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Regularity of distribution of $(n\alpha)$-sequences

by

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Dedicated to Prof. W. M. Schmidt on the occasion of his 75th birthday

1. Introduction. Let $\omega = (x_n)_{n\ge 1}$ be a sequence of real numbers and let $f : \mathbb{R} \to \mathbb{C}$ be periodic with period 1 and integrable over $[0,1]$. We say that f is *of bounded remainder* (*with respect to* ω) if the sequence

$$\Big(\sum_{n=1}^{N} f(x_n) - N\int_0^1 f(x)\,dx\Big)_{N\ge 1}$$

is bounded. In this paper we investigate the classical case $\omega = (n\alpha)_{n\ge 1}$, $\alpha \in [0,1]$ irrational, more closely.

Let c_A be the characteristic function of a set A and $\{x\} = x - [x]$ be the fractional part of the real number x. For N given define

$$D_N^*(\omega) = \sup_{0\le x\le 1}\Big|\sum_{n=1}^{N} c_{[0,x)}(\{x_n\}) - Nx\Big|,$$

the so-called $*$-*discrepancy* of the sequence $\omega = (x_n)_{n\ge 1}$. Let $f : \mathbb{R} \to \mathbb{C}$ be periodic with period 1 and of bounded variation V in $[0,1]$. Then a well known theorem by Koksma ([27, p. 143]) says that

$$\Big|\sum_{n=1}^{N} f(x_n) - N\int_0^1 f(x)\,dx\Big| \le V D_N^*(\omega).$$

For every sequence ω this inequality is best possible. On the other hand, there may exist, for ω given, a large class of functions f of bounded variation for which the left hand side is much smaller than the right hand side. Note that the right hand side is never bounded above for infinitely many N (except when f is constant); but the left hand side may be bounded.

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The concept of functions of bounded remainder was first introduced by P. Liardet in [29]. See also [2]. We refer the reader to [13] for the cases of van der Corput sequences and to [23] for a q-adic transformation sequence. For the concept of functions of bounded remainder for multi-dimensional $(n\alpha)$-sequences the interested reader may again consult [29]. Here we restrict ourselves entirely to the one-dimensional case of $(n\alpha)_{n\ge1}$-sequences, where $\alpha \in [0,1]$ is irrational. We say that a periodic function $f:\mathbb{R}\to\mathbb{C}$ with period 1 is *of bounded remainder with respect to* α if it is integrable over $[0,1]$ and

$$\sup_{N\ge1}\Big|\sum_{n=1}^{N} f(\{n\alpha\}) - N\int_0^1 f(x)\,dx\Big| < \infty.$$

We denote by B_f the set of all irrational α's for which f is of bounded remainder with respect to α. For a good overview of the whole subject for $(n\alpha)$-sequences the reader is referred to [21].

Let Ω denote the set of real irrational numbers. Throughout the paper we use the term "periodic" instead of "periodic with period 1". For $f:\mathbb{R}\to\mathbb{C}$ and $y\in\mathbb{R}$ let $L_y f(x) = f(x+y)$.

If f is an arbitrary function in L^∞, the question whether $\alpha\in\mathrm{B}_f$ does not make much sense: we could alter f at the countably many points $\{\alpha n\}$, $n\ge1$, thereby changing B_f, without changing the class of f. In order to exclude pathologies it is also desirable that $\mathrm{B}_f = \mathrm{B}_{L_x f}$ for all $x\in\mathbb{R}$; this condition comes from the fact that the sequences $(n\alpha)_{n\ge1}$ and $(n\alpha+x)_{n\ge1}$ have about the same discrepancy and hence for "reasonable" functions f with mean 0 over $[0,1]$ the sequences $(\sum_{n=1}^N f(n\alpha))_{N\ge1}$ and $(\sum_{n=1}^N f(n\alpha+x))_{N\ge1}$ should not differ to such an extent that one is bounded while the other is not. Hence we restrict ourselves to the smaller class of so-called regulated functions [10]. Recall that $f:\mathbb{R}\to\mathbb{C}$ is called *regulated* if there is a sequence of step functions which converges uniformly to f on all compact subsets of $\mathbb{R}$. In case f is periodic we may assume in addition that these step functions are again periodic. Equivalently, a function is regulated if and only if for every $x\in\mathbb{R}$ both limits $f(x-) := \lim_{t\to x,\, t<x} f(t)$ and $f(x+) := \lim_{t\to x,\, t>x} f(t)$ exist. The vector space of regulated periodic functions is a Banach space with the topology of uniform convergence. We denote by $\|\cdot\|_{\mathrm{u}}$ the norm on this space.

For $\alpha\in\Omega$ let $[a_0;a_1,a_2,\ldots]$ be the continued fraction expansion with convergents p_n/q_n, where $p_{-2}=0$, $p_{-1}=1$, $q_{-2}=1$, $q_{-1}=0$, $p_n = a_n p_{n-1}+p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ for $n\ge0$. Let us now consider the following example.

Example. For $\alpha\in\Omega$ and $x\in\mathbb{R}$ put

$$f(x) = \begin{cases} 1/m, & \{x\} = \{q_{2m}\alpha\} + 1/2,\\ 0, & \text{else.}\end{cases}$$

Then the function f is regulated and even continuous at $1/2$. Nevertheless, $\alpha \in \mathrm{B}_f \setminus \mathrm{B}_{L_{1/2}f}$, hence $\mathrm{B}_f \neq \mathrm{B}_{L_x f}$ in general.

This example shows that even within the class of regulated functions the concept of B_f is not quite appropriate. For this reason we have finally to restrict ourselves to periodic regulated functions with only finitely many discontinuities in $[0,1]$.

We note that if f and g are such that the set of all $x \in [0,1)$ with $f(x) \neq g(x)$ is finite, then $\mathrm{B}_f = \mathrm{B}_g$.

The aim of this paper is to determine the set B_f for a given regulated f with only finitely many discontinuities in $[0,1]$; this can be done in two steps. First, if f can be written as a sum of a periodic continuous function g and a periodic step function h then $\mathrm{B}_f = \mathrm{B}_g \cap \mathrm{B}_h$; otherwise $\mathrm{B}_f = \emptyset$. This is proved in the last section of this paper. There is also a simple (and almost obvious) criterion for the existence of such a decomposition. Hence the whole problem is reduced to step functions and to continuous functions. If f is a step function, B_f was first determined by Oren [33]. Corollary 3 in the next section provides a more transparent criterion. These results are not Diophantine in nature; roughly speaking, they tell us that $\alpha \in \mathrm{B}_f$ if and only if the lengths of the intervals where f is constant are in the additive group generated by 1 and α. This changes drastically if f is continuous. All known results suggest that whether $\alpha \in \mathrm{B}_f$ or not depends on approximation properties of α by rationals (i.e. on its continued fraction expansion). We know nothing about these approximation properties for general continuous functions f but there are some results for functions which are smooth in some sense. Several relevant references are given in Section 4. In that section we also develop a method by which one is able to find B_f if f is a primitive of a function of bounded variation. In Section 5 we test our method on some examples.

The whole matter is closely connected with the cylinder flow over an irrational rotation: let $\alpha \in \Omega$ and $S^1 = \mathbb{R}/\mathbb{Z}$ the one-dimensional torus, and let us identify α with its residue class $\alpha + \mathbb{Z}$ in S^1. The group $\mathbb{Z}$ acts on S^1 via $x.g = x + g\alpha$ ($x \in S^1$, $g \in \mathbb{Z}$). Let $f : S^1 \to \mathbb{C}$ be a Borel measurable function with mean 0. Then $v_f : S^1 \times \mathbb{Z} \to \mathbb{C}$ with $v_f(x,n) := \sum_{m<n} f(x + m\alpha)$ is a so-called *cocycle*, as for $g, h \in \mathbb{Z}$ and $x \in S^1$ we have the cocycle property $v_f(x,g) + v_f(x.g,h) = v_f(x, g+h)$. The function f is completely determined by v_f, as $v_f(x,1) = f(x)$ for all $x \in S^1$. Hence in our setting we may also call f a cocycle. A cocycle v_f (and the corresponding f) is called a *coboundary* if there exists a Borel measurable function $w : S^1 \to \mathbb{C}$ such that $v_f(x,g) = w(x.g) - w(x)$ for a.e. $(x,g) \in S^1 \times \mathbb{Z}$ (or, what is the same, $f(x) = w(x+\alpha) - w(x)$ for almost all $x \in S^1$). Two cocycles f_1, f_2 are called *α-cohomologous* if they differ by a coboundary only. In case f is a coboundary the corresponding function w is called a *transfer function*.

Associate to any such cocycle the skew product (cylinder flow) $\varphi_f : S^1 \times \mathbb{C} \to S^1 \times \mathbb{C}$, $\varphi_f(x, y) = (x + \alpha, y + f(x))$. Note that in the notation above, for $n \geq 0$,

$$\varphi_f^n(x, y) = \Big(x + n\alpha, y + \sum_{i=0}^{n-1} f(x + i\alpha)\Big) = (x.n, y + v_f(x, n)).$$

There is a vast literature on the question whether φ_f is ergodic. The interested reader may consult e.g. [1], [15], [21], [24], [30] or [40] and the references there. By a theorem in [40], if f_1 is α-cohomologous to f_2, then φ_{f_1} is ergodic if and only if φ_{f_2} is. For more general situations the reader may again consult [40]. It is easily seen that for f continuous and $\alpha \in \mathrm{B}_f$, φ_f cannot be ergodic.

A classical theorem by Gottschalk and Hedlund [16] in topological dynamics says (in our special case) that for periodic continuous f with mean 0 we have $\alpha \in \mathrm{B}_f$ if and only if f is a coboundary in the sense that there exists a periodic continuous transfer function g, that is, $f(x) = g(x + \alpha) - g(x)$ for all $x \in \mathbb{R}$.

Apart from the space of continuous functions there are other spaces for which such a coboundary theorem holds (that is, the transfer function lies in the same space as f). Assume that f is periodic, has mean 0, $f \in \mathrm{L}^p([0, 1])$ $(1 \leq p \leq \infty)$, $\alpha \in \Omega$, $F_N(x) := \sum_{n=0}^{N-1} f(x + n\alpha)$ and $\|F_N\|_p$ is bounded. Then there exists a periodic function $g \in \mathrm{L}^p([0, 1])$ such that $f(x) = g(x + \alpha) - g(x)$ almost everywhere. This has first been noticed by Browder [11] and by Browder and Petryshyn [12] in a more general setting. The reader is also invited to consult [1] and [29]. For the space of r-times differentiable functions f the reader is referred to the papers by Herman [25] and Veech [42]. If f is a periodic step function, the corresponding coboundary theorem has been proved first in [33] by an interesting abstract argument.

2. A coboundary theorem. The following proposition shows that if f has only finitely many discontinuities we have $\mathrm{B}_f = \mathrm{B}_{L_x f}$ (in order to avoid the unwanted example in Section 1). The first part of the following proposition—which is essentially taken from [29]—is based on the cocycle property.

PROPOSITION 1. *Let α be irrational, c, x_0 real numbers, $f : \mathbb{R} \to \mathbb{C}$ be periodic and Riemann integrable over* $[0, 1]$, $F_N : \mathbb{R} \to \mathbb{C}$,

$$F_N(x) := \sum_{n=1}^{N} f(x + n\alpha) - N \int_0^1 f(x)\, dx$$

and assume that $\sup_{N\geq 1} |F_N(x_0)| \leq c$. *Then for all* $p \geq 1$ (*including* $p = \infty$)

we have $\|F_N\|_p \le 2c$. Finally, if f is regulated with at most finitely many discontinuities in $[0,1)$, then $\|F_N\|_{\mathrm{u}}$ is uniformly bounded.

Proof. We may assume that $\int_0^1 f(x)\,dx = 0$. As $(x_0+\alpha m)_{m\ge 1}$ is uniformly distributed we get

$$\begin{aligned}\int_0^1 |F_N(x)|^p\,dx &= \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}|F_N(x_0+\alpha m)|^p\\ &= \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\Big|\sum_{n=1}^{N+m} f(x_0+n\alpha)-\sum_{n=1}^{m} f(x_0+n\alpha)\Big|^p\\ &= \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}|F_{N+m}(x_0)-F_m(x_0)|^p \le (2c)^p.\end{aligned}$$

Hence $(\|F_N\|_p)_{N\ge 1}$ is bounded independently of p. Passing to infinity we get the result also in the case $p=\infty$.

As for the last assertion we assume first that f is left continuous. Then $\|F_N\|_\infty = \|F_N\|_{\mathrm{u}}$ and we are done in this case. From this the general case is easily deduced.

Proposition 1 implies that for all $x\in\mathbb{R}$ and f as above, $\mathrm{B}_f = \mathrm{B}_{L_x f}$.

The following proof is a generalization of the corresponding proof in [16]. We note that the method applies to more general situations; the mapping θ in the proof below could be replaced—as long as f is periodic, regulated, right or left continuous and has only finitely many discontinuities in $[0,1]$—by any orientation-preserving homeomorphism of S^1 such that for all $x\in S^1$, $\{\theta^n(x)\mid n\in\mathbb{Z}\}$ is dense in S^1.

THEOREM 1. *Let $f:\mathbb{R}\to\mathbb{C}$ be a periodic regulated function which is left* (*resp. right*) *continuous and which has only finitely many discontinuities in* $[0,1]$. *The following assertions are equivalent*:

(1) $\alpha\in\mathrm{B}_f$.
(2) *There exists a periodic regulated function $g:\mathbb{R}\to\mathbb{C}$ which is left* (*resp. right*) *continuous and has only finitely many discontinuities in $[0,1]$ such that for all $x\in\mathbb{R}$,*

$$f(x)-\int_0^1 f(x)\,dx = g(x+\alpha)-g(x).$$

Any two periodic regulated solutions of this functional equation differ by a constant.

Proof. We may assume that $\int_0^1 f(x)\,dx = 0$.

As (2)⇒(1) is trivial we restrict ourselves to the converse assertion and assume that f is left continuous. Let $S^1 := \mathbb{R}/\mathbb{Z}$, $\bar f: S^1\to\mathbb{C}$, $\bar f(x+\mathbb{Z}) =$

$f(x)$, $\theta : S^1 \to S^1$, $\theta(x+\mathbb{Z}) = x+\alpha+\mathbb{Z}$, for $N \geq 1$, $F_N : S^1 \to \mathbb{C}$, $F_N(x) = \sum_{n=0}^{N-1} \bar{f}(\theta^n(x))$ and $\varphi : S^1\times\mathbb{C} \to S^1\times\mathbb{C}$, $\varphi(x,y) = (\theta(x), y+\bar{f}(x))$. Note that φ is a bijection and that $\varphi^n(x,y) = (\theta^n(x), y+F_n(x))$ for $n \geq 0$.

Let $(a_n)_{n\geq 1}$ be a sequence in S^1 convergent to a. We say that it *tends to a from the left* (resp. *right*) if for $n \geq 1$ there are $x_n \in a_n$ and $x \in a$ such that $(x_n)_{n\geq 1}$ tends to x and $x_n \leq x$ (resp. $x_n > x$). Note that this concept does not depend on the choice of x_n and x. If $(a_n)_{n\geq 1}$ tends to a from the left (resp. right), then $(\theta(a_n))_{n\geq 1}$ tends to $\theta(a)$ from the left (resp. right). If $(a_n)_{n\geq 1}$ converges to $a \in S^1$ from the left (resp. right and $x \in a$), then $(\bar{f}(a_n))_{n\geq 1}$ converges to $\bar{f}(a)$ (resp. to $\bar{f}(a+) := f(x+)$) independently of the choice of $(a_n)_{n\geq 1}$ (resp. and of x). For $n \geq 0$ let $F_n(a+) = \sum_{i=0}^{n-1} \bar{f}(\theta^i(a)+)$.

We say that a subset $B \subseteq S^1 \times \mathbb{C}$ has the *property* $(*)$ if $B \neq \emptyset$, B is compact and $(a,b) \in B$ implies that $(\theta^n(a), b+F_n(a)) \in B$ for all $n \geq 0$ or $(\theta^n(a), b+F_n(a+)) \in B$ for all $n \geq 0$.

For the reader's convenience we outline the plan of the proof and how it differs from the case when f is continuous. The closure of the (positive) orbit $B(x,y)$ of $(x,y) \in S^1 \times \mathbb{C}$ under φ and the closure of the orbit of the corresponding φ^+—when f is replaced by $x \mapsto f(x+)$—both have the property $(*)$. Zorn's lemma implies again the existence of a minimal subset B_0 with the property $(*)$ but in contrast to the continuous case it is no longer the graph of one function but the union of two graphs of functions g and h which differ only at the discontinuities of f. The function g has the desired properties, while h would satisfy $f(x+) = h(x+\alpha) - h(x)$.

Let us first prove that the closure $B(x,y)$ of $\{\varphi^n(x,y) \mid n \geq 0\}$ has the property $(*)$ for all $(x,y) \in S^1 \times \mathbb{C}$. Note that by our assumption on α and by Proposition 1 this set is compact and clearly not empty. Assume that $(a,b) \in B(x,y)$. Then there exists a non-decreasing sequence $(n_j)_{j\geq 1}$ of positive integers such that $a = \lim_{j\to\infty} \theta^{n_j}(x)$ and $b = y + \lim_{j\to\infty} F_{n_j}(x)$. There exists a subsequence $(n_{j_k})_{k\geq 1}$ such that $(\theta^{n_{j_k}}(x))_{k\geq 1}$ tends to a from the left or from the right. We may assume that this is the original sequence. Note that $\theta(a) = \lim_{j\to\infty} \theta^{n_j+1}(x)$ and $\lim_{j\to\infty} \bar{f}(\theta^{n_j}(x)) = \bar{f}(a)$ or $\bar{f}(a+)$. Hence

$$b + \bar{f}(a) = y + \lim_{j\to\infty} F_{n_j}(x) + \lim_{j\to\infty} \bar{f}(\theta^{n_j}(x)) = y + \lim_{j\to\infty} F_{n_j+1}(x)$$

or

$$b + \bar{f}(a+) = y + \lim_{j\to\infty} F_{n_j}(x) + \lim_{j\to\infty} \bar{f}(\theta^{n_j}(x)) = y + \lim_{j\to\infty} F_{n_j+1}(x).$$

Thus it is proved that $(\theta(a), y+\bar{f}(a)) \in B(x,y)$ or $(\theta(a), y+\bar{f}(a+)) \in B(x,y)$, where the first (resp. second) case happens if $(\theta^{n_j}(x))_{j\geq 1}$ tends to a from the left (resp. right). As $(\theta^{n_j+1}(x))_{j\geq 1}$ tends to $\theta(a)$ *from the same side* we can repeat the argument again and again.

Analogously the closure $B^+(x,y)$ of $\{(\theta^n(x), y+F_n(x+)) \mid n \geq 0\}$ has the property $(*)$.

Next we consider the set $\mathcal{B}$ of all subsets of $B(x,y)$ which have the property $(*)$. Let $\mathcal{B}'$ be a non-empty subset of $\mathcal{B}$ which is totally ordered with respect to inclusion and let $B' = \bigcap_{B\in\mathcal{B}'} B$. Then clearly B' is compact and again not empty. Let $(a,b) \in B'$, $\mathcal{B}_1 = \{B \in \mathcal{B}' \mid (\theta^n(a), b+F_n(a)) \notin B$ for some $n \geq 0\}$ and $\mathcal{B}_2 = \{B \in \mathcal{B}' \mid (\theta^n(a), b+F_n(a+)) \notin B$ for some $n \geq 0\}$. We prove that one of these two sets is empty. If not, choose $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. If $B_1 \subseteq B_2$, then $(\theta^n(a), b+F_n(a+)) \in B_1 \subseteq B_2$ for all $n \geq 0$, a contradiction. The other case is absurd for a similar reason and hence the assertion is proved. If $\mathcal{B}_1 = \emptyset$, then $(\theta^n(a), b+F_n(a)) \in B'$ for all $n \geq 0$. Otherwise $(\theta^n(a), b+F_n(a+)) \in B'$ for all $n \geq 0$. Zorn's lemma implies the existence of a minimal subset B_0 of $B(x,y)$ with the property $(*)$.

We note that for $(a,b) \in B_0$ we get either $B(a,b) \subseteq B_0$ or $B^+(a,b) \subseteq B_0$, hence by the minimality of B_0 either $B_0 = B(a,b)$ or $B_0 = B^+(a,b)$.

Next we prove that for all $a \in S^1$ there exists exactly one $b \in \mathbb{C}$ with $B(a,b) = B_0$. Let $(a_0, b_0) \in B_0$. There exists an n_0 such that $\bar{f}$ is continuous at $\theta^n(a_0)$ for all $n \geq n_0$. Replacing a_0 by $\theta^{n_0}(a_0)$ if need be we may assume that $\bar{f}$ is continuous at all the points $\theta^n(a_0)$. Then $B(a_0,b_0) = B^+(a_0,b_0) = B_0$. There exists a sequence $(n_j)_{j\geq 1}$ such that $(\theta^{n_j}(a_0))_{j\geq 1}$ tends to a from the left. The sequence $(F_{n_j}(a_0))_{j\geq 1}$, being bounded, has a convergent subsequence. We may assume that the original sequence converges. Put $b := b_0 + \lim_{j\to\infty} F_{n_j}(a_0)$. Then $(a,b) \in B(a_0,b_0) = B_0$. As $(\theta^{n_j+1}(a_0))_{j\geq 1}$ tends to $\theta(a)$ from the left, we can repeat this argument and get $B(a,b) \subseteq B_0$, hence $B_0 = B(a,b)$. Assume now that $B(a,b) = B(a,b+\beta) = B_0$. Let $\psi_\beta : S^1 \times \mathbb{C} \to S^1 \times \mathbb{C}$, $\psi_\beta(x,y) = (x, y+\beta)$. Then it is clear that $\psi_\beta \circ \varphi = \varphi \circ \psi_\beta$, hence $\psi_\beta(B(x,y)) = B(\psi_\beta(x,y))$ for all $(x,y) \in S^1 \times \mathbb{C}$. Our assumption implies $\psi_\beta(B_0) = B_0$ and hence $\psi_{n\beta}(B_0) = \psi_\beta^n(B_0) = B_0$ for all positive integers n, which is impossible for $\beta \neq 0$ as the union over n of the left hand side is unbounded.

Analogously for every $a \in S^1$ there exists exactly one $b \in \mathbb{C}$ with $B^+(a,b) = B_0$. Hence there are two functions $\overline{g}, \overline{h} : S^1 \to \mathbb{C}$ such that for all $x \in S^1$ $B_0 = B(x, \overline{g}(x)) = B^+(x, \overline{h}(x))$. Note that this implies $\overline{g}(\theta(x)) = \overline{g}(x) + \overline{f}(x)$ and $\overline{h}(\theta(x)) = \overline{h}(x) + \overline{f}(x+)$ for all $x \in S^1$. Define $g, h : \mathbb{R} \to \mathbb{C}$ by $g(x) = \overline{g}(x+\mathbb{Z})$ and $h(x) = \overline{h}(x+\mathbb{Z})$. Then g and h are periodic and $g(x+\alpha) = g(x) + f(x)$, $h(x+\alpha) = h(x) + f(x+)$ for $x \in \mathbb{R}$.

The set $D := \{\theta^n(a_0) \mid n \geq 0\}$ is dense in S^1 and has the following property: for all $a \in S^1$ and all sequences $(a_j)_{j\geq 1}$ in D which tend to a from the left, $(\overline{g}(a_j))_{j\geq 1}$ tends to $\overline{g}(a)$ (as this sequence cannot have two different accumulation points according to the above). This implies that for any sequence $(a_n)_{n\geq 1}$ which tends to a from the left, $(\overline{g}(a_n))_{n\geq 1}$ tends to

$\overline{g}(a)$. Hence g is left continuous. Similarly h is right continuous. Now again the complement D' of $\bigcup_{n\geq 0}\theta^{-n}(F)$, where F is the set of discontinuities of $\bar{f}$, has the property that it is dense and that $h|D' = g|D'$. Hence if $(a_n)_{n\geq 1}$ is any sequence in S^1 which tends to a given $a \in S^1$ from the right, $(\overline{g}(a_n))_{n\geq 1}$ tends to $\overline{h}(a)$. This implies that g is regulated. Similarly h is regulated.

Assume now that there are infinitely many $x_k \in S^1$ with $\delta_k := \overline{g}(x_k) - \overline{h}(x_k) \neq 0$. Then there exists an $n_k \in \mathbb{Z}_+$ with $\theta^{n_k}(x_k) \in F$. We may assume that k is so large that $\theta^{-n_k-n}(F) \cap F = \emptyset$ for $n \geq 0$. Note that $\bar{f}(\theta^{-1}(x)) = \overline{g}(x) - \overline{g}(\theta^{-1}(x))$ and $\bar{f}(\theta^{-1}(x)+) = \overline{h}(x) - \overline{h}(\theta^{-1}(x))$. Then for $n \geq 0$,

$$\bar{f}(\theta^{-n-1}(x_k)) = \overline{g}(\theta^{-n}(x_k)) - \overline{g}(\theta^{-n-1}(x_k)),$$
$$\bar{f}(\theta^{-n-1}(x_k)+) = \overline{h}(\theta^{-n}(x_k)) - \overline{h}(\theta^{-n-1}(x_k)),$$

and hence $\overline{g}(\theta^{-n}(x_k)) - \overline{h}(\theta^{-n}(x_k)) = \overline{g}(\theta^{-n-1}(x_k)) - \overline{h}(\theta^{-n-1}(x_k))$. Therefore $\delta_k = \overline{g}(\theta^{-n}(x_k)) - \overline{h}(\theta^{-n}(x_k))$ for $n \geq 0$, which implies that $g(x) - h(x) = \delta_k$ for all x in a dense set. This is a contradiction.

Finally, let us prove uniqueness. Assume that g and h are two such functions with $f(x) = g(x+\alpha) - g(x) = h(x+\alpha) - h(x)$. Then $g - h$ has periods 1 and α, hence the group of periods contains $\mathbb{Z} + \alpha\mathbb{Z}$ and so is dense. As $g - h$ is regulated, $g - h$ is constant.

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic regulated function which is left* (*resp. right*) *continuous and which has only finitely many discontinuities in* $[0,1]$, *and let $g : \mathbb{R} \to \mathbb{C}$ be a regulated periodic function such that $f(x) - \int_0^1 f(x)\,dx = g(x+\alpha) - g(x)$ for all $x \in \mathbb{R}$. Then g has only finitely many discontinuities in* $[0,1]$.

Proof. This follows immediately from Theorem 1.

Corollary 2. *Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic regulated function which has only finitely many discontinuities in* $[0,1]$. *The following assertions are equivalent*:

(1) $\alpha \in \mathrm{B}_f$.
(2) *There exists a periodic*, *bounded and integrable function $g : \mathbb{R} \to \mathbb{C}$ such that $f(x) - \int_0^1 f(x)\,dx = g(x+\alpha) - g(x)$ for all $x \in \mathbb{R}$.*

Proof. We may assume that $\int_0^1 f(x)\,dx = 0$.

As (2)⇒(1) is again trivial, we prove the converse. Put $\bar{f}(x) = f(x-)$. Then $\bar{f}$ is left continuous, regulated and $f(x) = \bar{f}(x)$ for all x with at most finitely many exceptions in $[0,1]$. Hence $\alpha \in \mathrm{B}_{\bar{f}}$. By Theorem 1 there exists a left continuous regulated function $\overline{g} : \mathbb{R} \to \mathbb{C}$ such that $\bar{f}(x) =$

$\overline{g}(x+\alpha)-\overline{g}(x)$. Put

$$g(x)=\overline{g}(x)-\sum_{i=0}^{\infty}(f(x+i\alpha)-\bar{f}(x+i\alpha)).$$

As f has only finitely many discontinuities, the sum is finite for every x and

$$g(x+\alpha)=\overline{g}(x+\alpha)-\sum_{i=1}^{\infty}(f(x+i\alpha)-\bar{f}(x+i\alpha)).$$

This implies

$$g(x+\alpha)-g(x)=\overline{g}(x+\alpha)-\overline{g}(x)+(f(x)-\bar{f}(x))=f(x).$$

If c is an upper bound for $|f|$, $|\overline{g}|$ and for the number of discontinuities of f, then $2c^2+c$ is an upper bound for $|g|$.

REMARKS. (1) Uniqueness in Corollary 2 is no longer true, as together with g also $g+h$ is a solution of the functional equation $f(x)-\int_0^1 f(x)\,dx = g(x+\alpha)-g(x)$, when h is any bounded function with periods 1 and α.

(2) Corollary 2 is no longer true if we demand that g should also be regulated. For example let $x_0\in[0,1)$ and let $a\in\mathbb{C}^{\times}$. Put

$$f(x)=\begin{cases} a, & \{x\}=x_0,\\ 0, & \text{else.}\end{cases}$$

Clearly $\mathrm{B}_f=\Omega$. Assume that $g:\mathbb{R}\to\mathbb{C}$ is regulated and that in addition $f(x)=g(x+\alpha)-g(x)$. Then $g(n\alpha+x)-g(x)=a\sum_{k=0}^{n-1}c_{x_0-k\alpha+\mathbb{Z}}(x)$ and hence for $n\ge n_x$,

$$g(n\alpha+x)-g(x)=\begin{cases} a, & x\in x_0-\alpha\mathbb{Z}_+ +\mathbb{Z},\\ 0, & x\notin x_0-\alpha\mathbb{Z}_+ +\mathbb{Z}.\end{cases}$$

If we let $\{n\alpha\}$ tend to $1-x$ from the left we see that g attends exactly two values, both on dense sets, which is impossible if g is regulated.

Nevertheless, $g=-ac_{x_0-\alpha\mathbb{Z}_+}$ is bounded and has the property that $g(x+\alpha)-g(x)=f(x)$.

In the next corollary we prove that Theorem 1 also implies generalizations of various known results. Hecke [19] and Kesten [26] have proved that for $f(x)=c_{[\beta,\gamma)}(\{x\})$ $(0\le\beta\le\gamma\le1)$, B_f consists of all α's for which $\gamma-\beta\in\mathbb{Z}+\mathbb{Z}\alpha$. It has also been noticed in [14] that for $f=c_{[\gamma,\gamma+\beta)+\mathbb{Z}}-c_{[\gamma',\gamma'+\beta)+\mathbb{Z}}$, $\alpha\in\mathrm{B}_f$ if and only if $\beta\in\mathbb{Z}+\alpha\mathbb{Z}$ or $\gamma-\gamma'\in\mathbb{Z}+\alpha\mathbb{Z}$. More generally, Oren [33] was the first to find a necessary and sufficient condition for $\alpha\in\mathrm{B}_f$ if f is a step function. He proved that $\alpha\in\mathrm{B}_f$ if and only if $\sum_{k\in\mathbb{Z}}(f(x+k\alpha+)-f(x+k\alpha-))=0$ for all $x\in\mathbb{R}$. Here we prove another such equivalence. The reader is also invited to consult [32, Theorem 3.1] and for ergodicity the papers [1], [15, Section 1.5] and [34].

Corollary 3. *Assume that $\alpha \in \Omega$. The complex vector space of periodic step functions with $\alpha \in \mathrm{B}_f$ is generated by the functions of the form $c_{I+\mathbb{Z}}$, where $I \subseteq [0,1)$ is an interval whose length is in $\mathbb{Z}+\alpha\mathbb{Z}$.*

Proof. Let $G := \mathbb{Z}+\alpha\mathbb{Z}$ and assume that $I \subseteq [0,1)$ is an interval whose length is in G. Then by the Hecke theorem, $\alpha \in \mathrm{B}_{c_I}$. For completeness we give a short proof. We may assume that I is of the form $[0,\beta)$ and $\beta = \{n\alpha\}$ for some $n > 0$ (otherwise consider $1 - c_{I+\mathbb{Z}}$). With $g(x) = -\sum_{i=1}^{n}\{x - i\alpha\}$ we have $g(x+\alpha) - g(x) = c_{I+\mathbb{Z}}(x) - \beta$.

Assume that conversely $f(x) = \sum_{i=0}^{m-1} a_i c_{I_i+\mathbb{Z}}(x)$, where $I_i \subseteq [0,1)$ are pairwise disjoint intervals. We may assume that f is right continuous, for f differs from a right continuous step function only by a linear combination of step functions of the form $c_{\beta+\mathbb{Z}}$. Then $f = \sum_{i=0}^{m-1} a_i c_{[\beta_i,\beta_{i+1})+\mathbb{Z}}$, where $\beta_0 = 0 < \beta_1 < \cdots < \beta_m = 1$. Put $a_{-1} = a_{m-1}$. Then $f = a_{m-1} + \sum_{i=0}^{m-1}(a_{i-1} - a_i)c_{[0,\beta_i)+\mathbb{Z}}$ and

$$f(x) - f(x-) = \sum_{i=0}^{m-1} a_i(c_{\beta_i+\mathbb{Z}}(x) - c_{\beta_{i+1}+\mathbb{Z}}(x)) = \sum_{i=0}^{m-1}(a_i - a_{i-1})c_{\beta_i+\mathbb{Z}}(x).$$

Let $T \subseteq \mathbb{R}$ be a complete system of representatives of $\mathbb{R}/G$ with $0 \in T$, and let g be periodic and regulated with $f(x) - f(x-) = g(x+\alpha) - g(x)$ (note that $\alpha \in \mathrm{B}_f \cap \mathrm{B}_{f(\cdot\,-)}$). Then for $k, n > 0$,

$$\sum_{i=0}^{m-1}(a_i - a_{i-1})c_{\beta_i - \alpha(\mathbb{Z}\cap[-k,n-1])+\mathbb{Z}}(x) = g(x+n\alpha) - g(x-k\alpha).$$

If we let $\{n\alpha\}$ tend to y from the right and $\{k\alpha\}$ to z from the left we get

$$g(x+y+) - g(x-z+) = \sum_{i=0}^{m-1}(a_i - a_{i-1})c_{\beta_i+G}(x) \quad \text{for all } x,y,z \in \mathbb{R}.$$

This implies that the right hand side is in fact zero. For $t \in T$ let $J_t := \{i \mid 0 \le i < m, \beta_i \in t+G\}$. Then

$$0 = \sum_{i=0}^{m-1}(a_i - a_{i-1})c_{\beta_i+G} = \sum_{t\in T}\sum_{i\in J_t}(a_i - a_{i-1})c_{t+G}.$$

As the sets $t+G$ are pairwise disjoint we get $\sum_{i\in J_t}(a_i - a_{i-1}) = 0$. (This is more or less Oren's condition; but we can go one step further.)

For $t \in T$ put $f_t = \sum_{i\in J_t}(a_i - a_{i-1})c_{[0,\beta_i)+\mathbb{Z}}$. Then clearly f_t is a right continuous periodic step function and $f_t = 0$ with at most finitely many exceptions $t \in T$. Furthermore, $f = a_{m-1} - \sum_{t\in T} f_t$. We prove that if β, β' are two discontinuities of f_t, then $\beta - \beta' \in G$. We distinguish two cases.

Assume first that $t \ne 0$. Then the condition $\sum_{i\in J_t}(a_i - a_{i-1}) = 0$ tells us that f_t is continuous at 0. Hence $\beta = \beta_i$, $\beta' = \beta_j$ for some $i, j \in J_t$, and hence $\beta - \beta' \in G$. Therefore f_t can be written as a linear combination of step functions of the form c_I where I has length in G.

If $t = 0$, then $\beta, \beta' \in G$ (possibly $= 0$) and hence again $\beta - \beta' \in G$.

REMARK. One must not think that if f is a periodic step function of the form $f = \sum_{i=1}^{n} a_i c_{I_i + \mathbb{Z}}$ with pairwise disjoint intervals I_i, and if $\alpha \in \mathrm{B}_f$, then the lengths of the I_i have to be in $\mathbb{Z} + \alpha\mathbb{Z}$. Consider the example $f = c_{[\gamma,\gamma+\beta)+\mathbb{Z}} - c_{[\gamma',\gamma'+\beta)+\mathbb{Z}}$ in [14], when $0 < \gamma' - \gamma \in \mathbb{Z} + \mathbb{Z}\alpha$. In that case f can be rewritten as $c_{[\gamma,\gamma')+\mathbb{Z}} - c_{[\gamma+\beta,\gamma'+\beta)+\mathbb{Z}}$.

3. On functions of bounded remainder with respect to all irrationals. We now investigate the case $\mathrm{B}_f = \Omega$ more closely. For the special case of f analytic the corresponding multiplicative problem was attacked in [3]. For a complete version of the proof the reader is referred to [4]. For $f \in \mathrm{C}^{1+\delta}$ the reader may consult [28]. If f is a C^1-function and f' is Lipschitz continuous see [23].

Note that if $f(x) = e^{2\pi i h x}$ and $h \neq 0$ is an integer, then $|\sum_{n=1}^{N} f(n\alpha)| \leq 1/|\sin \pi h\alpha|$ for all $\alpha \in \Omega$ and hence $\mathrm{B}_f = \Omega$. More generally, $\mathrm{B}_f = \Omega$ for every trigonometric polynomial f. In this section we prove the converse.

PROPOSITION 2. *Let $f : \mathbb{R} \to \mathbb{C}$ be periodic, integrable over $[0,1]$ with $\int_0^1 f(x)\,dx = 0$, let $F_N(x,\alpha) := \sum_{n=0}^{N-1} f(x + n\alpha)$, and assume that the set of $\alpha \in \Omega$ for which $\|F_N(\cdot,\alpha)\|_1$ is unbounded has cardinality less than that of the continuum. Then there exists a trigonometric polynomial t with $f = t$ almost everywhere.*

Proof. We may assume that f is real-valued. Let $(c_h)_{h\in\mathbb{Z}}$ be the sequence of Fourier coefficients of f. We have to prove that $c_h = 0$ for h large. We argue by contradiction.

Suppose $A := \{h > 0 \mid c_h \neq 0\}$ is infinite and let B be the set of all irrational α's with continued fraction expansion $[0; a_1, a_2, \ldots]$ and convergents p_n/q_n such that there exist two sequences $(m_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ of nonnegative integers—the first strictly increasing, the second consisting of positive numbers—with $a_{m_t+1}|c_{g_t q_{m_t}}| > g_t$. It is clear that if $B \neq \emptyset$, then B has the cardinality of the continuum, as at the infinitely many indices $m_t + 1$ we can replace a_{m_t+1} by any integer $a'_{m_t+1} > a_{m_t+1}$. We now prove that $B \neq \emptyset$.

We construct $(m_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ by induction on t. Let $m_0 = 0$, choose $g_0 \in A$ and let a_1 be any positive integer with $a_1|c_{g_0}| > g_0$. Assume now that $m_0, \ldots, m_t, g_0, \ldots, g_t$ and $a_1, \ldots, a_{m_t+1}$ are already defined. For $0 \leq k \leq m_t + 1$ let $p_k/q_k = [0; a_1, \ldots, a_k]$. Let $h \in A$ be chosen such that

$$h > q_{m_t+1}(q_{m_t+1} + q_{m_t}).$$

The inequality $h > q_{m_t} q_{m_t+1}$ implies that there are positive integers u, v with $h = u q_{m_t+1} + v q_{m_t}$. We may assume that $v < u$: the interval $\left(\frac{v-u}{q_{m_t}+q_{m_t+1}}, \frac{v}{q_{m_t}}\right)$

has length

$$\frac{v}{q_{m_t+1}} - \frac{v-u}{q_{m_t}+q_{m_t+1}} = \frac{h}{q_{m_t+1}(q_{m_t}+q_{m_t+1})} > 1$$

and therefore contains an integer w. If we put $u' = u + wq_{m_t}$ and $v' = v - wq_{m_t+1}$ we get $0 < v' < u'$ and $h = u'q_{m_t+1} + v'q_{m_t}$.

Define m_{t+1} and $a_{m_t+2}, \ldots, a_{m_{t+1}}$ by $v/u = [0; a_{m_t+2}, \ldots, a_{m_{t+1}}]$. Then

$$\begin{aligned}\frac{p_{m_{t+1}}}{q_{m_{t+1}}} &= [0; a_1, \ldots, a_{m_{t+1}}] = [0; a_1, \ldots, a_{m_t+1}, u/v] \\ &= \frac{p_{m_t+1}\frac{u}{v} + p_{m_t}}{q_{m_t+1}\frac{u}{v} + q_{m_t}} = \frac{p_{m_t+1}u + p_{m_t}v}{h}.\end{aligned}$$

Hence there is some $g_{t+1} > 0$ with $g_{t+1}q_{m_{t+1}} = h \in A$. Finally, choose $a_{m_{t+1}+1}$ such that $a_{m_{t+1}+1}|c_h| > g_{t+1}$ to complete the construction of an element in B.

As B has the cardinality of the continuum there exists an $\alpha \in B$ such that $\|F_N(\cdot,\alpha)\|_1$ is bounded. But then there exists a periodic integrable function $g : \mathbb{R} \to \mathbb{R}$ with $f(x) = g(x+\alpha) - g(x)$ almost everywhere. Let $(d_h)_{h\in\mathbb{Z}}$ be the sequence of Fourier coefficients of g. Then $c_h = d_h(e^{2\pi i h\alpha} - 1)$ and hence, as $(d_h)_{h\in\mathbb{Z}}$ tends to 0 for $|h| \to \infty$, we get $|c_h| \le 2|\sin \pi h\alpha|$ for $|h|$ large. In particular,

$$\begin{aligned}\frac{g_t}{a_{m_t+1}} &< |c_{g_t q_{m_t}}| \le 2|\sin \pi g_t(q_{m_t}\alpha - p_{m_t})| \\ &\le 2\pi g_t|q_{m_t}\alpha - p_{m_t}| \le \frac{2\pi g_t}{a_{m_t+1}q_{m_t}},\end{aligned}$$

and if t is large this is a contradiction.

Corollary 4. *Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic, regulated, left or right continuous function with only finitely many discontinuities in $[0,1)$. The following assertions are equivalent*:

(1) $\mathrm{B}_f = \Omega$.
(2) $\Omega \setminus \mathrm{B}_f$ *has cardinality less than that of the continuum.*
(3) f *is a trigonometric polynomial.*

Proof. It is clearly enough to prove that (2) implies (3) if $\int_0^1 f(x)\,dx = 0$. For $\alpha \in \mathbb{R}$ and positive integers N let $F_N(x,\alpha) = \sum_{i=0}^{N-1} f(x+i\alpha)$. Then by Proposition 1, $\|F_N(\cdot,\alpha)\|_1$ is bounded for $\alpha \in \mathrm{B}_f$. Hence by Proposition 2 there exists a trigonometric polynomial t with $f = t$ almost everywhere. As f is left or right continuous we get $f = t$.

It cannot happen that the remainder of f is uniformly bounded in α, except when f is constant. The assumptions on f can even be weakened:

PROPOSITION 3. *Let* $f:\mathbb{R}\to\mathbb{C}$ *be Riemann integrable over* $[0,1]$ *and periodic. Assume that*

$$g(\alpha):=\sup_{N\ge 1}\Big|\sum_{n=1}^{N} f(n\alpha)-N\int_0^1 f(x)\,dx\Big|$$

defines a quadratic integrable function g. *Then* f *is constant almost everywhere.*

Proof. Clearly we may assume that $\int_0^1 f(x)\,dx=0$. Let $F_N(x,\alpha)=\sum_{k=1}^{N} f(x+k\alpha)$ for $(x,\alpha)\in\mathbb{R}^2$. By Proposition 1,

$$\int_0^1 |F_N(x,\alpha)|^2\,dx\le 4g(\alpha)^2.$$

On the other hand, the left hand side is

$$\sum_{k=1}^{N}\sum_{l=1}^{N}\int_0^1 f(x+k\alpha)\overline{f(x+l\alpha)}\,dx=\sum_{k=1}^{N}\sum_{l=1}^{N}\int_0^1 f(x)\overline{f(x+(l-k)x)}\,dx$$
$$=\int_0^1 f(x)\sum_{u=1-N}^{N-1}(N-|u|)\overline{f(x+u\alpha)}\,dx.$$

Now we integrate both sides over α. We note that $\int_0^1\overline{f(x+u\alpha)}\,d\alpha=0$ except for $u=0$; hence $N\int_0^1|f(x)|^2\,dx\le 4\int_0^1 g(\alpha)^2\,d\alpha$. This implies that $f=0$ almost everywhere.

In particular, if such an f is of bounded remainder uniformly with respect to all irrational α's then f is constant almost everywhere.

4. The case when f is sufficiently smooth. We have seen that for step functions the set B_f can be described in terms of the lengths of the continuity intervals of f. But B_f is of a different nature when f is smooth in some sense. Whether $\alpha\in\mathrm{B}_f$ or not depends now on approximation properties of α by rationals.

For the converse question on how the vector space of smooth functions f for which $\alpha\in\mathrm{B}_f$ looks like the reader may consult e.g. [5].

We denote by $B_k(x)$ the periodic continuation of the kth Bernoulli polynomial in $[0,1)$ (the so-called kth Bernoulli function). We note that B_k is $k-2$-times continuously differentiable.

There are several papers which prove sufficient conditions for $\alpha\in\mathrm{B}_f$ when f is sufficiently smooth. The case $f=B_2$ has been settled in [41]. The general case $f=B_k$ has been solved in [39]. For the case where f is differentiable and f' is Lipschitz continuous the reader is referred to [23],

[15], [8, Théorème 1.2] and [37]. For estimates of B_f from below and from above one should also consult [20].

Let $\alpha \in \Omega$, $0 < \alpha < 1$. We need some known facts on the so-called Ostrowski expansion of a positive integer N to base α: there is exactly one sequence $(b_n)_{n\ge 0}$ of non-negative integers such that $b_0 < a_1$, for $i \ge 1$ we have $b_i \le a_{i+1}$ and $b_i = a_{i+1} \Rightarrow b_{i-1} = 0$, and $N = b_0q_0 + b_1q_1 + \cdots + b_mq_m$. This representation of N is called the *Ostrowski expansion* of N to base α. For a systematic investigation of Ostrowski expansions, their metric, number-theoretic and topological properties the reader is referred to [6].

In this section we restrict ourselves to the case when $f : \mathbb{R} \to \mathbb{C}$ is periodic and is a primitive of a function $g : \mathbb{R} \to \mathbb{C}$ of bounded variation on $[0,1]$. We expand $\sum_{n<N} f(n\alpha) - N\int_0^1 f(x)\,dx$ for N large asymptotically with an $O(1)$ error term (independent of α and N) and a main term written entirely in the digits $b_0,\ldots,b_m$ of N to base α. This formula enables one to determine B_f for such functions f.

THEOREM 2. *Let α be an irrational number with continued fraction expansion $[0;a_1,a_2,\ldots]$ and convergents p_n/q_n, let N be a positive integer with Ostrowski expansion $N = \sum_{i=0}^m b_iq_i$ to base α, and assume that $f : \mathbb{R} \to \mathbb{C}$ is periodic and is a primitive of a function $g : \mathbb{R} \to \mathbb{C}$ with variation $\le V$ in $[0,1]$. Then*

$$\sum_{n=1}^{N} f(n\alpha) - N\int_0^1 f(x)\,dx$$
$$= \sum_{k=0}^{m}(-1)^k a_{k+1}q_k\int_0^1 B_1(q_kx)\left(f\left(x+\frac{(-1)^kb_k}{a_{k+1}q_k}\right)-f(x)\right)dx + O(V)$$
$$= -\frac{1}{2}\sum_{k=0}^{m}(-1)^k a_{k+1}\int_0^1 B_2(q_kx)\left(g\left(x+\frac{(-1)^kb_k}{a_{k+1}q_k}\right)-g(x)\right)dx + O(V).$$

The O-constant is absolute.

For the proof we need some lemmas. Some of the ideas were already used in [22] and [23].

LEMMA 1. *There is a positive constant c with the following property: if r/s is a rational number represented in its lowest terms, if $r/s = [0;a_1,\ldots,a_t]$ is a continued fraction expansion (no matter which of the two existing ones), and if $D(r/s)$ is the discrepancy of the two-dimensional sequence $(n/s, \{rn/s\})_{0\le n<s}$ (resp. $(n/s,\{-rn/s\})_{0\le n<s}$), then*

$$D\left(\frac{r}{s}\right) \le c\sum_{i=1}^{t} a_i.$$

Proof. The first case was proved in [23, Lemma 2.1]. The same proof can be used to prove the other result.

Lemma 2. *There is a positive constant c with the following property: if $g:\mathbb{R}\to\mathbb{C}$ is a periodic function of bounded variation V on $[0,1]$ with $\int_0^1 g(x)\,dx=0$, if α is an irrational number with continued fraction expansion $\alpha=[0;a_1,a_2,\ldots]$ and convergents p_n/q_n, and if $x\in[0,1]$, then*

$$\Bigg|\sum_{t=0}^{q_m-1} c_{[x,1)}\bigg(\bigg\{-\frac{q_{m-1}t}{q_m}\bigg\}\bigg) g\bigg(x(q_m\alpha-p_m)+\frac{t}{q_m}\bigg)\Bigg|\le cV\sum_{i=1}^m a_i \quad \textit{for } m \textit{ even},$$

$$\Bigg|\sum_{t=1}^{q_m} c_{[x,1)}\bigg(\bigg\{\frac{q_{m-1}t}{q_m}\bigg\}\bigg) g\bigg(x(q_m\alpha-p_m)+\frac{t}{q_m}\bigg)\Bigg|\le cV\sum_{i=1}^m a_i \quad \textit{for } m \textit{ odd}.$$

Proof. Assume first that m is even. Note that $\int_0^1 g(x)\,dx=0$ implies that $\|g\|_{\mathrm{u}}\le V$. Let

$$h_x:[0,1]^2\to\mathbb{C},\qquad h_x(u,v)=c_{[x,1)}(u)g(x(q_m\alpha-p_m)+v).$$

Let us first estimate the total variation of h_x in the sense of Hardy and Krause (see [27, p. 147] for this concept). We have

$$h_x(0,v)=\begin{cases}0, & x\ne 0,\\ g(v), & x=0,\end{cases}$$

and hence $V_0^1(h_x(0,\cdot))\le V$. Furthermore

$$h_x(1,v)=0 \quad\text{and}\quad h_x(u,0)=c_{[x,1)}(u)g(x(q_m\alpha-p_m)),$$

hence $V_0^1(h_x(\cdot,0))\le V$. Analogously $V_0^1(h_x(\cdot,1))\le V$. Finally, let $0=u_0<\cdots<u_k=1$ and $0=v_0<\cdots<v_l=1$ be two finite sequences and choose r such that $u_r<x\le u_{r+1}$. Then

$$\begin{aligned}\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}&|h_x(u_{i+1},v_{j+1})-h_x(u_i,v_{j+1})-h_x(u_{i+1},v_j)+h_x(u_i,v_j)|\\ &=\sum_{j=0}^{l-1}|h_x(u_{r+1},v_{j+1})-h_x(u_r,v_{j+1})-h_x(u_{r+1},v_j)+h_x(u_r,v_j)|\\ &=\sum_{j=0}^{l-1}|g(x(q_m\alpha-p_m)+v_{j+1})-g(x(q_m\alpha-p_m)+v_j)|\le V.\end{aligned}$$

This implies that h_x has total variation $\le 4V$.

Furthermore, we have

$$\int_0^1\!\!\int_0^1 h_x(u,v)\,du\,dv=\int_x^1\!\!\int_0^1 g(x(q_m\alpha-p_m)+v)\,dv\,du=(1-x)\int_0^1 g(v)\,dv=0.$$

Hence we get the result by the Koksma–Hlawka inequality ([27, Chapter 2, Theorem 5.5]), by $q_{m-1}/q_m = [0; a_m, \ldots, a_1]$ and by Lemma 1.

The second inequality is proved similarly.

LEMMA 3. *Let $g : [0,1] \to \mathbb{C}$ be a regulated function with a primitive f, let $q \geq 1$ be an integer and for $0 \leq n < q$ let $\delta_n \in [0,1)$. Then with*

$$T(x) = \sum_{\substack{n+\delta_n \leq x \\ n \geq 0}} 1, \qquad \textit{resp.} \quad T(x) = \sum_{\substack{n-\delta_n \leq x \\ n \geq 1}} 1,$$

the following formula holds:

$$\sum_{n=0}^{q-1} f\left(\frac{n+\delta_n}{q}\right) = \frac{1}{2}(f(1)-f(0)) + \int_0^q f\left(\frac{x}{q}\right) dx + \frac{1}{q}\int_0^q \left(x - T(x) - \frac{1}{2}\right) g\left(\frac{x}{q}\right) dx,$$

resp.

$$\sum_{n=1}^{q} f\left(\frac{n-\delta_n}{q}\right) = \frac{1}{2}(f(1)-f(0)) + \int_0^q f\left(\frac{x}{q}\right) dx + \frac{1}{q}\int_0^q \left(x - T(x) - \frac{1}{2}\right) g\left(\frac{x}{q}\right) dx.$$

Proof. We have

$$\begin{aligned}
\int_0^q \left(x - T(x) - \frac{1}{2}\right) g\left(\frac{x}{q}\right) dx &= \int_0^q \left(x - \frac{1}{2}\right) g\left(\frac{x}{q}\right) dx - \sum_{n=0}^{q-1} \int_{n+\delta_n}^{q} g\left(\frac{x}{q}\right) dx \\
&= q\left(x - \frac{1}{2}\right) f\left(\frac{x}{q}\right)\Big|_0^q - q\int_0^q f\left(\frac{x}{q}\right) dx - \sum_{n=0}^{q-1} q\left(f(1) - f\left(\frac{n+\delta_n}{q}\right)\right) \\
&= q\left(q - \frac{1}{2}\right) f(1) + \frac{1}{2}\, q f(0) - q\int_0^q f\left(\frac{x}{q}\right) dx - q^2 f(1) + q\sum_{n=0}^{q-1} f\left(\frac{n+\delta_n}{q}\right) \\
&= \frac{1}{2}\, q(f(0) - f(1)) - q\int_0^q f\left(\frac{x}{q}\right) dx + q\sum_{n=0}^{q-1} f\left(\frac{n+\delta_n}{q}\right).
\end{aligned}$$

The second statement is proved similarly.

LEMMA 4. *Let $\alpha = [0; a_1, a_2, \ldots]$ be an irrational number with convergents p_n/q_n, $f : \mathbb{R} \to \mathbb{C}$ periodic and a primitive of a function $g : \mathbb{R} \to \mathbb{C}$ with variation $\leq V$ on $[0,1]$, m a positive integer, $s_m := q_m\alpha - p_m$, and suppose that $0 \leq k \leq a_{m+1}$. Then for m even,*

$$\sum_{n=0}^{q_m-1} \left(f\left(\frac{n}{q_m} + s_m\left\{-\frac{q_{m-1}n}{q_m}\right\} + ks_m\right) - f\left(\frac{n}{q_m} + ks_m\right)\right) = O\left(s_m V \sum_{i=1}^{m} a_i\right),$$

and for m odd,

$$\sum_{n=1}^{q_m} \left(f\left(\frac{n}{q_m} + s_m\left\{\frac{q_{m-1}n}{q_m}\right\} + ks_m\right) - f\left(\frac{n}{q_m} + ks_m\right)\right) = O\left(|s_m| V \sum_{i=1}^{m} a_i\right).$$

The O-constants are absolute.

Proof. Assume first that m is even, let $\delta_{n,m} = q_m s_m\{-q_{m-1}n/q_m\}$ and $T_m(x) = \sum_{n+\delta_{n,m}\le x} 1$. Note that

$$T_m(x) = [x] + 1 - c_{(\{x\},1)}(\delta_{[x],m}).$$

By Lemma 3 and the periodicity of f the left hand side is equal to

$$\frac{1}{q_m}\int_0^{q_m}\left(x - T_m(x) - \frac{1}{2} - x + [x] + 1 + \frac{1}{2}\right) g\left(\frac{x}{q_m} + ks_m\right) dx$$
$$= \frac{1}{q_m}\int_0^{q_m} c_{[\{x\},1)}(\delta_{[x],m})\, g\left(\frac{x}{q_m} + ks_m\right) dx$$
$$= \frac{1}{q_m}\sum_{t=0}^{q_m-1}\int_t^{t+1} c_{[\{x\},1)}(\delta_{t,m})\, g\left(\frac{x}{q_m} + ks_m\right) dx$$
$$= \frac{1}{q_m}\sum_{t=0}^{q_m-1}\int_0^{q_m s_m} c_{[x,1)}(\delta_{t,m})\, g\left(\frac{x+t}{q_m} + ks_m\right) dx$$
$$= \frac{1}{q_m}\int_0^{q_m s_m}\sum_{t=0}^{q_m-1} c_{[x/q_m s_m,1)}\left(\left\{-\frac{q_{m-1}t}{q_m}\right\}\right) g\left(\frac{x+t}{q_m} + ks_m\right) dx$$
$$= \frac{q_m s_m}{q_m}\int_0^{1}\sum_{t=0}^{q_m-1} c_{[x,1)}\left(\left\{-\frac{q_{m-1}t}{q_m}\right\}\right) g\left(xs_m + \frac{t}{q_m} + ks_m\right) dx.$$

By Lemma 2 applied to $t \mapsto g(t + ks_m)$ $(t \in \mathbb{R})$—which has the same total variation as g—we deduce that the inner sum is $O(V\sum_{i=1}^{m} a_i)$. Hence we get the result above.

The proof for m odd is similar.

Remark 1. Let $g : [0,1] \to \mathbb{R}$ be a function of variation $\le V$. Then

$$\int_0^{t}\left(\{x\} - \frac{1}{2}\right) g\left(\frac{x}{t}\right) dx = O(V + \|g\|_{\mathrm{u}}),$$

where the O-constant is absolute.

Assume first that g is increasing. Then by the mean value theorem there is an $x_t \in [0,t]$ such that the above integral is

$$g(0)\int_0^{x_t}\left(\{x\} - \frac{1}{2}\right) dx + g(1)\int_{x_t}^{t}\left(\{x\} - \frac{1}{2}\right) dx = O(|g(0)| + |g(1)|).$$

In the general case $g = g_1 - g_2$ for some increasing functions g_1, g_2, where $g_1 = O(V)$ and $g_2 = O(V)$. Hence the result.

Remark 2. Let $g:\mathbb{R}\to\mathbb{C}$ be periodic and assume that g has bounded variation V on $[0,1]$. Let t,q be positive integers and y,z real numbers. Then

$$\int_0^1 B_t(qx)(g(x+y)-g(x+z))\,dx = O(V|y-z|).$$

The O-constant depends at most on t.

For the proof we may assume that $g:\mathbb{R}\to\mathbb{R}$. We first prove that

$$\int_0^1 (B_t(qx-qy)-B_t(qx-qz))g(x)\,dx = O(V|y-z|)$$

whenever g is of bounded variation V on $[0,1]$, periodic or not.

For this purpose we first assume that g is increasing on $[0,1]$. Then by the second mean value theorem there is a $u\in[0,1]$ such that the integral in question is equal to

$$\begin{aligned}
&g(0)\int_0^u (B_t(qx-qy)-B_t(qx-qz))\,dx + g(1)\int_u^1 (B_t(qx-qy)-B_t(qx-qz))\,dx\\
&= \frac{g(0)}{q}\int_0^{qu} (B_t(x-qy)-B_t(x-qz))\,dx + \frac{g(1)}{q}\int_{qu}^{q} (B_t(x-qy)-B_t(x-qz))\,dx\\
&= \frac{g(0)}{(t+1)q}(B_{t+1}(qu-qy)-B_{t+1}(qu-qz)-B_{t+1}(-qy)+B_{t+1}(-qz))\\
&\quad+\frac{g(1)}{(t+1)q}(B_{t+1}(-qy)-B_{t+1}(-qz)-B_{t+1}(qu-qy)+B_{t+1}(qu-qz))\\
&= \frac{g(0)-g(1)}{(t+1)q}(B_{t+1}(qu-qy)-B_{t+1}(qu-qz))\\
&\quad+\frac{g(1)-g(0)}{(t+1)q}(B_{t+1}(-qy)-B_{t+1}(-qz)).
\end{aligned}$$

As B_{t+1} is Lipschitz continuous, there is a $c=c_t>0$ such that $|B_{t+1}(a)-B_{t+1}(b)|\le c|a-b|$ for all $a,b\in\mathbb{R}$. Inserting this above we get the result in this case.

In the general case let $g_1(x)=V_0^x(g)$ and $g_2=g_1-g$. Then $g=g_1-g_2$ and g_1,g_2 are both increasing on $[0,1]$. Furthermore, $g_1(1)-g_1(0)=V$ and $g_2(1)-g_2(0)=V-(g(1)-g(0))\le 2V$. Applying the result for g_1 and g_2 separately, we deduce it for g itself.

Now if g is periodic we have

$$\int_0^1 (B_t(qx-qy)-B_t(qx-qz))g(x)\,dx = \int_0^1 B_t(qx)(g(x+y)-g(x+z))\,dx.$$

Proof of Theorem 2. Note that both sides of the conclusion of the theorem remain unchanged if we replace f by $f + c$, where c is a constant. Hence we may assume that $\int_0^1 f(x)\,dx = 0$. Note that $\int_0^1 g(t)\,dt = 0$ implies $\|g\|_{\mathrm{u}} \le V$.

Let $N_k = \sum_{i=0}^k b_i q_i$ and $s_k := q_k\alpha - p_k$. The left hand side of the conclusion is equal to

$$\sum_{k=0}^{m} \sum_{n=N_{k-1}+1}^{N_k} f(n\alpha) = \sum_{k=0}^{m} \sum_{n=1}^{b_k q_k} f((n+N_{k-1})\alpha).$$

Let

$$A_{k,r} := \sum_{n=1}^{b_k q_k} f((n+r)\alpha) \quad \text{for } r < q_k.$$

Then

$$\begin{aligned} A_{k,r} - A_{k,r-1} &= \sum_{n=r+1}^{r+b_k q_k} f(n\alpha) - \sum_{n=r}^{r-1+b_k q_k} f(n\alpha) \\ &= f((r+b_k q_k)\alpha) - f(r\alpha) = f(\{r\alpha\} + b_k s_k) - f(\{r\alpha\}) \end{aligned}$$

and hence

$$\begin{aligned} A_{k,N_{k-1}} - A_{k,0} &= \sum_{r=1}^{N_{k-1}} (A_{k,r} - A_{k,r-1}) = \sum_{r=1}^{N_{k-1}} (f(\{r\alpha\} + b_k s_k) - f(\{r\alpha\})) \\ &= \sum_{r=1}^{N_{k-1}} \int_0^{b_k s_k} g(\{r\alpha\} + x)\,dx = \int_0^{b_k s_k} \sum_{r=1}^{N_{k-1}} g(\{r\alpha\} + x)\,dx. \end{aligned}$$

For all $x \in \mathbb{R}$, V is the total variation of $y \mapsto g(x+y)$ $(0 \le y \le 1)$. The discrepancy of the finite sequence $(\{r\alpha\})_{1\le r\le N_{k-1}}$ is $O(\sum_{i=1}^k a_i)$ and $\int_0^1 g(y+x)\,dy = 0$. Hence the Koksma inequality implies

$$\Big|\sum_{r=1}^{N_{k-1}} g(\{r\alpha\} + x)\Big| = O\Big(V \sum_{i=1}^{k} a_i\Big).$$

Therefore

$$A_{k,N_{k-1}} - A_{k,0} = O\Big(V b_k |s_k| \sum_{i=1}^{k} a_i\Big) = O\Big(\frac{V}{q_k} \sum_{i=1}^{k} a_i\Big).$$

Summing up we get

$$\begin{aligned} \sum_{n=1}^{N} f(n\alpha) - \sum_{k=0}^{m} A_{k,0} &= O\Big(\sum_{k=0}^{m} \frac{V}{q_k} \sum_{i=1}^{k} a_i\Big) = O\Big(V \sum_{i=1}^{m} a_i \sum_{k=i}^{m} \frac{1}{q_k}\Big) \\ &= O\Big(V \sum_{i=1}^{m-1} \frac{a_i}{q_i}\Big) = O(V). \end{aligned}$$

Assume now that $0 \le k \le m$ and let σ_k be the permutation of $\{1, \dots, b_k q_k\}$ such that $\{\sigma_k(n)\alpha\} < \{\sigma_k(n+1)\alpha\}$ for $1 \le n < b_k q_k$. Then for k even, by what has been proved in [7, Proposition 1],

$$\{\sigma_k(n)\alpha\} = \begin{cases} n s_k, & 1 \le n \le b_k, \\ \dfrac{1}{q_k}\left[\dfrac{n-1}{b_k}\right] + s_k\left(b_k\left\{\dfrac{n-1}{b_k}\right\} + \left\{-\dfrac{q_{k-1}}{q_k}\left[\dfrac{n-1}{b_k}\right]\right\}\right), & b_k < n \le b_k q_k. \end{cases}$$

Therefore

$$\begin{aligned} A_{k,0} &= \sum_{n=1}^{b_k} f(ns_k) + \sum_{t=1}^{q_k-1} \sum_{m=tb_k+1}^{(t+1)b_k} f(\{\sigma_k(m)\alpha\}) \\ &= \sum_{n=1}^{b_k} f(ns_k) + \sum_{t=1}^{q_k-1} \sum_{m=0}^{b_k-1} f\left(\frac{t}{q_k} + s_k\left(m + \left\{-\frac{q_{k-1}t}{q_k}\right\}\right)\right) \\ &= \sum_{t=0}^{q_k-1} \sum_{m=0}^{b_k-1} f\left(\frac{t}{q_k} + s_k\left(m + \left\{-\frac{q_{k-1}t}{q_k}\right\}\right)\right) + f(b_k s_k) - f(0) \\ &= \sum_{t=0}^{q_k-1} \sum_{m=0}^{b_k-1} f\left(\frac{t}{q_k} + ms_k\right) + O\left(V \sum_{m=0}^{b_k-1} s_k \sum_{i=1}^{k} a_i\right) + O\left(\frac{V}{q_k}\right) \end{aligned}$$

by Lemma 4 and as f is Lipschitz continuous. The remainder term is $O((V/q_k)\sum_{i=1}^{k} a_i)$. Similarly, if k is odd we get

$$\begin{aligned} A_{k,0} &= \sum_{t=1}^{q_k} \sum_{m=0}^{b_k-1} f\left(\frac{t}{q_k} + ms_k\right) + O\left(\frac{V}{q_k} \sum_{i=1}^{k} a_i\right) \\ &= \sum_{t=0}^{q_k-1} \sum_{m=0}^{b_k-1} f\left(\frac{t}{q_k} + ms_k\right) + O\left(\frac{V}{q_k} \sum_{i=1}^{k} a_i\right) \end{aligned}$$

as $f(1 + ms_k) = f(m(q_k\alpha - p_k))$. The sum of the remainder terms is again

$$V \sum_{k=0}^{m} \frac{1}{q_k} \sum_{i=1}^{k} a_i = O(V).$$

Hence

$$\sum_{n=1}^{N} f(n\alpha) = \sum_{k=0}^{m} \sum_{n=0}^{q_k-1} \sum_{m=0}^{b_k-1} f\left(\frac{n}{q_k} + ms_k\right) + O(V).$$

We now have

$$\begin{aligned} \sum_{n=0}^{q_k} f\left(\frac{n}{q_k} + x\right) &= \frac{1}{2} f(x) + \frac{1}{2} f(1+x) \\ &\quad + \int_0^{q_k} f\left(\frac{y}{q_k} + x\right) dy + \frac{1}{q_k} \int_0^{q_k} \left(\{y\} - \frac{1}{2}\right) g\left(\frac{y}{q_k} + x\right) dy \end{aligned}$$

and hence

$$\sum_{n=0}^{q_k-1} f\Big(\frac{n}{q_k}+x\Big) = \frac{1}{q_k}\int_0^{q_k}\Big(\{y\}-\frac{1}{2}\Big)g\Big(\frac{y}{q_k}+x\Big)\,dy = O\Big(\frac{V}{q_k}\Big)$$

by Remark 1, where the O-constant does not depend on x. Furthermore,

$$\sum_{m=0}^{b_k} f\Big(\frac{n}{q_k}+ms_k\Big) = \frac{1}{2}f\Big(\frac{n}{q_k}+b_ks_k\Big)+\frac{1}{2}f\Big(\frac{n}{q_k}\Big) + \int_0^{b_k} f\Big(\frac{n}{q_k}+ys_k\Big)\,dy + s_k\int_0^{b_k}\Big(\{y\}-\frac{1}{2}\Big)g\Big(\frac{n}{q_k}+ys_k\Big)\,dy.$$

But $\sum_{n=0}^{q_k-1} g(n/q_k+ys_k) = O(V)$ as the discrepancy of the sequence $(n/q_k)_{0\le n<q_k}$ is $O(1)$ and the O-constant does not again depend on y or k. Therefore

$$s_k\sum_{n=0}^{q_k-1}\int_0^{b_k}\Big(\{y\}-\frac{1}{2}\Big)g\Big(\frac{n}{q_k}+ys_k\Big)\,dy = O(Vb_k|s_k|) = O\Big(\frac{V}{q_k}\Big).$$

Altogether this results in

$$\begin{aligned}\sum_{n=0}^{q_k-1}\sum_{m=0}^{b_k-1} f\Big(\frac{n}{q_k}+ms_k\Big) &= \sum_{n=0}^{q_k-1}\int_0^{b_k} f\Big(\frac{n}{q_k}+ys_k\Big)\,dy + O\Big(\frac{V}{q_k}\Big)\\ &= \frac{1}{q_k}\int_0^{b_k}\int_0^{q_k}\Big(\{x\}-\frac{1}{2}\Big)g\Big(\frac{x}{q_k}+ys_k\Big)\,dx\,dy + O\Big(\frac{V}{q_k}\Big)\\ &= \int_0^{b_k}\int_0^{1}\Big(\{q_kx\}-\frac{1}{2}\Big)g(x+ys_k)\,dx\,dy + O\Big(\frac{V}{q_k}\Big)\\ &= \frac{1}{s_k}\int_0^{1}\Big(\{q_kx\}-\frac{1}{2}\Big)(f(x+b_ks_k)-f(x))\,dx + O\Big(\frac{V}{q_k}\Big).\end{aligned}$$

Collecting everything we get

$$\sum_{n=1}^{N} f(\{n\alpha\}) = \sum_{k=0}^{m}\frac{1}{s_k}\int_0^1 B_1(q_kx)(f(x+b_ks_k)-f(x))\,dx + O(V),$$

and by integration by parts the main term is equal to

$$-\frac{1}{2}\sum_{k=0}^{m}\frac{1}{s_kq_k}\int_0^1 B_2(q_kx)(g(x+b_ks_k)-g(x))\,dx.$$

We prove the second formula first. The other follows again by integration by parts.

Remark 2 implies (with $\alpha_k = [a_k; a_{k+1}, \ldots])$

$$\int_0^1 B_2(q_k x)\Big(g(x + b_k(q_k\alpha - p_k)) - g\Big(x + \frac{(-1)^k b_k}{a_{k+1}q_k}\Big)\Big)\,dx$$

$$= O\Big(Vb_k\Big|\frac{1}{a_{k+1}q_k} - |q_k\alpha - p_k|\Big|\Big) = O\Big(Vb_k\Big(\frac{1}{a_{k+1}q_k} - \frac{1}{\alpha_{k+1}q_k + q_{k-1}}\Big)\Big)$$

$$= O\Big(Vb_k\,\frac{(\alpha_{k+1} - a_{k+1})q_k + q_{k-1}}{q_{k+1}^2}\Big) = O(Vb_k q_k/q_{k+1}^2) = O(V/q_{k+1}).$$

Hence

$$-\frac{1}{2}\sum_{k=0}^{m}\frac{1}{q_k(q_k\alpha - p_k)}\int_0^1 B_2(\{q_k x\})(g(x + b_k(q_k\alpha - p_k)) - g(x))\,dx$$

$$+\frac{1}{2}\sum_{k=0}^{m}\frac{1}{q_k(q_k\alpha - p_k)}\int_0^1 B_2(\{q_k x\})\Big(g\Big(x + \frac{(-1)^k b_k}{a_{k+1}q_k}\Big) - g(x)\Big)\,dx$$

$$= O\Big(V\sum_{k=0}^{m} a_{k+1}/q_{k+1}\Big) = O(V).$$

Furthermore, $\frac{1}{q_k(q_k\alpha - p_k)} - (-1)^k a_{k+1} = O(1)$. Therefore, again by Remark 2,

$$-\frac{1}{2}\sum_{k=0}^{m}\Big(\frac{1}{q_k(q_k\alpha-p_k)} - (-1)^k a_{k+1}\Big)\int_0^1 B_2(\{q_k x\})\Big(g\Big(x+\frac{(-1)^k b_k}{a_{k+1}q_k}\Big) - g(x)\Big)\,dx$$

$$= O\Big(\sum_{k=0}^{m} V\frac{b_k}{a_{k+1}q_k}\Big) = O(V).$$

5. Corollaries, applications and examples

Remark 3. Assume that f is periodic and is a primitive of a function $g : \mathbb{R} \to \mathbb{C}$ of bounded variation on $[0,1]$. Then $\alpha \in \mathrm{B}_f$ if and only if

$$\sum_{k=0}^{m}(-1)^k a_{k+1}q_k\int_0^1 B_1(q_k x)(f(x + (-1)^k x_k/q_k) - f(x))\,dx$$

is bounded in m and in $x_0, \ldots, x_m$, where $x_k \in [0,1)$, $0 \le k \le m$. This can be seen by defining $b_k = [x_k a_{k+1}]$ for $0 \le k \le m$, by noting that $0 \le b_k < a_{k+1}$, $\sum_{k=0}^{m} b_k q_k$ is the Ostrowski expansion of $N_m := \sum_{k=0}^{m} b_k q_k$ and by using $b_k/a_{k+1} = x_k + O(1/a_{k+1})$ together with Remark 2.

Corollary 5. *Let t be a positive integer, α an irrational number with continued fraction expansion $[0; a_1, a_2, \ldots]$ and convergents p_n/q_n, and N be a positive integer with Ostrowski expansion $N = \sum_{i=0}^{m} b_i q_i$ to base α. Assume that $f : \mathbb{R} \to \mathbb{C}$ is a $t-1$-times differentiable periodic function, and*

$f^{(t-1)}$ is a primitive of a function g of bounded variation. Then

$$\sum_{n=1}^{N} f(n\alpha) - N\int_0^1 f(x)\,dx$$
$$= \frac{(-1)^t}{(t+1)!}\sum_{k=0}^{m}\frac{(-1)^k a_{k+1}}{q_k^{t-1}}\int_0^1 B_{t+1}(q_k x)\Big(g\Big(x+\frac{(-1)^k b_k}{a_{k+1}q_k}\Big)-g(x)\Big)\,dx + O(1).$$

The O-constant depends at most on f.

Proof. Let $f^{(t)} := g$. By induction on j, $0 \le j \le t$, we have for positive integers q and $y \in \mathbb{R}$,

$$\int_0^1 B_1(qx)(f(x+y)-f(x))\,dx = \frac{(-1)^j}{(j+1)!q^j}\int_0^1 B_{j+1}(qx)(f^{(j)}(x+y)-f^{(j)}(x))\,dx.$$

Putting $j = t$ we get the result by Theorem 2.

Corollary 6. *Let t be a positive integer, α an irrational number with continued fraction expansion $[0; a_1, a_2, \ldots]$ and convergents p_n/q_n, and N, m positive integers with $q_m \le N < q_{m+1}$. Assume that $f : \mathbb{R} \to \mathbb{C}$ is $t-1$-times differentiable, periodic and $f^{(t-1)}$ is a primitive of a function g of bounded variation. Then*

$$\sum_{n=1}^{N} f(n\alpha) - N\int_0^1 f(x)\,dx = O\Big(\sum_{k=0}^{m}\frac{a_{k+1}}{q_k^t}\Big).$$

The O-constant depends at most on f.

Proof. This is an immediate consequence of Corollary 5.

Clearly Corollary 6 implies that for any periodic $f : \mathbb{R} \to \mathbb{C}$, which is a primitive of a function $g : \mathbb{R} \to \mathbb{C}$ of bounded variation on $[0, 1]$, $\Omega \setminus \mathrm{B}_f$ is a set of measure 0, and that (e.g. by Roth's theorem) B_f contains the real algebraic irrationals.

Corollary 7. *Let α be an irrational number with continued fraction expansion $[0; a_1, a_2, \ldots]$ and convergents p_n/q_n, and let N be a positive integer with Ostrowski expansion $N = \sum_{i=0}^{m} b_i q_i$ to base α. Assume that $f : \mathbb{R} \to \mathbb{C}$ is periodic and a primitive of a function g of bounded variation V on $[0, 1]$ and has Fourier coefficients $(c_h)_{h\in\mathbb{Z}}$. Then*

$$\sum_{n=1}^{N} f(n\alpha) - N\int_0^1 f(x)\,dx$$
$$= \frac{1}{2\pi i}\sum_{k=0}^{m}(-1)^k a_{k+1}q_k\sum_{h\ne 0}\frac{1}{h}c_{hq_k}(e^{2\pi i h(-1)^k b_k/a_{k+1}} - 1) + O(V).$$

Proof. This follows from the fact that for positive integers q and for $y \in \mathbb{R}$,

$$\int_0^1 B_1(qx)(f(x+y)-f(x))\,dx = -\frac{1}{2\pi i}\sum_{h\neq 0}\frac{1}{h}\,c_{-hq}(e^{-2\pi i q h y}-1),$$

and from Theorem 2.

Remark 2. In view of Remark 1 we have $\alpha \in \mathrm{B}_f$ if and only if

$$\frac{1}{2\pi i}\sum_{k=0}^{m}(-1)^k a_{k+1}q_k\sum_{h\neq 0}\frac{1}{h}\,c_{hq_k}(e^{2\pi i h(-1)^k x_k}-1)$$

is bounded in $m \geq 0$ and in $x_k \in [0,1),\ 0 \leq k \leq m$.

The corollaries above are now best suited to determine B_f for functions f as considered in the last section. To illustrate our method we present two examples. The first corollary has already been proved in [39] by different methods and by using special properties of the Bernoulli polynomials.

Corollary 8. *Let $t \geq 1$ and let $\alpha = [0; a_1, a_2, \ldots]$ be the continued fraction expansion of α with convergents p_n/q_n. Then*

$$\mathrm{B}_{B_t} = \Big\{\alpha \in \Omega \;\Big|\; \sum_{k=0}^{\infty} a_{k+1}/q_k^{t-1} < \infty\Big\}.$$

Proof. Let $\alpha \in \mathrm{B}_{B_t}$. We have

$$B_t(x) = -\frac{t!}{(2\pi i)^t}\sum_{h\neq 0}\frac{1}{h^t}\,e^{2\pi i h x}.$$

Therefore $c_h = -t!/(2\pi i h)^t$ and hence for positive integers q and for $y \in \mathbb{R}$ we have

$$\sum_{h\neq 0}\frac{1}{h}\,c_{hq}(e^{2\pi i h y}-1) = \frac{2\pi i}{t+1}\,q^{-t}(B_{t+1}(y)-B_{t+1}(0)).$$

Furthermore, $B_{t+1}((-1)^k x_k) - B_{t+1}(0) = (-1)^{k(t+1)}(B_{t+1}(x_k)-B_{t+1}(0))$. Hence

$$\sum_{k=0}^{m}(-1)^{kt}a_{k+1}q_k^{1-t}(B_{t+1}(x_k)-B_{t+1}(0)) = O(1).$$

From this point onward the argument is the same as in [39]; we repeat it for completeness. Choose $x_0 \in (0,1)$ with $B_{t+1}(x_0) \neq B_{t+1}(0)$ and $\varepsilon \in \{0,1\}$. Put $x_k = \frac{1}{2}(1+(-1)^{kt+\varepsilon})x_0$. Then

$$\begin{aligned}B_{t+1}(x_k) &= B_{t+1}\Big(\frac{1}{2}\,(1+(-1)^{kt+\varepsilon})x_0\Big)\\ &= \frac{1}{2}\,(1+(-1)^{kt+\varepsilon})B_{t+1}(x_0) + \frac{1}{2}\,(1-(-1)^{kt+\varepsilon})B_{t+1}(0).\end{aligned}$$

This implies

$$\begin{aligned}(B_{t+1}(x_0) - B_{t+1}(0)) \sum_{k=0}^{m} \frac{1}{2}\,((-1)^{kt} + (-1)^{\varepsilon}) a_{k+1} q_k^{1-t} \\ = \sum_{k=0}^{m} (-1)^{kt} \frac{1}{2}\,(1 + (-1)^{kt+\varepsilon})(B_{t+1}(x_0) - B_{t+1}(0)) a_{k+1} q_k^{1-t} \\ = \sum_{k=0}^{m} (-1)^{kt} (B_{t+1}(x_k) - B_{t+1}(0)) a_{k+1} q_k^{1-t} = O(1)\end{aligned}$$

for $\varepsilon \in \{0,1\}$. If we choose $\varepsilon = 0$ we get $\sum_{2|k} a_{k+1}/q_k^{t-1} < \infty$. If we choose $\varepsilon \equiv t \pmod 2$ we get $\sum_{2\nmid k} a_{k+1}/q_k^{t-1} < \infty$. Hence $\sum_{k=0}^{\infty} a_{k+1}/q_k^{t-1} < \infty$.

The converse follows immediately from Corollary 6.

Next we present an example of an analytic f:

COROLLARY 9. *Let a be a complex number with $|a| < 1$ and let*

$$f(x) = \frac{ae^{2\pi i x}}{1 - ae^{2\pi i x}}.$$

Then

$$\mathrm{B}_f = \Big\{ \alpha \in \Omega \;\Big|\; \sum_{k=0}^{\infty} a_{k+1} q_k |a|^{q_k} < \infty \Big\}.$$

Proof. Note that $f(x) = \sum_{h=1}^{\infty} a^h e^{2\pi h i x}$ and hence $c_h = a^h$ for $h > 0$, and $c_h = 0$ for $h \le 0$. Then $\alpha \in \mathrm{B}_f$ if and only if

$$\begin{aligned}\frac{1}{2\pi i} \sum_{k=0}^{m} (-1)^k a_{k+1} q_k \sum_{h=1}^{\infty} \frac{1}{h} a^{h q_k} (e^{2\pi i h (-1)^k x_k} - 1) \\ = \frac{1}{2\pi i} \sum_{k=0}^{m} (-1)^k a_{k+1} q_k (\log(1 - a^{q_k}) - \log(1 - a^{q_k} e^{2\pi i (-1)^k x_k}))\end{aligned}$$

is bounded in $m \ge 0$ and in $x_k \in [0,1),\ 0 \le k \le m$.

First assume that $\alpha \in \mathrm{B}_f$ and $\varphi \in [0,1)$ is such that $a = |a| e^{2\pi i \varphi}$. The equation $\cos 2\pi x = |a|$ has exactly two solutions $c, d \in [0,1)$ and we may assume that $0 \le c < 1/2 < d$. Then $\sin 2\pi c = \sqrt{1 - |a|^2}$ and $\sin 2\pi d = -\sqrt{1 - |a|^2}$. Put

$$u_k^{(0)} = \begin{cases} c, & 2 \mid k, \\ d, & 2 \nmid k, \end{cases} \qquad u_k^{(1)} = \begin{cases} d, & 2 \mid k, \\ c, & 2 \nmid k. \end{cases}$$

Then $\sin 2\pi u_k^{(\varepsilon)} = (-1)^{k+\varepsilon} \sqrt{1 - |a|^2}$ for $\varepsilon \in \{0,1\}$. Next choose $x_k^{(\varepsilon)} \in [0,1)$ such that $\varphi q_k + (-1)^k x_k^{(\varepsilon)} \equiv u_k^{(\varepsilon)} \pmod 1$. Then, as the arguments of the logarithms have positive real part, we have

$$
\begin{aligned}
&\Im(\log(1 - a^{q_k} e^{2\pi i(-1)^k x_k^{(0)}}) - \log(1 - a^{q_k} e^{2\pi i(-1)^k x_k^{(1)}})) \\
&\quad = -\arctan |a|^{q_k} \frac{\sin 2\pi(\varphi q_k + (-1)^k x_k^{(0)})}{1 - |a|^{q_k} \cos 2\pi(\varphi q_k + (-1)^k x_k^{(0)})} \\
&\qquad + \arctan |a|^{q_k} \frac{\sin 2\pi(\varphi q_k + (-1)^k x_k^{(1)})}{1 - |a|^{q_k} \cos 2\pi(\varphi q_k + (-1)^k x_k^{(1)})} \\
&\quad = -\arctan |a|^{q_k} \frac{(-1)^k \sqrt{1 - |a|^2}}{1 - |a|^{q_k+1}} + \arctan |a|^{q_k} \frac{(-1)^{k+1} \sqrt{1 - |a|^2}}{1 - |a|^{q_k+1}} \\
&\quad = -2(-1)^k \arctan |a|^{q_k} \frac{\sqrt{1 - |a|^2}}{1 - |a|^{q_k+1}}.
\end{aligned}
$$

This implies that

$$\sum_{k=0}^{\infty} a_{k+1} q_k \arctan |a|^{q_k} \frac{\sqrt{1 - |a|^2}}{1 - |a|^{q_k+1}} < \infty.$$

As

$$\arctan |a|^{q_k} \frac{\sqrt{1 - |a|^2}}{1 - |a|^{q_k+1}} \gg |a|^{q_k},$$

we get the assertion.

The converse statement follows immediately from Corollary 6.

In view of Corollaries 8 and 9 one might think that $\alpha \in \mathrm{B}_f$ if and only if $\sum_{k=0}^{\infty} a_{k+1} q_k |c_{q_k}| < \infty$. We present a counterexample even if f is analytic.

Example. Let $a_1 = 4$ and assume that positive integers $a_1, \ldots, a_k$ are already defined. Let p_k, q_k be positive and coprime and such that $p_k/q_k = [0; a_1, \ldots, a_k]$. Put $a_{k+1} = 10^{6q_k}$. Then the sequence $(a_k)_{k \geq 1}$ defines an irrational $\alpha := [0; a_1, a_2, \ldots]$. Put

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{a_{k+1} q_k} e^{4\pi i q_k x}.$$

It is easily seen that f is analytic. Note that $c_h = 0$ except when h is of the form $h = 2q_k$. If $1 \leq h \leq a_{k+1}$ the equation $hq_k = 2q_u$ has the only solution $h = 2, u = k$. Clearly $\sum_{k=0}^{\infty} a_{k+1} q_k |c_{q_k}| = 0 < \infty$.

We have $\alpha \notin \mathrm{B}_f$, as for

$$x_k = \begin{cases} 1/4, & 2 \mid k, \\ 0, & 2 \nmid k, \end{cases}$$

we get

$$\sum_{h=1}^{\infty}\frac{1}{h}\,c_{hq_k}(e^{2\pi ih(-1)^k x_k}-1)$$
$$=\sum_{h\le a_{k+1}}\frac{1}{h}\,c_{hq_k}(e^{2\pi ih(-1)^k x_k}-1)+O\Big(\sum_{h>a_{k+1}}\frac{1}{h}\,c_{hq_k}\Big)$$
$$=-\frac{1}{2}\,(1+(-1)^k)\,\frac{1}{a_{k+1}q_k}+O\Big(\frac{1}{a_{k+1}}\sum_{r\ge q_k}c_r\Big)$$
$$=-\frac{1}{2}\,(1+(-1)^k)\,\frac{1}{a_{k+1}q_k}+O\Big(\frac{1}{a_{k+1}^2}\Big).$$

This implies that

$$\sum_{k=0}^{m}(-1)^k a_{k+1}q_k\sum_{h=1}^{\infty}\frac{1}{h}\,c_{hq_k}(e^{2\pi ih(-1)^k x_k}-1)=-\sum_{2|k\le m}1+O(1),$$

and this tends to $-\infty$ for m large.

6. General regulated functions. The two methods presented in this paper can be combined to determine B_f for a large class of regulated f with only finitely many discontinuities. This follows from the theorem below. The part containing the equivalence has clearly been noticed by many authors and is more or less obvious.

THEOREM 3. *Let $f:\mathbb{R}\to\mathbb{C}$ be a periodic regulated function with only finitely many discontinuities in $[0,1]$. The following assertions are equivalent*:

(1) *There are periodic $u,v:\mathbb{R}\to\mathbb{C}$ such that u is continuous, v is a step function and $f=u+v$.*
(2) $\sum_{x\in[0,1)}(f(x+)-f(x-))=0$.

If these conditions are satisfied, then u and v are uniquely determined up to an additive constant and $\mathrm{B}_f=\mathrm{B}_\mathrm{u}\cap\mathrm{B}_v$. Otherwise $\mathrm{B}_f=\emptyset$.

Proof. We may assume that f is right continuous, as changing f at finitely many points affects neither (1) nor (2). Let $I:=[0,1)$.

(1)⇒(2). As v is right continuous, it is of the form

$$v(x)=\sum_{i=0}^{m-1}a_i c_{[\beta_i,\beta_{i+1})+\mathbb{Z}}(x).$$

Then $a_i = v(\beta_i)$ and $a_{i-1} = v(\beta_i -)$ with $a_{-1} = a_{m-1}$. Hence

$$\sum_{x\in I}(f(x) - f(x-)) = \sum_{x\in I}(v(x) - v(x-)) = \sum_{i=0}^{m-1}(a_i - a_{i-1}) = 0.$$

(2)⇒(1). We have

$$f(x) - f(x-) = \sum_{\beta\in I}(f(\beta) - f(\beta-))c_{\beta+\mathbb{Z}}(x).$$

We put $v = \sum_{\beta\in I}(f(\beta) - f(\beta-))c_{[\beta,1)+\mathbb{Z}}$. Clearly v is a periodic step function. By our assumption $v(1-) = \sum_{\beta\in I}(f(\beta) - f(\beta-)) = 0$ and $v(1) = v(0) = f(0) - f(0-)$. The relation

$$v(x) - v(x-) = \sum_{\beta\in I}(f(\beta) - f(\beta-))c_{\beta+\mathbb{Z}}(x) = f(x) - f(x-)$$

is also true if x is not an integer. Therefore $f(x) - v(x) = f(x-) - v(x-)$, that is, $f - v$ is also left continuous, hence continuous and clearly periodic.

Uniqueness and $\mathrm{B}_u \cap \mathrm{B}_v \subseteq \mathrm{B}_f$ are trivial. Assume now that $\alpha \in \mathrm{B}_f$. We prove that (2) holds and that $\alpha \in \mathrm{B}_u \cap \mathrm{B}_v$.

As $x \mapsto f(x) - f(x-)$ $(x \in \mathbb{R})$ is a difference of a right and left continuous function both of bounded remainder with respect to α and both with only finitely many discontinuities in $[0, 1]$, there exists a regulated function $g : \mathbb{R} \to \mathbb{C}$ with $f(x) - f(x-) = g(x + \alpha) - g(x)$. For positive integers m, n we have

$$\begin{aligned} g(x + m\alpha) - g(x - n\alpha) &= \sum_{\beta\in I}(f(\beta) - f(\beta-)) \sum_{k=-n}^{m-1} c_{\beta+\mathbb{Z}}(x + k\alpha) \\ &= \sum_{\beta\in I}(f(\beta) - f(\beta-))c_{\beta-\alpha([-n,m)\cap\mathbb{Z})+\mathbb{Z}}(x). \end{aligned}$$

If we let $\{m\alpha\}$ tend to some y from the right and $\{n\alpha\}$ to some z from the left we get

$$g(x + y+) - g(x - z+) = \sum_{\beta\in I}(f(\beta) - f(\beta-))c_{\beta+G}(x),$$

where G denotes the group $\mathbb{Z} + \alpha\mathbb{Z}$. This is only possible if

$$\sum_{\beta\in I}(f(\beta) - f(\beta-))c_{\beta+G}(x) = 0.$$

Let $T \subseteq I$ be a complete system of representatives for $\mathbb{R}/G$ with $0 \in T$. Then

$$\sum_{t\in T}\ \sum_{\beta\in(t+G)\cap I}(f(\beta) - f(\beta-))c_{t+G} = 0$$

and hence $\sum_{\beta\in(t+G)\cap I}(f(\beta)-f(\beta-))=0$ for all $t\in T$. Summing over $t\in T$ we get (2) and hence u and v exist.

For $t\in T$ let

$$f_t=\sum_{\beta\in(t+G)\cap I}(f(\beta)-f(\beta-))c_{[\beta,1)+\mathbb{Z}}.$$

Then clearly $v=\sum_{t\in T}f_t$. We prove that $\alpha\in \mathrm{B}_{f_t}$ for all $t\in T$. Assume first that $t\neq 0$. Then $f_t(0)=0$ and

$$f_t(0-)=f_t(1-)=\sum_{\beta\in(t+G)\cap I}(f(\beta)-f(\beta-))=0.$$

Hence f_t is continuous at 0. If β,β' are any discontinuities of f_t, then $\beta,\beta'\in t+G$, hence $\beta-\beta'\in G$. By Corollary 3, $\alpha\in \mathrm{B}_{f_t}$. Further, if β,β' are discontinuities of f_0, then $\beta,\beta'\in G$ (possibly $=0$) and so again $\beta-\beta'\in G$. Corollary 3 implies again $\alpha\in\mathrm{B}_{f_0}$. Therefore $\alpha\in\mathrm{B}_v$. Finally, $u=f-v$ implies $\alpha\in\mathrm{B}_u$.

If f is piecewise Lipschitz continuous and the Fourier coefficients $(c_h)_{h\in\mathbb{Z}}$ of f satisfy $|c_h|\gg|h^{-1}|$ for sufficiently many h, then $\mathrm{B}_f=\emptyset$; this has been quantitatively improved by Perelli and Zannier [38]. See also [31] for more recent quantitative statements.

If f has only one discontinuity in $[0,1)$ then $\mathrm{B}_f=\emptyset$ by Theorem 3. This applies e.g. to $f(x)=\{x\}-1/2$. See e.g. [8], [9], [17], [18], [19] and [35] for qualitative improvements. For functions which are continuously differentiable except at one point in $[0,1)$ we refer to [22]–[24], and for ergodicity to the papers [1], [36] and [37].

The problem of what B_f looks like if f is continuous but otherwise wild remains open.

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