s+1} : s \le u \le 105,\ u < v \le 180\}.$$

Using Mathematica, we find that $[p_{s+2}, 1100] \cap 2\mathbb{Z} \subseteq H(s)$ for $2 \le s \le 182$. Thus, for $p_{s+2} < 2n \le 1100$, $\sum_{i=2}^{s+1} p_i + 2n$ can be expressed as the sum of s distinct odd primes.

Let h_s be the largest even number $2n \le 1100$ such that $\sum_{i=2}^{s+1} p_i + 2n$ cannot be expressed as the sum of s distinct integers > 1 which are pairwise relatively prime. Noting that $p_{s+2} > 1100$ for $s \ge 183$, by the above arguments we have $h_s \le \min\{1100, p_{s+2}\}$ for all $s \ge 2$.

We will use induction on s to prove that $\tau'_s = h_s$ for all $s \ge 2$.

For every even $\ell > 6$, we have $\phi(\ell) > 2$, where $\phi(\ell)$ is Euler's totient function. Hence there exists an integer n with $2 \le n \le \ell - 2$ and $(n, \ell) = 1$. So

$$\ell = n + (\ell - n), \quad (n, \ell - n) = 1, \quad n \ge 2,\ \ell - n \ge 2.$$

Thus $\tau'_2 = -2 = h_2$. Suppose that $\tau'_s = h_s$. Now we prove that $\tau'_{s+1} = h_{s+1}$.

Let ℓ be an integer which has the same parity as $s+1$. Then we can write

$$\ell = \sum_{i=2}^{s+2} p_i + 2n.$$

By the definition of τ'_{s+1} and h_{s+1}, it is enough to prove that if $2n > 1100$, then $\sum_{i=2}^{s+2} p_i + 2n$ can be expressed as the sum of $s+1$ distinct integers > 1 which are pairwise relatively prime.

Assume that $2n > 1100$. Write $2t = 2n - \tau'_s$. As $\tau'_s = h_s \le p_{s+2}$ we have $p_{s+2} + 2t = p_{s+2} + 2n - \tau'_s \ge 2n > 1100$. By Lemma 2.1 there exists an odd prime q with $\frac{2}{3}(p_{s+2} + 2t) < q < p_{s+2} + 2t$. Then

$$\ell - q > \ell - p_{s+2} - 2t = \sum_{i=2}^{s+1} p_i + \tau'_s.$$

Since

$$\ell - q \equiv s \pmod 2,$$

by the induction hypothesis we have

$$\ell - q = n_1 + \dots + n_s,$$

where $1 < n_1 < \dots < n_s$ and $(n_i, n_j) = 1$ for $1 \le i, j \le s$, $i \ne j$. Since $\ell - q \equiv s \pmod 2$ and $(n_i, n_j) = 1$ for $1 \le i, j \le s$, $i \ne j$, we have $2 \nmid n_i$ for $1 \le i \le s$.

If $q > n_s$, we are done. Now we assume that $q \le n_s$. As $\ell - q = n_1 + \cdots + n_s$, we have

$$\ell \ge 2q + p_2 + \cdots + p_s > \tfrac{4}{3}p_{s+2} + \tfrac{8}{3}t + p_2 + \cdots + p_s. \tag{3.1}$$

By (3.1), since

$$\ell = \sum_{i=2}^{s+2} p_i + 2t + \tau_s',$$

we obtain

$$\tfrac{1}{3}p_{s+2} - p_{s+1} + \tfrac{2}{3}t < \tau_s'. \tag{3.2}$$

Noting that $\tau_s' \le p_{s+2}$, by (3.2) we have

$$2n = 2t + \tau_s' < 4\tau_s' + 3p_{s+1} - p_{s+2} < 6p_{s+2}. \tag{3.3}$$

Since $2n > 1100$, Lemma 2.2 yields

$$2n = \sum_{i=2}^{\infty}(p_i^{t_i} - p_i), \qquad t_i \ge 1,\ i = 2, 3, \ldots. \tag{3.4}$$

For $i \ge s+3$, by (3.3) and (3.4) we have

$$p_{s+3}^{t_i} - p_{s+3} \le p_i^{t_i} - p_i \le 2n < 6p_{s+2}.$$

Since $p_{s+3} - 1 \ge p_5 - 1 = 10$, it follows that $t_i = 1$ for all $i \ge s+3$. Hence

$$\ell = \sum_{i=2}^{s+2} p_i + 2n = \sum_{i=2}^{s+2} p_i + \sum_{i=2}^{s+2}(p_i^{t_i} - p_i) = \sum_{i=2}^{s+2} p_i^{t_i}.$$

Thus we have proved that if $\ell = \sum_{i=2}^{s+2} p_i + 2n$ cannot be expressed as the sum of $s+1$ distinct integers > 1 which are pairwise relatively prime, then $2n \le 1100$. By the definition of h_{s+1} and τ_{s+1}', we have $\tau_{s+1}' = h_{s+1}$. Therefore, $\tau_s' = h_s$ for all $s \ge 2$.

Thus we have proved that $\tau_s' = h_s$ is the largest even number $2n \le 1100$ such that $\sum_{i=2}^{s+1} p_i + 2n$ cannot be expressed as the sum of s distinct integers > 1 which are pairwise relatively prime, and $\tau_s' = h_s \le \min\{1100, p_{s+2}\}$.

In order to prove Theorem 1.6, it is enough to prove that $\tau_s' \notin U + V_s$ and if $2n$ is an even number with $\tau_s' < 2n \le \min\{1100, p_{s+2}\}$, then $2n \in U + V_s$.

Assume $\tau_s' < 2n \le \min\{1100, p_{s+2}\}$. Now we prove that $2n \in U + V_s$. By Lemma 2.3 and the definition of τ_s', we have

$$p_2 + \cdots + p_{s+1} + 2n = p_{l_1}^{\alpha_1} + \cdots + p_{l_s}^{\alpha_s},$$

where $2 \le l_1 < \cdots < l_s$ and $\alpha_i \ge 1$ $(1 \le i \le s)$. If $l_1 \ge s+2$, then

$l_i \ge s+1+i$ ($1 \le i \le s$). Thus $l_s \ge 2s+1 \ge 5$ and $p_{l_s} \ge p_5 = 11$. Hence

$$\begin{aligned}2n &= p_{l_1}^{\alpha_1} + \cdots + p_{l_s}^{\alpha_s} - (p_2 + \cdots + p_{s+1}) \\ &\ge p_{s+2} + \cdots + p_{2s+1} - (p_2 + \cdots + p_{s+1}) \\ &\ge p_{s+2} + \cdots + p_{2s} + 11 - (p_2 + \cdots + p_{s+1}) \\ &> p_{s+2},\end{aligned}$$

in contradiction with $2n \le \min\{1100, p_{s+2}\}$. So $l_1 \le s+1$. Let r be the largest index with $l_r \le s+1$. If $r = s$, then $l_i = i+1$ ($1 \le i \le s$). Thus

$$2n = (p_2^{\alpha_1} - p_2) + \cdots + (p_{s+1}^{\alpha_s} - p_{s+1}). \tag{3.5}$$

If $r < s$, let

$$\{2, 3, \ldots, s+1\} = \{l_1, \ldots, l_r\} \cup \{j_1, \ldots, j_{s-r}\}$$

with $j_1 < \cdots < j_{s-r}$. Hence

$$2n = (p_{l_1}^{\alpha_1} - p_{l_1}) + \cdots + (p_{l_r}^{\alpha_r} - p_{l_r}) + p_{l_{r+1}}^{\alpha_{r+1}} + \cdots + p_{l_s}^{\alpha_s} - p_{j_1} - \cdots - p_{j_{s-r}}. \tag{3.6}$$

For $1 \le i \le r$, if $\alpha_i \ge 2$, then by (3.5) and (3.6) we have

$$p_{l_i}(p_{l_i} - 1) \le 2n \le 1100.$$

Thus $p_{l_i} \le 31$ and $l_i \le 11$. Hence

$$(p_{l_1}^{\alpha_1} - p_{l_1}) + \cdots + (p_{l_r}^{\alpha_r} - p_{l_r}) \in U. \tag{3.7}$$

For $r < i \le s$, if $\alpha_i \ge 2$, then

$$\begin{aligned}&p_{l_{r+1}}^{\alpha_{r+1}} + \cdots + p_{l_s}^{\alpha_s} - p_{j_1} - \cdots - p_{j_{s-r}} \\ &\qquad \ge p_{s+2}^2 + (s-r-1)p_{s+3} - (s-r)p_{s+1} > p_{s+2} \ge 2n,\end{aligned}$$

a contradiction. So $\alpha_i = 1$ for all $r < i \le s$. By (3.6) we have

$$p_{l_{r+1}}^{\alpha_{r+1}} + \cdots + p_{l_s}^{\alpha_s} - p_{j_1} - \cdots - p_{j_{s-r}} \le 2n \le 1100.$$

Hence

$$\begin{aligned}&p_{l_{r+1}}^{\alpha_{r+1}} + \cdots + p_{l_s}^{\alpha_s} - p_{j_1} - \cdots - p_{j_{s-r}} \\ &\qquad = p_{l_{r+1}} + \cdots + p_{l_s} - p_{j_1} - \cdots - p_{j_{s-r}} \in V_s.\end{aligned} \tag{3.8}$$

By (3.5)–(3.8) we have $2n \in U + V_s$.

It remains to prove that $\tau'_s \notin U + V_s$. Suppose that $\tau'_s \in U + V_s$. Then

$$\tau'_s = \sum_{i=2}^{11} (p_i^{\beta_i} - p_i) + p_{i_1} + \cdots + p_{i_l} - p_{w_1} - \cdots - p_{w_l},$$

where β_i ($2 \le i \le 11$) are positive integers and $w_1 < \cdots < w_l \le s+1 < i_1 < \cdots < i_l$. Let

$$\sum_{i=2}^{11} (p_i^{\beta_i} - p_i) = \sum_{i=1}^{m} (p_{e_i}^{d_i} - p_{e_i}),$$

where $2 \le e_1 < \cdots < e_m \le 11$ and $d_i \ge 2$ $(1 \le i \le m)$. Since

$$p_{e_m}(p_{e_m}-1) \le p_{e_m}^{d_m} - p_{e_m} \le \tau_s' \le p_{s+2},$$

we have $e_m \le s+1$. If $w_1 \le e_m$, then

$$\begin{aligned}\tau_s' &= \sum_{i=1}^{m}(p_{e_i}^{d_i} - p_{e_i}) + p_{i_1} + \cdots + p_{i_l} - p_{w_1} - \cdots - p_{w_l} \\ &\ge p_{e_m}^{d_m} - p_{e_m} - p_{w_1} + p_{s+2} \ge p_{e_m}(p_{e_m}-2) + p_{s+2} > p_{s+2},\end{aligned}$$

a contradiction as $\tau_s' \le \min\{1100, p_{s+2}\}$. Hence $e_m < w_1$. Thus

$$2 \le e_1 < \cdots < e_m < w_1 < \cdots < w_l \le s+1 < i_1 < \cdots < i_l.$$

Let

$$\{f_1, \ldots, f_{s-m-l}\} = \{2, \ldots, s+1\} \setminus \{e_1, \ldots, e_m, w_1, \ldots, w_l\}.$$

Then

$$p_2 + \cdots + p_{s+1} + \tau_s' = \sum_{i=1}^{m} p_{e_i}^{d_i} + p_{f_1} + \cdots + p_{f_{s-m-l}} + p_{i_1} + \cdots + p_{i_l}.$$

Since $e_1, \ldots, e_m, f_1, \ldots, f_{s-m-l}, i_1, \ldots, i_l$ are pairwise distinct, this contradicts the definition of τ_s' and completes the proof of Theorem 1.6.

4. Proofs of Theorem 1.1 and Corollary 1.3. It is easy to see that $c_2 = -2$ and $\{0, 2, 4, 6\} \in V_2$. Thus, as $0 \in U$, all even numbers $2n$ with $-2 < 2n \le \min\{1100, p_{2+2}\}$ are in $U + V_2$. So the conclusion of Theorem 1.1 is true for $s = 2$.

Now we assume that $s > 2$.

In order to prove Theorem 1.1, by Theorem 1.6 it is enough to prove that for any odd number $2k+1 > \tau_s'$, $p_2 + \cdots + p_{s+1} + 2k + 1$ can be expressed as the sum of s distinct integers > 1 which are pairwise relatively prime. Since $\tau_s' \ge -2$, we have $k \ge -1$. If $k = -1$, then

$$p_2 + \cdots + p_{s+1} + 2k + 1 = p_1 + p_3 + p_4 + \cdots + p_{s+1}.$$

If $k = 0$, then

$$p_2 + \cdots + p_{s+1} + 2k + 1 = p_1^2 + p_3 + p_4 + \cdots + p_{s+1}.$$

Now we assume that $k \ge 1$. By Theorem 1.6 we have $p_{s+1} + 2k - 1 > \tau_{s-1}'$. Hence

$$p_2 + \cdots + p_s + (p_{s+1} + 2k - 1) = n_1 + \cdots + n_{s-1},$$

where $1 < n_1 < \cdots < n_{s-1}$ and $(n_i, n_j) = 1$ for $1 \le i, j \le s-1$, $i \ne j$. Since $p_2 + \cdots + p_s + (p_{s+1} + 2k - 1) \equiv s - 1 \pmod 2$ and $(n_i, n_j) = 1$ for $1 \le i, j \le s-1$, $i \ne j$, we have $2 \nmid n_i$ for $1 \le i \le s-1$. Thus

$$p_2 + \cdots + p_s + (p_{s+1} + 2k + 1) = 2 + n_1 + \cdots + n_{s-1}$$

is the required form.

This completes the proof of Theorem 1.1.

Proof of Corollary 1.3. Suppose that $p_{s+2} - p_{s+1} > 1100$. Then $V_s = \{0\}$. Since $1100 \notin U$, we have $1100 \notin U + V_s$. By Theorem 1.1 we have $c_s = 1100$. This proves the first part of Corollary 1.3. The second part follows from the fact that the number of primes $p \le x$ such that $p + k$ is prime, is bounded above by $cx/\log^2 x$, where c depends only on k (Brun [1], Sándor, Mitrinović and Crstici [4, p. 238], Wang [6]). ∎

5. Final remarks. Let $A = ([2, 1100] \cap 2\mathbb{N}) \setminus U$ and for $t < s$, let

$$V_s(t) = \{p_{s+2+i} - p_{s+1-j} \mid 0 \le i, j \le t\} \cup \{p_{s+2+i} + p_{s+2+j} - p_{s+1-u} - p_{s+1-v} \mid 0 \le i < j \le t,\ 0 \le u < v \le t\}.$$

Let $a(s, t) = \max(A \setminus (U + V_s(t)))$. If

$$a(s, t) < \min\{p_{s+2+t} - p_{s+1}, p_{s+2} - p_{s+1-t}, p_{s+3} + p_{s+2} - p_{s+1} - p_s\},$$

then

$$a(s, t) = \max(A \setminus (U + V_s)).$$

Noting that $A = ([2, 1100] \cap 2\mathbb{N}) \setminus U$, by Theorem 1.1 we have $c_s = a(s, t)$. Taking $t = 5$, using Mathematica we find that $c_{500} = 16$, $c_{900} = 14$, $c_{1000} = 8$, $c_{2000} = 22$, etc.

Appendix

$U = \{$0, 6, 20, 24, 26, 42, 44, 48, 62, 66, 68, 78, 86, 98, 110, 116, 120, 126, 130, 134, 136, 140, 144, 152, 154, 156, 158, 162, 168, 172, 176, 178, 180, 182, 186, 188, 196, 198, 200, 204, 208, 218, 222, 224, 230, 234, 236, 240, 242, 250, 254, 260, 266, 272, 276, 278, 282, 286, 290, 292, 296, 298, 300, 302, 308, 310, 314, 316, 318, 320, 324, 328, 332, 334, 336, 338, 340, 342, 344, 348, 350, 352, 354, 356, 358, 360, 362, 364, 366, 368, 370, 380, 382, 384, 386, 388, 390, 392, 396, 398, 402, 404, 406, 408, 410, 412, 414, 416, 420, 424, 426, 428, 430, 434, 438, 440, 444, 446, 448, 450, 452, 454, 456, 458, 460, 462, 464, 466, 468, 470, 472, 476, 478, 480, 482, 486, 490, 492, 494, 496, 498, 500, 502, 504, 506, 508, 510, 512, 514, 516, 518, 520, 522, 524, 526, 528, 530, 532, 534, 536, 538, 540, 542, 544, 546, 548, 550, 554, 558, 560, 562, 564, 566, 568, 570, 572, 574, 576, 578, 580, 582, 584, 586, 590, 592, 596, 600, 602, 604, 606, 608, 612, 614, 616, 618, 620, 622, 624, 626, 628, 632, 634, 636, 638, 640, 642, 644, 646, 650, 652, 656, 658, 660, 662, 664, 666, 668, 670, 674, 676, 678, 680, 682, 684, 686, 688, 690, 692, 694, 696, 698, 700, 702, 704, 706, 710, 712, 714, 718, 722, 724, 726, 728, 730, 732, 734, 736, 738, 740, 742, 744, 746, 748, 750, 752, 754, 756, 758, 760, 762, 764, 766, 768, 770, 772, 776, 778, 780, 782, 784, 786, 788, 790, 792, 794, 796, 798, 800, 802, 804, 806, 808, 810, 812, 814, 816, 818, 820, 822, 824, 826, 830, 832, 834, 836, 838, 840, 842, 844, 846, 848, 850, 852, 854, 856, 858, 860, 862, 864, 866, 868, 870, 872, 874, 876, 878, 880, 882, 884, 886, 888, 890, 892, 894, 896, 898, 900, 902, 904, 906, 908, 910, 912, 914, 916, 918, 920, 922, 924, 926, 928, 930, 932, 934, 936, 938, 940, 942, 944, 946, 948, 950, 952, 954, 956, 958, 960, 962, 964, 966, 968, 970, 972, 974, 976, 978, 980, 982, 984, 986, 988, 990, 992, 994, 996, 998, 1000, 1002, 1004, 1006, 1008, 1010, 1012, 1014, 1016, 1018, 1020, 1022, 1024, 1026, 1028, 1030, 1032, 1034, 1036, 1038, 1040, 1042, 1044, 1046, 1048, 1050, 1052, 1054, 1056, 1058, 1060, 1062, 1064, 1066, 1068, 1070, 1072, 1074, 1076, 1078, 1080, 1082, 1084, 1086, 1088, 1090, 1092, 1094, 1096, 1098$\}$.

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