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Sequences with bounded l.c.m. of each pair of terms, III

by

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1. Introduction. Let A_x be a set of positive integers with the least common multiple of each pair of terms not exceeding x and $|A_x|$ being the largest. In 1951, P. Erdős [5] (see also Guy [7]) proposed the following problem: what is the value of $|A_x|$? It is known that

$$\sqrt{\frac{9}{8}x} + O(1) \le |A_x| \le \sqrt{4x} + O(1).$$

For a proof see Erdős [6]. Choi [2] improved the upper bound to $1.638\sqrt{x}$, and later [3] to $1.43\sqrt{x}$. Let B_x be the union of the set of positive integers not exceeding $\sqrt{x/2}$ and the set of even integers between $\sqrt{x/2}$ and $\sqrt{2x}$. It is clear that the least common multiple of each pair of terms of B_x does not exceed x. By calculation we have

$$|B_x| = \sqrt{\frac{9}{8}x} + O(1).$$

Chen [1] gave an asymptotic formula for $|A_x|$ and showed that A_x is almost the same as B_x, namely

$$|A_x \setminus B_x| = o(\sqrt{x}).$$

In particular,

$$|A_x| = |B_x| + o(\sqrt{x}) = \sqrt{\frac{9}{8}x} + o(\sqrt{x}).$$

Dai and Chen [4] gave an explicit bound of the remainder for $|A_x|$:

$$|A_x| = \sqrt{\frac{9}{8}x} + R(x),$$

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where

$$-2 \le R(x) \le \sqrt{\frac{9}{8}x} + 45\sqrt{\frac{x}{\log x}}\log\log x.$$

On the other hand, it is natural to ask whether $R(x) = O(1)$.

Let C_x be a set of positive integers with the least common multiple of each pair of terms not exceeding x, $B_x \subseteq C_x$ and $|C_x|$ being the largest. Write

$$|C_x| = |B_x| + R_1(x).$$

If $a \in C_x \setminus B_x$, then $a \notin B_x$ and $[a, k] \le x$ for all positive integers k not exceeding $\sqrt{x/2}$ and all even integers k between $\sqrt{x/2}$ and $\sqrt{2x}$. Intuitively, this seems impossible for sufficiently large x. A more interesting question is whether $R_1(x) = O(1)$.

For any positive real number x we define the function $\operatorname{loc} x$ to be the nonnegative integer r with

$$0 \le \underbrace{\log\log\cdots\log}_{r} x < 1.$$

In this paper the following results are proved.

Theorem 1.

(i) $R_1(x) = 0$ *for infinitely many positive integers* x;
(ii) $R_1(x) \ge \operatorname{loc} x - 2$ *for infinitely many positive integers* x.

From Theorem 1 we have immediately

Corollary 1. $R(x) \ge \operatorname{loc} x - 2$ *for infinitely many positive integers* x.

In order to study the properties of $R_1(x)$, we introduce the following notation.

Definition. Let u be a positive real number. Two positive integers s, t are *u-compromise* if there exist primes p_i $(i = 0, 1, \ldots, [us])$ and primes q_j $(j = 0, 1, \ldots, [ut])$ such that

$$p_i \mid s + i, \quad i = 0, 1, \ldots, [us],$$
$$q_j \mid t + j, \quad j = 0, 1, \ldots, [ut],$$

and $p_i \mid s - t$ when p_i is equal to one of q_j $(0 \le i \le [us],\ 0 \le j \le [ut])$.

It is clear that if s, t are u-compromise, then they are also v-compromise for any $0 < v \le u$.

Theorem 2. *If there are three real numbers* $0 < u < 1$, $\tau > 0$, $T > 0$ *and a positive integer* r *such that for any two u-compromise integers* s, t *with* $t > s \ge T$ *we always have*

$$\underbrace{\log\log\cdots\log}_{r} t \ge \underbrace{\log\log\cdots\log}_{r} s + \tau,$$

then

$$R_1(x) = O(\underbrace{\log\log\cdots\log}_{r+1} x).$$

COROLLARY 2. $R_1(x) = O(\log\log x)$.

THEOREM 3. *If there are two real numbers* $0 < u < 1$, $T > 0$ *and a positive integer* r *such that for any two u-compromise integers* s, t *with* $t > s \geq T$ *we always have*

$$\underbrace{\log\log\cdots\log}_{r} t \geq \frac{1}{2}\underbrace{\log\log\cdots\log}_{r-1} s,$$

then

$$R_1(x) \leq 2\,\mathrm{loc}\, x + O(1).$$

We pose the following problems.

PROBLEM 1. *Given any positive integer* r, *are there three real numbers* $0 < u < 1$, $\tau > 0$ *and* $T > 0$ *such that for any two u-compromise integers* s, t *with* $t > s \geq T$ *we always have*

$$\underbrace{\log\log\cdots\log}_{r} t \geq \underbrace{\log\log\cdots\log}_{r} s + \tau\,?$$

It is easy to prove that Problem 1 is true for $r = 1$ (see the proof of Lemma 4 in the next section).

PROBLEM 2. *Are there two real numbers* $0 < u < 1$, $T > 0$ *and a positive integer* r *such that for any two u-compromise integers* s, t *with* $t > s \geq T$ *we always have*

$$\underbrace{\log\log\cdots\log}_{r} t \geq \frac{1}{2}\underbrace{\log\log\cdots\log}_{r-1} s\,?$$

It is clear that Problem 2 is stronger than Problem 1.

2. Proof of theorems

LEMMA 1. *Let* q *be a prime with* $3 \leq q \leq \sqrt{x/2}$ *and* $4q(q-2) > x$. *Then*

$$C_x \subseteq \{2l \mid l \in \mathbb{N},\, l \leq x/(2q)\} \cup \{l \mid l \in \mathbb{N},\, l \leq x/(2q),\, 2 \nmid l\}.$$

Proof. Let $a \in C_x$. Since $q \leq \sqrt{x/2}$, we have $q \leq x/(2q)$. Thus we need only consider $a \neq q, 2q$. Since $2q, 2(q-1), 2(q-2) \in B_x \subseteq C_x$, we have

$$[a, 2q] \leq x, \quad [a, 2(q-1)] \leq x, \quad [a, 2(q-2)] \leq x.$$

CASE 1: $2 \nmid a$ and $q \nmid a$. As $2aq = [a, 2q] \leq x$ we have $a \leq x/(2q)$.

CASE 2: $2 \mid a$ and $q \nmid a$. As $aq = [a, 2q] \leq x$ we have $a/2 \leq x/(2q)$.

Case 3: $q \mid a$. Let $a = qbt$, where $t = 1$ if $2 \nmid a$ and $t = 2$ if $2 \mid a$. Then

$$[a, 2(q-1)] = [qbt, 2(q-1)] = 2q[b, q-1],$$
$$[a, 2(q-2)] = [qbt, 2(q-2)] = 2q[b, q-2].$$

Since $a \neq q, 2q$, we have $b > 1$. Hence either $[b, q-1] \neq q-1$ or $[b, q-2] \neq q-2$. Thus

$$\max\{[a, 2(q-1)], [a, 2(q-2)]\} \geq 4q(q-2) > x,$$

a contradiction. This completes the proof of Lemma 1.

Lemma 2. *Let u be a real number with $0 < u < 1$, k be an integer with $k \leq \sqrt{x/2} < k+1$ and s be an integer such that*

$$\frac{4}{1-u} + \frac{1}{u} < s < \frac{1-u}{2u} k$$

and either $k+s \in C_x$ with $2 \nmid k+s$ or $2(k+s) \in C_x$. Then there exist primes p_i $(i = 0, 1, \dots, [us])$ such that

$$p_i \mid s+i, \quad p_i \mid k+s, \quad i = 0, 1, \dots, [us].$$

Proof. Let $a = k+s$ if $k+s \in C_x$ with $2 \nmid k+s$, otherwise let $a = 2(k+s)$. Let i be an integer with $0 \leq i \leq us$. Then $2(k-i) \in B_x \subseteq C_x$. Hence $[a, 2(k-i)] \leq x$. Since

$$\frac{4}{1-u} + \frac{1}{u} < s < \frac{1-u}{2u} k,$$

we have

$$k > \frac{8u}{(1-u)^2} + \frac{2}{1-u}.$$

Hence

$$2(k+s)(k-i) \geq 2(k+s)(k-us) > 2\left(k + \frac{4}{1-u} + \frac{1}{u}\right)\left(k - \frac{4u}{1-u} - 1\right)$$
$$> 2(k+1)^2 > x.$$

Noting that $[a, 2(k-i)] \leq x$ and

$$[a, 2(k-i)] = 2[k+s, k-i] = \frac{2(k+s)(k-i)}{(k+s, k-i)},$$

we have $(k+s, k-i) > 1$. Thus $(k+s, s+i) > 1$. Therefore, for each i with $0 \leq i \leq us$ we may choose a prime p_i with $p_i \mid k+s$ and $p_i \mid s+i$. This completes the proof of Lemma 2.

Lemma 3. *Let s be a positive integer and k be an integer with $k \leq \sqrt{x/2} < k+1$. Then $s = O(\log x)$ if $k+s \in C_x$ with $2 \nmid k+s$ or if $2(k+s) \in C_x$.*

Proof. By a result on the distribution of primes and Lemma 1 we have $s = O(x^\theta)$, where θ is a positive constant with $\theta < 1/2$, for example we can take $\theta = 7/24$ (see Huxley [8]). Thus we may assume that $10 < s < k/2$. By Lemma 2 there exist primes p_i $(i = 0, 1, \ldots, [s/2])$ such that

$$p_i \mid s + i, \quad p_i \mid k + s, \quad i = 0, 1, \ldots, [s/2].$$

Thus

$$\prod_{s \le p \le 3s/2} p \,\Big|\, k + s$$

and so

$$\prod_{s \le p \le 3s/2} p \le k + s \le x,$$

where the product is taken over all primes p in the interval $[s, 3s/2]$. Therefore $s = O(\log x)$. This completes the proof of Lemma 3.

Lemma 4. *Let k be an integer with $k \le \sqrt{x/2} < k + 1$ and s, t be two integers with $10 < s < t < k/2$ such that either $k + s \in C_x$ with $2 \nmid k + s$ or $2(k + s) \in C_x$, and either $k + t \in C_x$ with $2 \nmid k + t$ or $2(k + t) \in C_x$. Then $t \ge 5s/4$ for $s \ge M$, where M is a positive constant.*

Proof. By the proof of Lemma 3 we have

$$\prod_{s \le p \le 3s/2} p \,\Big|\, k + s, \qquad \prod_{t \le p \le 3t/2} p \,\Big|\, k + t.$$

Hence

$$\prod_{t \le p \le 3s/2} p \,\Big|\, t - s.$$

Thus

$$\prod_{t \le p \le 3s/2} p \le t - s.$$

If $t < 5s/4$, then

$$\prod_{5s/4 \le p \le 3s/2} p \le s/4.$$

This cannot hold for s large enough. This completes the proof of Lemma 4.

Lemma 5. *For any positive integer m we have*

$$m + \prod_{p \le m} p \le 2^{3m},$$

where the product is taken over all primes p less than m.

Proof. We use induction on m. It is easy to verify the assertion for $m \le 5$. Suppose that it is true for all positive integers less than m. If $m \ge 6$, then

$$[m/2] + 1 + \prod_{p \le [m/2]+1} p \le 2^{3[m/2]+3}.$$

Since

$$m + \prod_{[m/2]+1<p\le m} p \le m + \binom{m}{[m/2]} \le 2^m,$$

we have

$$m + \prod_{p\le m} p \le 2^{3[m/2]+3+m} \le 2^{3m}.$$

This completes the proof of Lemma 5.

Proof of Theorem 1

(i) Take $x = 2q^2$, where q is an odd prime. By Lemma 1 we have $C_x \setminus B_x = \emptyset$. Hence $R_1(x) = 0$.

(ii) Let $d_1 = 2$ and

$$d_{n+1} = d_n + \prod_{p\le 2d_n-1} p, \quad n = 1, 2, \ldots,$$

where the product is taken over all primes p less than $2d_n - 1$. Then $2 \mid d_n$ for all $n \ge 1$. Let

$$k_n = -d_n + \prod_{p\le 2d_n-1} p, \quad x_n = 2k_n^2, \quad n = 1, 2, \ldots.$$

By Bertrand's postulate and $2 \mid d_n$ we have

$$k_n \ge -d_n + \frac{1}{2}\, d_n(d_n+1) \ge 3d_n, \quad n \ge 2. \tag{1}$$

From (1) and $k_1 = 4$, $d_1 = 2$, $x_1 = 32$, we have $k_n + d_n \le x_n$ $(n \ge 1)$. It is clear that

$$B_{x_n} = \{2h \mid 1 \le h \le k_n,\, h \in \mathbb{Z}\} \cup \{l \mid 1 \le l \le k_n,\, l \in \mathbb{Z},\, 2 \nmid l\}.$$

Now we show that $[a,b] \le x_n$ for any

$$a, b \in B_{x_n} \cup \{2(k_n+d_1),\, 2(k_n+d_2), \ldots, 2(k_n+d_n)\}.$$

It is clear for $n = 1$. Now we assume that $n \ge 2$.

Case 1: $a, b \in \{2(k_n+d_1), 2(k_n+d_2), \ldots, 2(k_n+d_n)\}$. Let

$$a = 2(k_n+d_i), \quad b = 2(k_n+d_j).$$

From $2 \mid d_i$, $2 \mid d_j$, $2 \mid k_n$ and (1) we have

$$[a,b] \le (k_n+d_i)(k_n+d_j) \le \frac{16}{9}\, k_n^2 < x_n.$$

Case 2: $a = 2(k_n+d_i)$ $(1 \le i \le n)$ and $b \in B_{x_n}$. Without loss of generality, we may assume that $b \in \{2h \mid 1 \le h \le k_n,\, h \in \mathbb{Z}\}$. Write $b = 2(k_n - j)$.

If $j \ge d_i$, then $[a,b] \le \frac{1}{2}ab \le 2(k_n^2 - d_i^2) < 2k_n^2 \le x$.

If $0 \le j \le d_i - 1$, let p be a prime with $p \mid d_i + j$; then $p \le 2d_i - 1$. Hence

$$k_n \equiv -d_n \equiv -d_{n-1} \equiv \cdots \equiv -d_i \equiv j \pmod p.$$

Thus

$$(a,b) = 2(k_n + d_i, k_n - j) \ge 2p. \tag{2}$$

By (1) and (2) we have

$$[a,b] = \frac{ab}{(a,b)} \le \frac{1}{2p}\,ab \le (k_n + d_i)(k_n - j) \le \frac{4}{3}\,k_n^2 < x_n.$$

Therefore $[a,b] \le x_n$ for any

$$a, b \in B_{x_n} \cup \{2(k_n + d_1),\, 2(k_n + d_2), \ldots, 2(k_n + d_n)\}.$$

To complete the proof, it is enough to prove that $n \ge \operatorname{loc} x_n - 2$. By Lemma 5 we have $d_{i+1} \le 2^{5d_i}$ $(i \ge 1)$. Thus $\log d_{i+1} \le 5d_i$ $(i \ge 1)$. Hence

$$\log x_n = \log 2 + 2\log k_n \le \log 2 + 2\log d_{n+1} \le 11d_n,$$

$$\log\log x_n \le \log 11 + \log d_n \le 7d_{n-1}.$$

Continuing this procedure, we have

$$\underbrace{\log\log\cdots\log}_{i} x_n \le 7d_{n+1-i}.$$

Since $\operatorname{loc}(7d_1) = 2$, we have $\operatorname{loc} x_n \le n + 2$. This completes the proof of Theorem 1.

Proof of Theorem 2. Assume that x is large enough. Without loss of generality, we may assume that

$$\underbrace{\log\log\cdots\log}_{r} T > 0.$$

Let k be an integer with $k \le \sqrt{x/2} < k + 1$ and let $t_1, \ldots, t_l$ be positive integers with

$$\max\left\{T,\ \frac{4}{1-u} + \frac{1}{u}\right\} < t_1 < \cdots < t_l$$

and either $k + t_i \in C_x$ with $2 \nmid k + t_i$ or $2(k + t_i) \in C_x$ $(1 \le i \le l)$. By Lemma 3 we have $t_l = O(\log x)$. Hence we may assume that $t_l < (1-u)k/(2u)$. By Lemma 2 and the definition of u-compromise we see that t_i, t_{i+1} are u-compromise $(1 \le i \le l - 1)$. Hence

$$\underbrace{\log\log\cdots\log}_{r} t_{i+1} \ge \underbrace{\log\log\cdots\log}_{r} t_i + \tau, \quad 1 \le i \le l - 1.$$

Thus

$$\underbrace{\log\log\cdots\log}_{r} t_l \ge \underbrace{\log\log\cdots\log}_{r} t_1 + (l-1)\tau \ge (l-1)\tau.$$

Noting that $t_l = O(\log x)$, we have

$$l = O(\underbrace{\log\log\cdots\log}_{r+1} x).$$

Therefore

$$R_1(x) = O(\underbrace{\log\log\cdots\log}_{r+1} x).$$

This completes the proof of Theorem 2.

Corollary 2 follows from Lemma 4 and Theorem 2 immediately.

Proof of Theorem 3. The initial part is as in the proof of Theorem 2. Then

$$\underbrace{\log\log\cdots\log}_{r} t_{i+1} \geq \frac{1}{2}\underbrace{\log\log\cdots\log}_{r-1} t_i, \quad 1 \leq i \leq l-1.$$

Without loss of generality, we may assume that

$$\underbrace{\log\log\cdots\log}_{r-1} T > 4\log 4.$$

Thus

$$\underbrace{\log\log\cdots\log}_{r-1} t_i > 4\log 4, \quad 1 \leq i \leq l.$$

Hence

$$\begin{aligned}\underbrace{\log\log\cdots\log}_{r+1} t_l &\geq \log\frac{1}{2} + \underbrace{\log\log\cdots\log}_{r} t_{l-1}\\ &\geq \log\frac{1}{2} + \frac{1}{2}\underbrace{\log\log\cdots\log}_{r-1} t_{l-2}\\ &\geq \frac{1}{4}\underbrace{\log\log\cdots\log}_{r-1} t_{l-2}.\end{aligned}$$

Continuing this procedure we have

$$\underbrace{\log\log\cdots\log}_{r+l-2} t_l \geq \frac{1}{4}\underbrace{\log\log\cdots\log}_{r-1} t_1 \geq \frac{1}{4}\underbrace{\log\log\cdots\log}_{r-1} T \geq 1.$$

Hence $\operatorname{loc} t_l \geq r+l-2$. Since $t_l \leq x$, we have

$$\operatorname{loc} x \geq r+l-2.$$

Therefore

$$R_1(x) \leq 2l + O(1) \leq 2\operatorname{loc} x + O(1).$$

References

[1] Y. G. Chen, *Sequences with bounded l.c.m. of each pair of terms*, Acta Arith. 84 (1998), 71–95.

[2] S. L. G. Choi, *The largest subset in* $[1, n]$ *whose integers have pairwise l.c.m. not exceeding* n, Mathematika 19 (1972), 221–230.

[3] —, *The largest subset in* $[1, n]$ *whose integers have pairwise l.c.m. not exceeding* n, *II*, Acta Arith. 29 (1976), 105–111.

[4] L. X. Dai and Y. G. Chen, *Sequences with bounded l.c.m. of each pair of terms*, *II*, ibid. 124 (2006), 315–326.

[5] P. Erdős, *Problem*, Mat. Lapok 2 (1951), 233.

[6] —, *Extremal problems in number theory*, in: Theory of Numbers, Proc. Sympos. Pure Math. 8, Amer. Math. Soc., 1965, 181–189.

[7] R. K. Guy, *Unsolved Problems in Number Theory*, 2nd ed., Springer, New York, 1994.

[8] M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. 15 (1972), 164–170.

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