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Integer points close to convex hypersurfaces

by

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1. Introduction. Let C be the boundary surface of a strictly convex bounded d-dimensional body. Strictly convex means that if P and Q are points on C, then points on the line segment PQ between P and Q lie in the convex body, but not on its boundary C. Let MC denote the dilation of C by a factor M. Andrews [1], [2], proved that the number of points of the integer lattice on MC is

$$O(M^{d(d-1)/(d+1)}), \tag{1}$$

as M tends to infinity. Strict convexity is necessary because a part of a $(d-1)$-dimensional hyperplane in the boundary C can give as many as a constant times M^{d-1} integer points for infinitely many values of M.

We consider the integer points within a distance δ of the hypersurface MC. The two-dimensional case has been well studied ([12], [5], [9], [6], [10] and [11]). More recently the author [15] has examined the three-dimensional case. Introducing δ requires some uniform approximability condition on the surface C, usually expressed in terms of upper and lower bounds for derivatives and determinants of derivatives. Let A be the $(d-1)$-dimensional volume of C. The search region has d-dimensional volume

$$(2A\delta + O(\delta^2))M^{d-1}, \tag{2}$$

and this is known to be the number of integer points on average over translations of the surface MC. To obtain an asymptotic formula one considers the Fourier transform of the convex body, with conditions at least as far as the $6d$th derivatives in order to estimate the multiple exponential integrals. Hlawka [8] obtained an asymptotic formula with error of size (1); see also Krätzel [13]. Under the C^∞ hypothesis of a convergent Taylor series, the error term in the asymptotic formula has been improved, most recently by Müller [18].

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We derive an upper bound for the number of integer points within a distance δ of the hypersurface. We require only that C has a tangent hyperplane at every point, and that any two-dimensional cross-section through the normal at some point P consists (in a neighbourhood of P) of a plane curve C' with continuous radius of curvature bounded away from zero and infinity.

Curvature Condition (with size parameter M). For any point P on C and any two-plane Π through the normal to C at P, let $C(\Pi, P)$ be the closed plane curve $C \cap \Pi$. Then $C(\Pi, P)$ is a twice differentiable plane curve with radius of curvature ϱ lying in the range

$$c_0 M + 1/2 \le \varrho \le c_1 M - 1/2, \tag{3}$$

where the constants c_0, c_1 and δ satisfy

$$1/M < c_0 \le 1 \le c_1 \quad \text{and} \quad \delta < 1/4. \tag{4}$$

Local Curvature Condition. There is a constant κ such that for $C(\Pi, P)$ defined as above, the points Q of $C(\Pi, P)$ with $PQ \le \kappa M$ form a twice differentiable plane curve with radius of curvature satisfying (3).

In order to state our results, we set up some notation. Let C_0 be the locus of points at distance δ from C measured along the interior normals to C, and let C_1 be the locus of points at distance δ measured along the exterior normals. Let E be the d-dimensional shell bounded by C_0 and C_1 so that E has thickness 2δ. Let S be the set of integer points in E, and let H be the convex hull of S, so that H is a d-dimensional convex polytope ([3], [4], [14], [17] and [16]). All points of S lie in H, but not all integer points on the boundary of H lie in S.

By Lemma 2.1 of [15], the boundary surfaces C_0 and C_1 of the shell E have a tangent hyperplane at each point Q, and their two-dimensional cross-sections $C(\Pi, Q)$ in planes normal to the tangent hyperplanes are twice differentiable, with radius of curvatures in the range

$$c_0 M \le \varrho \le c_1 M. \tag{5}$$

Under the Curvature Condition, the shell E containing S, the set of integer points, lies in a d-hypersphere of radius $R = c_1 M$. The volume V_d and surface content S_d of this sphere is given by the formulae (see [19])

$$V_d = \alpha_d R^d, \quad S_d = d\alpha_d R^{d-1}, \tag{6}$$

where

$$\alpha_{2k} = \frac{\pi^k}{k!}, \quad \alpha_{2k+1} = \frac{2^{2k+1}\pi^k k!}{(2k+1)!}, \quad \alpha_d \le 6, \quad \frac{\alpha_d}{\alpha_{d-1}} \le \pi, \tag{7}$$

and for $d \ge 2$,

$$d\alpha_d \le (2\pi)^{d-1}. \tag{8}$$

We can now state our results.

THEOREM 1.1. *Suppose that C is a convex hypersurface in d-dimensional Euclidean space $\mathbb{E}^d$ $(d \geq 3)$, satisfying the Curvature Condition at size M (so that C is contained in a hypersphere of radius $c_1 M$). Then the total number, N, of integer points lying either on C, or within a distance δ of C, is bounded by*

$$N \leq \frac{2^{3d^2+5d-7} d!}{\alpha_{d-1}} \left(\frac{c_1}{c_0}\right)^{d-1} ((c_1 M)^{d(d-1)/(d+1)} + 2^9 \delta (c_1 M)^{d-1}). \tag{9}$$

THEOREM 1.2. *Suppose that C is a convex hypersurface in d-dimensional Euclidean space $\mathbb{E}^d$ $(d \geq 3)$, satisfying the Local Curvature Condition at size M (so that C is contained in a hypersphere of radius $c_1 M$), with*

$$M \geq 100 \delta c_1 / \kappa^2. \tag{10}$$

Then N, the total number of integer points lying either on C, or within a distance δ of C, satisfies the same bound (9) *as in Theorem* 1.1.

2. Major arcs

DEFINITION. It is helpful in many problems to separate "major arcs", regions where there is good Diophantine approximation, from "minor arcs", regions where there is not. In this paper a major arc can be described informally as a region U of the shell E such that the convex hull of all the integer points in U is contained in the intersection of E with some hyperplane. Hence U can be of dimension j, with $j = 1, \ldots, d-1$.

For each major arc we are interested in the integer points which lie within a distance δ from the hypersurface C. In the preceding paper [15] we showed that the integer points lie in clusters around the vertices of the convex hull H, which we call components of a major arc. We also observed that at most two one-dimensional components can lie on the same straight line. Higher dimensional components are, however, not as simple and for $d-1 \geq j \geq 2$, there can exist many j-dimensional components on the same j-dimensional plane.

Each j-dimensional component of a major arc has maximum diameter equal to the maximum length of a component of a one-dimensional major arc. By Lemma 4.1 of [15] this is

$$\leq 4\sqrt{\delta c_1 M}. \tag{11}$$

Hence a j-dimensional component is contained within a j-dimensional hypercube of volume

$$\leq (4\sqrt{\delta c_1 M})^j. \tag{12}$$

LEMMA 2.1. *Let Π be a hyperplane with equation*

$$\mathbf{n} \cdot \mathbf{x} = D,$$

where $\mathbf{n}$ *is a primitive integer vector, and* D *is an integer. Then the integer points of* Π *form a lattice with determinant* $|\mathbf{n}|$.

Proof. This is Lemma 4.4 of [15]. ∎

LEMMA 2.2. *Let* Λ *be a* j*-dimensional lattice of determinant* n, $1 \le j \le d$. *Let* U *be a convex set in the* j*-plane of* Λ, *with* j*-dimensional volume* V, *containing* K *points of the lattice* Λ. *Then one of the following two cases holds*:

(1) *Major arc case*: *All the points of* Λ *in the set* U *lie on a* $(j-1)$*-dimensional plane.*
(2) *Minor arc case*:

$$K \le j!\,\frac{V}{n} + j \le (j+1)!\,\frac{V}{n}.$$

Proof. This is Lemma 4.5 of [15]. ∎

3. Vertex components. For each point P in our shell E, there exists a normal to the hypersurface C, meeting the outer boundary surface C_1 normally at a point R_1 and the inner boundary surface C_0 normally at a point R_0. We call R_0 and R_1 the *normal projections* of P onto C_0 and C_1 respectively. The vertices of our convex polytope H must, by definition, lie in E and for every other non-vertex integer point in E there must exist a nearest vertex. This argument follows the account in [15] of the 3-dimensional case.

DEFINITION. Let P be a point of S in the shell E and R_1 the normal projection of P onto C_1. Let V be a vertex of the convex hull H and E' the plane sectional strip of E containing V, P and R_1. If the line segment R_1V lies entirely within the closed strip E', then we say that P lies in the *vertex component* $S(V)$ of S.

LEMMA 3.1. *Every point* P *of* S *belongs to some vertex component* $S(V)$.

Proof. The line segment PR_1 cuts the boundary of the convex hull H at some point Q between P and R_1 inside E, so that Q lies in some hyperplane face F of H. If Q is a vertex of H then P belongs to $S(Q)$ as QR_1 will lie on the line segment R_0R_1 inside E.

We now assume that the point Q is not a vertex of H and triangulate the facet F of H containing Q so that Q lies in some simplex $W = V_1 \ldots V_d$. If the line segment QV_i does not enter the interior of the convex set bounded by C_0 then neither does R_1V_i, implying that P lies in $S(V_i)$.

If P lies in no $S(V_i)$ then each line segment QV_i on F cuts the interior of C_0 in some point Q_i also on F but not in E. The whole convex simplex $Q_1 \ldots Q_d$ therefore lies strictly inside C_0 and contains Q. Hence, Q is not in E, which is impossible, since Q lies on the line segment R_0R_1, which is strictly inside E. This contradiction shows that for some i, the line segment

V_iQ lies in E and so V_iR_1 lies in E and P is in the component corresponding to V_i. ∎

LEMMA 3.2 (Spacing lemma). *Let V be a vertex of the convex hull H. Let P be a point of S not in the component $S(V)$ of V. Let R_1 and R_2 be the respective normal projections of P and V onto C_1. Then*

$$R_1R_2 > \sqrt{c_0\delta M} \tag{13}$$

and the angle between the normals to C_1 at R_1 and R_2 is

$$> \frac{1}{c_1}\sqrt{\frac{c_0\delta}{M}}. \tag{14}$$

Proof. This is Lemma 5.2 of [15]. The number of dimensions does not affect the geometry of the two-dimensional section. ∎

As each integer point P in S belongs to at least one component $S(V)$ labelled by some vertex V of the convex hull H, components labelled by different vertices may well overlap and different vertices of the convex hull may be close together. We pick a well-spaced set of vertices of H as follows. Pick a vertex V_1, and let the *enlarged component* $S'(V_1)$ be the union of all components $S(V)$ with V in $S(V_1)$.

Now pick a vertex V_2 not in $S'(V_1)$, and form the enlarged component $S'(V_2)$. We pick V_{i+1} not in $S'(V_1), \ldots, S'(V_i)$, and so on until all of the vertices V of the convex hull H lie in some enlarged component.

LEMMA 3.3 (Thickness lemma). *Let $S'(V)$ be an enlarged component and let R_2 be the normal projection of V onto C_1. Let P be a point in $S'(V)$ and let R_1 be the normal projection of P onto C_1. Then the distance h of P from the tangent plane at R_2 satisfies*

$$h \le 52\delta c_1/c_0, \tag{15}$$

and

$$R_1R_2 \le 10\sqrt{\delta c_1 M}. \tag{16}$$

Proof. This is Lemma 5.3 of [15]. The number of dimensions does not affect the geometry of the two-dimensional section. ∎

REMARK. As with the three-dimensional case in [15], we are ultimately working towards a shelling argument. This uses the property that if we can obtain a bound valid for δ sufficiently small, then we can deduce a possible weaker bound for large δ by dividing the shell E into concentric shells E_r, $1 \le r \le R$, of thickness δ_0, bounded by shrunken copies of the exterior hypersurface C_1 of E. By inequality (5), we have a uniform upper bound of c_1M for the sectional radius of curvature at any point on each shell E_r. Hence, when regarding maximum sectional radius of curvatures, we can work within the general shell boundary C_1, whose sectional radius of curvature is also $\leq c_1 M$.

LEMMA 3.4 (Flatness lemma). *Let $S'(V)$ be an enlarged vertex component of our convex hull H. If*

$$\delta < \delta_0 = \left(\frac{c_0}{2^{2d} 5^{d-1} 13 d! c_1} \right)^{2/(d+1)} (c_1 M)^{-(d-1)/(d+1)}, \tag{17}$$

then all the points of $S'(V)$ lie on a hyperplane through the vertex V.

Proof. Let P be a point of $S'(V)$ and let R_1 and R_2 be the normal projections of P and V onto C_1. All points P of $S'(V)$ lie within a distance $52\delta c_1/c_0$ from the tangent hyperplane at R_2, and by (16),

$$PV \leq R_1 R_2 \leq 10\sqrt{\delta c_1 M}.$$

Hence, the set of integer points $S'(V)$ all lie within a rectangular box L, of d-dimensional volume $\operatorname{Vol}(L)$, with

$$\operatorname{Vol}(L) \leq \frac{52\delta c_1}{c_0} (20\sqrt{\delta c_1 M})^{d-1} < \frac{1}{d!}, \tag{18}$$

where we have used the assumption (17). Therefore, by Lemma 2.2, the major arc case holds, and all points of the enlarged vertex component $S'(V)$, including V itself, lie on a hyperplane. ∎

LEMMA 3.5 (Approximate tangency). *Let $S'(V)$ be an enlarged component. Let T be the point of C_1 closest to V. Let P be another point of $S'(V)$, and let $\mathbf{g}$ be the integer vector VP. Then the angle α between VP and the normal to C_1 at T satisfies*

$$|\cos\alpha| \leq \frac{52\delta c_1}{c_0 |\mathbf{g}|}. \tag{19}$$

Proof. This is Lemma 5.5 of [15]. The number of dimensions does not affect the geometry of the two-dimensional section. ∎

LEMMA 3.6 (Sums of reciprocal vector lengths). *For $j = 1, \ldots, d-1$ we have*

$$\sum_{1 \leq |\mathbf{e}| \leq E} \frac{1}{|\mathbf{e}|^j} \leq 2^{2d+j} E^{d-j}. \tag{20}$$

Proof. Applying the Cauchy condensation method, we divide the normal vectors into ranges

$$F/2 < |\mathbf{e}| \leq F, \quad F = 1, 2, 4, \ldots, 2^K,$$

where 2^K is the largest power of 2 less than or equal to E. The number of integer vectors in this range is

$$\begin{aligned}&\le (2F+1)^d-(F+1)^d \le \sum_{j=0}^{d}\binom{d}{j}(2^{d-j}-1)F^{d-j}\\&\le F^d(3^d-2^d)\le 2^{2d-1}F^d,\end{aligned}$$

so that

$$\sum_{F/2<|\mathbf{e}|\le F}\frac{1}{|\mathbf{e}|^j}\le 2^{2d-1}F^d\left(\frac{2}{F}\right)^j = 2^{2d+j-1}F^{d-j}.$$

Summing over the ranges for F, we have

$$\begin{aligned}\sum_{1\le|\mathbf{e}|\le F}\frac{1}{|\mathbf{e}|^j}&\le 2^{2d+j-1}(1+(2^1)^{d-j}+(2^2)^{d-j}+\cdots+(2^K)^{d-j})\\&=2^{2d+j-1}\,\frac{(2^{d-j})^{K+1}-1}{2^{d-j}-1}\le 2^{2d+j}2^{(d-j)K}\le 2^{2d+j}E^{d-j}. \ \blacksquare\end{aligned}$$

Definition. Let R be the normal projection of V onto the outer surface C_1. We define the *reach*, $U(V)$, of the enlarged vertex component $S'(V)$ to be the set of points on C_1 such that for all points $P\in U(V)$ we have

$$PR\le 10\sqrt{\delta c_1 M}. \tag{21}$$

By (16), if Q is an integer point in $S'(V)$, the normal projection R_1 of Q onto the surface C_1 lies in $U(V)$.

Lemma 3.7 (Enlarged vertex components and the Local Curvature Condition). *If*

$$M\ge 100\delta c_1/\kappa^2, \tag{22}$$

then the Local Curvature Condition with respect to R holds at all points R_1 in the reach of $S'(V)$.

Proof. Let P be a point of C_1 in $U(V)$. By (21) and (22),

$$PR\le 10\sqrt{\delta c_1 M}\le \kappa M,$$

which is the threshold for the Local Curvature Condition. $\blacksquare$

Lemma 3.8. *In d-dimensional space, the number of integer points of S in E that lie strictly inside the convex hull H of S is*

$$\le 2\delta d!\alpha_d d(c_1 M)^{d-1}. \tag{23}$$

Proof. This is Lemma 4.3 of [15]. $\blacksquare$

Let $S(H)$ be the set of integer points in S that lie on the boundary of the convex hull H. The rest of this paper is devoted to the study of $S(H)$. The points of $S(H)$ fall into enlarged vertex components, where an enlarged vertex component, $S'(V)$, of $S(H)$ is either full d-dimensional or it lies strictly on some j-dimensional hyperplane that contains the vertex V with $0 \le j \le d-1$.

LEMMA 3.9. *Let f_{d-1} be the number of $(d-1)$-dimensional hyperplane faces of the convex hull H. Then*

$$f_{d-1} \le 2(3\alpha_d d!)^{d/(d+1)}(c_1 M)^{d(d-1)/(d+1)} \le 36d!(c_1 M)^{d(d-1)/(d+1)}. \tag{24}$$

Proof. This is Theorem 3.4 of [15], where we have used (7) to obtain the second inequality. ∎

LEMMA 3.10. *For $0 \le j \le d-2$, let f_j be the number of j-faces* (*that is, j-dimensional faces*) *of the convex hull H. Then*

$$\begin{aligned} f_j &\le 2(3\alpha_d d!)^{d/(d+1)}(2(j+1)c_1 M)^{d(d-1)/(d+1)} \\ &\le 36d!(2(j+1)c_1 M)^{d(d-1)/(d+1)}. \end{aligned} \tag{25}$$

Proof. This is Theorem 3.6 of [15]. ∎

LEMMA 3.11. *Let $R = c_1 M$ and let F be a facet or hyperplane face of H that lies in a hyperplane Ψ with outward normal $\mathbf{n}$. Let X be the point of C_1 at which $\mathbf{n}$ is the outward normal. Let h be the distance from X along the inward normal to the nearest point Y on the hyperplane Ψ. Let E' be the $(d-1)$-dimensional section of E contained in Ψ, so that E' contains all parts of the face F that lie in the shell E. Then the $(d-1)$-dimensional volume V of E' is bounded above by*

$$V \le 2^{(d+9)/2} d\delta (c_1 M)^{(d-1)/2} h^{(d-3)/2}. \tag{26}$$

Proof. This is Lemma 4.2 of [15]. ∎

4. Boundary components. Let $S^\star(V_i)$ be the subset of $S'(V_i)$ consisting of integer points on the boundary of H. We will call this a *boundary component*. We have shown that for each enlarged vertex component $S'(V_i)$, if δ is sufficiently small then $S'(V_i)$ lies in a hyperplane and so $S^\star(V_i)$ lies in the same hyperplane.

The *dimension* of the integer point set $S^\star(V_i)$ is defined to be the least e for which $S^\star(V_i)$ lies in an e-dimensional hyperplane, and $|S^\star(V_i)|$ to be the number of elements of $S^\star(V_i)$ in S. When $e = 0$ we merely have to count the vertices of H. When $e = d$, the points of the enlarged vertex component lie on two or more hyperfaces of H, and we use a volume argument (Lemma 4.3 below). When $e = d-1$ we have a straightforward but complicated estimation (Lemma 4.2 below). For intermediate dimensions $1 \le e \le d-2$ we consider "girdles" of parallel planes and use a solid angle spacing argument. This takes its simplest form when $e = 1$ (Lemma 4.1 below). The cases $2 \le e \le d-2$ require more combinatorial geometry and will be considered in the next section.

We define a *one-dimensional girdle* to be the set of all the boundary components $S^\star(V)$ of H which are one-dimensional and which lie parallel to some primitive integer vector $\mathbf{e}$. When considering the j-dimensional boundary components with $j \le d-2$, we must also take into account the possibility that many of these components may be parallel. To clarify the parallel condition in higher dimensions, we introduce the idea of degrees of parallelism as described in [19].

Definition. Let Π_1 and Π_2 be two planes of dimension p and q ($p \ge q$) respectively in $\mathbb{E}^d$ that have no point in common. Let Ψ be the plane of least dimension d that contains both Π_1 and Π_2. Let $r = p + q - d$. Then Π_1 and Π_2 intersect in an r-plane at infinity and we say that Π_1 and Π_2 are *$(r+1)/q$-parallel.*

If $p = q$ and $r = p - 1$, then $d = p + 1$, and Π_1 and Π_2 are contained in the $(p+1)$-plane Ψ. We say that Π_1 and Π_2 are *completely parallel.* When this occurs, then through each point O in Ψ there is a unique line in Ψ that is normal to both Π_1 and Π_2. If two normals are drawn through two points O, O', cutting Π_1 and Π_2 in A, B and A', B' then $ABB'A'$ is a rectangle and $AB = A'B'$. The distance AB is called the *distance between the completely parallel p-planes.*

We deduce that if two completely parallel p-planes share a common point, then they are in fact the same p-plane.

In contrast to complete parallelism, we again refer to [19] in order that we may clarify complete orthogonality in higher dimensions.

Definition. Through any point O in $\mathbb{E}^d$ we can find d lines that are all mutually perpendicular. We begin with a line l_1. All lines perpendicular to l_1 through O form a $(d-1)$-plane Π_1 whose normal vector at O is l_1. Let l_2 be one of these lines and let Π_2 be the $(d-1)$-plane whose normal vector at O is l_2. Then all lines perpendicular to both l_1 and l_2 at O lie in the $(d-2)$-plane that is the intersection of Π_1 and Π_2. Let l_3 be one of these lines. Continuing in this manner we create a system of d lines $l_1, \ldots, l_d$ that are all mutually perpendicular. Any p of these lines determine a p-plane Ψ_p, and the remaining $d-p$ lines determine a $(d-p)$-plane Ψ_{d-p}. These two planes only intersect at O and have the property that every line of Ψ_p through O is perpendicular to every line of Ψ_{d-p} through O. The two planes Ψ_p and Ψ_{d-p} are said to be *completely orthogonal.*

We deduce that for Ψ_p, defined as above and containing the point O, there exists a unique $(d-p)$-plane Ψ_{d-p} that is completely orthogonal to Ψ_p through O. Hence for a given system of d mutually orthogonal lines in $\mathbb{E}^d$ and any point O, for each partition of the lines into two sets containing p and $d-p$ lines there exists a *unique pair of completely orthogonal planes*, Ψ_p and Ψ_{d-p}, that intersect only at O.

LEMMA 4.1. *The number of integer points on one-dimensional boundary components is estimated by*

$$\sum_{\dim S^\star(V_i)=1} |S^\star(V_i)| \le \frac{2^{6d-1}3^3 c_1^{(d-1)/2}\pi^{d-1}}{\alpha_{d-1}c_0^{(d+1)/2}}\,\delta(c_1M)^{d-1}. \tag{27}$$

Proof. In the proof of Lemma 6.1 of [15] we noted that at most two one-dimensional boundary components can lie on the same straight line.

We consider all the boundary components $S^\star(V_i)$ which are one-dimensional lying parallel to some primitive integer vector $\mathbf{e}$. Suppose that the component contains l points of $S(H)$, where

$$L+1 \le l \le 2L \tag{28}$$

for some L equal to a power of two. We can take $\mathbf{g} = (l-1)\mathbf{e}$ in Lemma 3.5, with

$$|\mathbf{g}| \ge (l-1)|\mathbf{e}| \ge L|\mathbf{e}|.$$

In Lemma 3.5 the angle α between the vector $\mathbf{e}$ and the normal to C_1 at T, the point of C_1 nearest to V, satisfies

$$|\cos\alpha| \le \frac{52\delta c_1}{c_0L|\mathbf{e}|}.$$

Hence

$$\left|\frac{\pi}{2}-\alpha\right| \le \frac{26c_1\pi\delta}{c_0L|\mathbf{e}|}. \tag{29}$$

We want to discuss the spacing of the vertices V_i that label the enlarged components $S'(V_i)$ and so the boundary components $S^\star(V_i)$. Each V_i has a normal projection T_i on C_1. Consider a d-dimensional sphere B of radius c_1M. We map T_i on C_1 to the point W_i on B where the outward normal $\mathbf{n}$ to B is parallel to the outward normal to C_1 at T_i.

Let V_i and V_j be distinct vertices labelling enlarged vertex components. Since $V_j \notin S(V_i)$, we have

$$T_iT_j > \sqrt{c_0\delta M},$$

by (13) of Lemma 3.2. Since C_1 has sectional radii of curvature at most c_1M,

$$W_iW_j \ge T_iT_j > \sqrt{c_0\delta M}.$$

Hence d-dimensional balls B_i of radii $\frac{1}{2}\sqrt{c_0\delta M}$, centred at the points W_i on B, are disjoint.

The d-ball B_i meets the surface of the d-sphere B in a $(d-1)$-dimensional set A_i which contains the centre W_i of B_i and is a $(d-1)$-ball in spherical geometry. As the B_i are disjoint, the $(d-1)$-dimensional volumes of the sets A_i, on the boundary surface of the d-sphere B_i, are also disjoint and do not overlap. Hence different sets $S'(V_i)$ correspond to disjoint sets A_i, with centre W_i, on the surface of the d-sphere B. The $(d-1)$-volume of A_i is greater than the $(d-1)$-volume of the intersection of a hyperplane through W_i with B_i, which is

$$\alpha_{d-1}(\sqrt{c_0\delta M/4})^{d-1}. \tag{30}$$

As $V_i \in S^\star(V_i)$ and $S^\star(V_i) \subseteq S'(V_i)$, different sets $S^\star(V_i)$ also correspond to disjoint sets A_i, with centre W_i, on the surface of the sphere B.

For each vector $\mathbf{e}$, there is an equatorial hyperplane of the d-sphere B at right angles to $\mathbf{e}$. By (29) the point W on the surface of B, where the normal is parallel to the normal $\mathbf{n}$ to C_1 at T, lies

$$\le \frac{26\pi\delta c_1 M}{c_0 L|\mathbf{e}|}$$

from the equatorial hyperplane measured along the surface of B. As stated, the set A_i is the intersection of the surface of B with a d-ball of radius $\frac{1}{2}\sqrt{c_0\delta M}$, so it forms a $(d-1)$-ball in the spherical geometry of the surface of B, whose radius in spherical geometry is

$$\le \frac{\pi}{2}\sqrt{\frac{c_0\delta M}{4}} \le \pi\sqrt{\frac{c_0\delta M}{16}}\,\frac{4\sqrt{\delta c_1 M}}{L|\mathbf{e}|} = \frac{\pi\delta c_1 M}{L|\mathbf{e}|}\left(\frac{c_0}{c_1}\right)^{1/2} \le \frac{\pi\delta c_1 M}{c_0 L|\mathbf{e}|},$$

by (4) and (11).

Hence, each point of A_i lies within a distance

$$\le \frac{26\pi\delta c_1 M}{c_0 L|\mathbf{e}|} + \frac{\pi\delta c_1 M}{c_0 L|\mathbf{e}|} = \frac{27\pi\delta c_1 M}{c_0 L|\mathbf{e}|}$$

from the equatorial hyperplane, measured along the surface of the d-sphere B.

We consider the "girdle" of one-dimensional boundary components $S^\star(V_i)$ which are parallel to the fixed vector $\mathbf{e}$. The components in the girdle satisfying (28) correspond to points W_i and sets A_i on the surface of B, such that every point of A_i lies close to the equatorial hyperplane perpendicular to $\mathbf{e}$. The sets A_i lie in a $(d-1)$-annulus whose volume in spherical geometry is at most

$$(2\pi c_1 M)^{d-2}\left(\frac{54\pi\delta c_1 M}{c_0 L|\mathbf{e}|}\right) = \frac{27(2\pi)^{d-1}\delta(c_1 M)^{d-1}}{c_0 L|\mathbf{e}|}.$$

By (30) the number of disjoint sets A_i in the girdle is at most

$$\frac{2^{d-1}}{\alpha_{d-1}(c_0\delta M)^{(d-1)/2}}\,\frac{27(2\pi)^{d-1}\delta(c_1 M)^{d-1}}{c_0 L|\mathbf{e}|} = \frac{27(4\pi c_1)^{d-1}M^{d-1/2}}{\alpha_{d-1}c_0^{(d+1)/2}\delta^{(d-3)/2}L|\mathbf{e}|} \tag{31}$$

so the boundary components $S^\star(V_i)$ in the girdle for which the number l of points is in the range (28) contribute at most

$$\frac{54(4\pi c_1)^{d-1}M^{(d-1)/2}}{\alpha_{d-1}c_0^{(d+1)/2}\delta^{(d-3)/2}|\mathbf{e}|} \tag{32}$$

integer points. The estimate (32) refers only to components in the girdle for which l lies in the range (28). We keep the condition (28), and sum over primitive integer vectors $\mathbf{e}$. Since the component is a straight line segment lying within the strip E, by (11) we have

$$L|\mathbf{e}| \le (l-1)|\mathbf{e}| \le 4\sqrt{\delta c_1 M}.$$

We note that if two boundary components lie on the same line, then the vertices V_i which label the boundary components $S^\star(V_i)$ must be different, so they are counted separately in this argument. We use the bound of Lemma 3.6 with $j=1$ to sum over $\mathbf{e}$, so that the number of points on one-dimensional boundary components with l in the range (28) is at most

$$\frac{54(4\pi c_1)^{d-1}M^{(d-1)/2}}{\alpha_{d-1}c_0^{(d+1)/2}\delta^{(d-3)/2}}\cdot 2^{2d+1}\left(\frac{4\sqrt{\delta c_1 M}}{L}\right)^{d-1} = \frac{2^{6d-2}3^3c_1^{(d-1)/2}\pi^{d-1}\delta(c_1M)^{d-1}}{\alpha_{d-1}c_0^{(d+1)/2}L^{d-1}}. \tag{33}$$

Finally, we remove the condition (28) by summing L through powers of 2, noting that

$$1+\frac{1}{2^k}+\frac{1}{4^k}+\frac{1}{8^k}+\cdots \le \frac{2^k}{2^k-1} \le 2.$$

Hence the total number of integer points of $S(H)$ which lie on one-dimensional boundary components is at most

$$\frac{2^{6d-1}3^3c_1^{(d-1)/2}\pi^{d-1}}{\alpha_{d-1}c_0^{(d+1)/2}}\,\delta(c_1M)^{d-1}. \ \blacksquare$$

LEMMA 4.2. *The number of integer points on $(d-1)$-dimensional boundary components, when $\delta \le \delta_0$, is estimated by*

$$\sum_{\dim S^\star(V_i)=d-1} |S^\star(V_i)| \tag{34}$$
$$\le d!(d+1)!2^{\frac{9d+17}{2}}\left(\frac{c_1}{c_0}\right)^{(d-1)/2}\left((c_1M)^{\frac{d(d-1)}{d+1}}+2\left(\frac{c_1}{c_0}\right)^{(d-1)/2}\delta_0(c_1M)^{d-1}\right).$$

Proof. Each $(d-1)$-dimensional boundary component $S^\star(V_i)$ is part of a hyperplane. The intersection of all such hyperplanes forms a convex polytope, $H^\star$, that is contained within the convex hull H, and the vertices of $H^\star$ are points of $S(H)$. Let Ψ be a hyperplane face of $H^\star$, with outward normal vector $\mathbf{n}$ with respect to $H^\star$ (a primitive integer vector). Let Z be the point of C at which the normal $\mathbf{m}$ to C is parallel to $\mathbf{n}$. Let $\mathbf{m}$ cut Ψ in Y and the boundary surfaces C_0 and C_1 in W and X respectively (Figure 1). Then $\mathbf{m}$ is also the outward normal to C_0 at W, to C_1 at X, and the boundary hyperplane Ψ of the convex hull $H^\star$ at Y. Let $h = XY$, $h' = WY$ be the heights of X above Ψ and of W above or below Ψ as depicted in Figure 1. Each component in the annulus $E \cap \Pi$ is convex. We apply

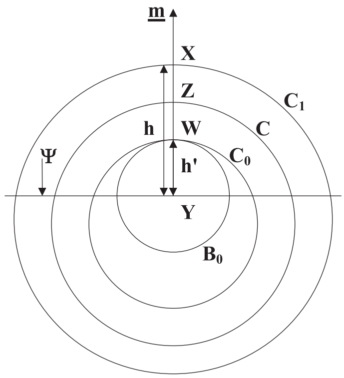

Fig. 1. Heights along the common normal $\mathbf{m}$

Lemma 2.2 with $j = d-1$. The set of points is strictly $(d-1)$-dimensional so we use the minor arc case of Lemma 2.2 with $j = d-1$, and lattice determinant $n = |\mathbf{n}|$ by Lemma 2.1. The volume V is estimated in Lemma 3.11, so we have an estimate for the number $N(\Psi)$ of integer points that lie in $E \cap \Psi$:

$$N(\Psi) \le \frac{(d-1)!V}{|\mathbf{n}|} + d - 1 \le \frac{d!V}{|\mathbf{n}|} \le \frac{d!2^{(d+9)/2} d\delta (c_1 M)^{(d-1)/2} h^{(d-3)/2}}{|\mathbf{n}|}. \tag{35}$$

We sum over all the outward normal vectors of the hyperplanes Ψ. We get the total number, N, of integer points on the $(d-1)$-boundary components to be

$$N \le \sum N(\Psi) \le d!2^{(d+9)/2} d\delta (c_1 M)^{(d-1)/2} h^{(d-3)/2} \sum \frac{1}{|\mathbf{n}|}. \tag{36}$$

We distinguish various cases according to the order of the points W, X, Y and Z on the normal $\mathbf{m}$. If $h > 2\delta$ then the point W lies between X and Y and $h' > 0$, as shown in Figure 1. By the Curvature Condition, a d-ball B_0 of radius $c_0 M$, touching C_0 at W, fits completely inside C_0. Since $h' > 0$, the hyperplane Ψ cuts both C_0 and B_0. A "cap" of the hypersurface C_0 lies above the hyperplane Ψ. The $(d-1)$-dimensional surface content A of the cap cut from C_0 is greater than the content A' of its projection onto the plane Ψ. If $h \le c_0 M + 2\delta$, then the equator of the d-ball B_0 lies below Ψ, and $A' \ge A''$, the $(d-1)$-dimensional content of $B_0 \cap \Psi$. This was calculated in the proof in [15] of our Lemma 3.11, so we have

$$A \ge A' \ge A'' = \alpha_{d-1}((2c_0 M - h')h')^{(d-1)/2}. \tag{37}$$

For given $h_0 \ge 4\delta$, let $Q(h_0)$ be the number of hyperplane faces of H with height in the range $h \ge h_0$. Let $h_0' = h_0 - 2\delta$ $(\ge 2\delta)$.

First we consider the extreme case

$$h \ge c_0 M + 2\delta. \tag{38}$$

The equatorial plane $\Psi^\star$ parallel to Ψ through the centre of B_0, cuts off a cap from C_0 of smaller $(d-1)$-dimensional content $A^\star$. Then $A^\star$ is greater than or equal to half the surface content of the ball B_0, which is greater than $B_0 \cap \Psi^\star$, so that

$$A \ge A^\star \ge \frac{1}{2} d\alpha_d (c_0 M)^{d-1} \ge B_0 \cap \Psi^\star = \alpha_{d-1}(c_0 M)^{d-1}. \tag{39}$$

The boundary content of C_0 is less than or equal to that of a d-sphere of radius $c_1 M$,

$$\le d\alpha_d (c_1 M)^{d-1}. \tag{40}$$

Let Q_E be the number of "extreme faces" satisfying (38). Dividing the upper bound (40) by the lower bound (39) gives

$$Q_E \le \frac{d\alpha_d (c_1 M)^{d-1}}{\alpha_{d-1}(c_0 M)^{d-1}} = \frac{d\alpha_d}{\alpha_{d-1}} \left(\frac{c_1}{c_0}\right)^{d-1} = \lambda_E, \tag{41}$$

say.

Secondly we consider the usual case

$$h \le c_0 M + 2\delta, \tag{42}$$

so that $h_0' = h_0 - 2\delta \le h - 2\delta \le c_0 M$. Then from (37),

$$A \ge \alpha_{d-1}((2c_0 M - h')h')^{(d-1)/2} \ge \alpha_{d-1}((2c_0 M - h_0')h_0')^{(d-1)/2}. \tag{43}$$

Let $Q_U(h_0)$ be the number of "usual" faces with height $h \ge h_0$ satisfying (42). Dividing the upper bound, (40), by the lower bound, (43), for this case gives

$$Q_U(h_0) \le \frac{d\alpha_d (c_1 M)^{d-1}}{\alpha_{d-1}((2c_0 M - h_0')h_0')^{(d-1)/2}}. \tag{44}$$

We simplify the upper bound (44). When $4\delta \le h_0 \le c_0 M + 2\delta$, then $2\delta \le$

$h_0' \le c_0 M$. This implies that

$$\frac{1}{2c_0M - h_0'} = \frac{1}{2c_0M - h_0 + 2\delta} \le \frac{1}{c_0 M}$$

and $1/h_0' \le 2/h_0$. Hence we can write

$$\begin{aligned}(45)\qquad Q_U(h_0) &\le \frac{2^{(d-1)/2} d\alpha_d (c_1 M)^{d-1}}{\alpha_{d-1}(c_0 M h_0)^{(d-1)/2}} \\ &\le \frac{d\alpha_d}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{(d-1)/2}\left(\frac{2c_1M}{h_0}\right)^{(d-1)/2} = \lambda_U\left(\frac{2c_1M}{h_0}\right)^{(d-1)/2},\end{aligned}$$

say.

Each face Ψ is contained within the outer shell boundary C_1, which itself is contained within a d-hypersphere of radius $c_1 M$. Therefore all heights are at most $2c_1M$, and we have

$$\begin{aligned}(46)\qquad Q(h_0) &\le Q_U(h_0) + Q_E \\ &\le (\lambda_E + \lambda_U)\left(\frac{2c_1M}{h_0}\right)^{(d-1)/2} \le \frac{2d\alpha_d}{\alpha_{d-1}}\left(\frac{\sqrt{2}\, c_1}{c_0}\right)^{d-1}\left(\frac{c_1M}{h_0}\right)^{(d-1)/2} \\ &\le 2^{(d+5)/2} d\left(\frac{c_1}{c_0}\right)^{d-1}\left(\frac{c_1M}{h_0}\right)^{(d-1)/2} = \lambda_1\left(\frac{c_1M}{h_0}\right)^{(d-1)/2},\end{aligned}$$

say, where we have used (7). This result is valid for all faces with height $h \ge h_0 \ge 4\delta$.

For a fixed height h_0, the sum in (36) is maximal when as many short vectors as possible are counted, up to the upper bound in (46). In the proof of Lemma 3.6 we saw that there are at most $2^{2d-1}F^d$ vectors in each of the partitions and the inequality (20) is calculated assuming this maximum.

The total number of faces counted is

$$\begin{aligned}2^{2d-1}((2^0)^d + (2^1)^d + (2^2)^d + \cdots + (2^k)^d) &= 2^{2d-1}\,\frac{(2^d)^{k+1} - 1}{2^d - 1} \\ &\ge 2^{d(k+1)+d-1} \ge 2^{d(k+1)}.\end{aligned}$$

Therefore, to ensure that all possible faces are counted, we require

$$2^{d(k+1)} \ge \lambda_1\left(\frac{c_1M}{h_0}\right)^{(d-1)/2},$$

which implies that

$$2^{dk} \ge \frac{\lambda_1}{2^d}\left(\frac{c_1M}{h_0}\right)^{(d-1)/2}.$$

Hence if

$$(47)\qquad E = \lambda_1^{1/d}\left(\frac{c_1M}{h_0}\right)^{(d-1)/2d} \ge 2^k \ge \left(\frac{\lambda_1}{2^d}\right)^{1/d}\left(\frac{c_1M}{h_0}\right)^{(d-1)/2d}$$

in Lemma 3.6 with $j=1$, then (36) is maximal. We have

$$\sum_{1\le|\mathbf{e}|\le 2^k}\frac{1}{|\mathbf{e}|}\le 2^{2d+1}\left(\lambda_1^{1/d}\left(\frac{c_1M}{h_0}\right)^{(d-1)/2d}\right)^{d-1}. \tag{48}$$

We now consider three cases.

CASE 1:

$$h\ge\frac{1}{(c_1M)^{(d-1)/(d+1)}}\ge 4\delta. \tag{49}$$

Let L be the total number of $(d-1)$-faces satisfying (49). We partition these $(d-1)$-faces into sets $G_1,\dots,G_n$, according to their respective heights $h_i, 1\le i\le n$, where $h_n>\cdots>h_1\ge 4\delta$. Let $L_i=|G_i|$, the number of hyperplane faces whose height is h_i; let $\mathbf{n}_{i,1},\dots,\mathbf{n}_{i,L_i}$ be the normal vectors of the faces in G_i and let

$$\sigma_i=\sum_{j=1}^{L_i}\frac{1}{|\mathbf{n}_{i,j}|}. \tag{50}$$

By (47) we have

$$\sum_{i=1}^{n}\sigma_i\le\sum_{1\le|\mathbf{e}|\le 2^k}\frac{1}{|\mathbf{e}|}\le 2^{2d+1}\left(\lambda_1^{1/d}\left(\frac{c_1M}{h_i}\right)^{(d-1)/2d}\right)^{d-1}.$$

Hence for each h_i, there exists a real number τ_i, $0<\tau_i\le 1$, with

$$\sigma_i=\tau_i 2^{2d+1}\left(\lambda_1^{1/d}\left(\frac{c_1M}{h_i}\right)^{(d-1)/2d}\right)^{d-1}, \tag{51}$$

and

$$0<\sum_{i=1}^{n}\tau_i\le 1. \tag{52}$$

Let $N(h_i)$ be the number of integer points lying in $G_i\cap E$. Then by (35) and (51), we have

$$\begin{aligned}N(h_i)&\le d!2^{(d+9)/2}d\delta(c_1M)^{(d-1)/2}h_i^{(d-3)/2}\sum_{j=1}^{L_i}\frac{1}{|\mathbf{n}_{i,j}|}\\&\le d!2^{(d+9)/2}d\delta(c_1M)^{(d-1)/2}h_i^{(d-3)/2}\tau_i 2^{2d+1}\left(\lambda_1^{1/d}\left(\frac{c_1M}{h_i}\right)^{(d-1)/2d}\right)^{d-1}\\&=\lambda_2\tau_i\delta(c_1M)^{(d-1)/2+(d-1)^2/2d}h_i^{(d-3)/2-(d-1)^2/2d},\end{aligned}$$

say. Summing over all heights h_i gives N_1, the total number of integer points contributed in this case, to be

$$\le \lambda_2\delta(c_1M)^{(d-1)(2d-1)/2d}\sum_{i=1}^{n}\tau_i h_i^{-(d+1)/2d}. \tag{53}$$

The exponent of h_i in (53) is negative, and as the h_i are positive, the sum is maximal when the h_i are as small as possible and the τ_i are as large as possible for the smallest h_i. Hence we take $\sum_{i=1}^{n}\tau_i = 1$ in (53), and

$$h_i = \frac{1}{(c_1M)^{(d-1)/(d+1)}}$$

for all i. Substituting for h_i in (53) gives the total number of integer points N_1 contributed to be

$$N_1 \le \lambda_2\delta(c_1M)^{\frac{(d-1)(2d-1)}{2d}+\frac{d-1}{2d}}\sum_{i=1}^{n}\tau_i = \lambda_2\delta(c_1M)^{d-1}. \tag{54}$$

CASE 2:

$$4\delta \le h \le \frac{1}{(c_1M)^{(d-1)/(d+1)}}. \tag{55}$$

By Lemma 3.9, the maximum possible number of faces is

$$\le 2(3\alpha_d d!)^{d/(d+1)}(c_1M)^{d(d-1)/(d+1)}.$$

Hence if

$$\begin{aligned}E &= 4(3\alpha_d d!)^{1/(d+1)}(c_1M)^{(d-1)/(d+1)}\\ &\ge 2^k \ge 2(3\alpha_d d!)^{1/(d+1)}(c_1M)^{(d-1)/(d+1)}\end{aligned}$$

in Lemma 3.6 with $j=1$, then (36) is maximal. We have

$$\sum_{1\le|\mathbf{e}|\le 2^k}\frac{1}{|\mathbf{e}|} \le 2^{2d+1}(4(3\alpha_d d!)^{1/(d+1)}(c_1M)^{(d-1)/(d+1)})^{d-1}. \tag{56}$$

Let N_2 be the total number of integer points in this case. Then substituting (56) into (36) yields

$$\begin{aligned}N_2 \le{}& d!2^{(d+9)/2}d\delta(c_1M)^{(d-1)/2}h^{(d-3)/2}\\ &\times 2^{2d+1}4^{d-1}(3\alpha_d d!)^{(d-1)/(d+1)}(c_1M)^{(d-1)^2/(d+1)}.\end{aligned} \tag{57}$$

Taking

$$h = \frac{1}{(c_1M)^{(d-1)/(d+1)}}$$

to maximise (57) we have

$$N_2 \le \lambda_3\delta(c_1M)^{\frac{(d-1)^2}{d+1}-\frac{(d-3)(d-1)}{2(d+1)}+\frac{d-1}{2}} = \lambda_3\delta(c_1M)^{d-1}. \tag{58}$$

CASE 3: $0 \le h \le 4\delta$. As in the previous case, we assume the maximum number of short vector faces and we take $h = 4\delta$ to maximise (57). Let N_3

be the total number of integer points in this case. Then

$$\begin{aligned} N_3 &\leq \lambda_3 \delta (4\delta)^{(d-3)/2} (c_1 M)^{\frac{(d-1)^2}{d+1} + \frac{d-1}{2}} \\ &= \lambda_3 4^{(d-3)/2} (\delta c_1 M)^{(d-1)/2} (c_1 M)^{(d-1)^2/(d+1)}. \end{aligned}$$

When

$$\begin{aligned} \delta \leq \delta_0 &= \left(\frac{c_0}{2^{2d} 5^{d-1} 13 d! c_1} \right)^{2/(d+1)} (c_1 M)^{-(d-1)/(d+1)} \\ &= \mu (c_1 M)^{-(d-1)/(d+1)}, \end{aligned}$$

we have the bound

$$\begin{aligned} N_3 &\leq \lambda_3 \mu^{(d-1)/2} 2^{d-3} ((c_1 M)^{2/(d+1)})^{(d-1)/2} (c_1 M)^{(d-1)^2/(d+1)} \\ &= \lambda_3 \mu^{(d-1)/2} 2^{d-3} (c_1 M)^{d(d-1)/(d+1)}. \end{aligned} \tag{59}$$

Finally, we add together the upper bounds for N_1, N_2 and N_3 in (54), (58) and (59) respectively. When $\delta = \delta_0$ this gives the total number, N, of integer points lying on the $(d-1)$-dimensional boundary components to be

$$N \leq (\lambda_2 + \lambda_3) \delta_0 (c_1 M)^{d-1} + \lambda_3 \mu^{(d-1)/2} 2^{d-3} (c_1 M)^{d(d-1)/(d+1)}.$$

After simplification we find that

$$\lambda_2 \leq d(d+1)! 2^{3d+8} \left(\frac{c_1}{c_0} \right)^{d-1}, \qquad \lambda_3 \leq d!(d+1)! 2^{(9d+17)/2},$$

and

$$2^{d-3} \mu^{(d-1)/2} \leq 1,$$

where we have used (6). Hence, if $\delta \leq \delta_0$ then N does not exceed

$$d!(d+1)! 2^{\frac{9d+17}{2}} \left(\frac{c_1}{c_0} \right)^{(d-1)/2} \left((c_1 M)^{\frac{d(d-1)}{d+1}} + 2 \left(\frac{c_1}{c_0} \right)^{(d-1)/2} \delta_0 (c_1 M)^{d-1} \right). \ \blacksquare$$

LEMMA 4.3. *The number of integer points on d-dimensional boundary components, when $\delta = \delta_0$, is estimated by*

$$\begin{aligned} \sum_{\dim S^\star(V_i) = d} |S^\star(V_i)| &\leq 2(d+1)(3\alpha_d d!)^{d/(d+1)} (2c_1 M)^{d(d-1)/(d+1)} \\ &\leq 36(d+1)!(2c_1 M)^{d(d-1)/(d+1)}. \end{aligned} \tag{60}$$

Proof. From (18), the d-dimensional boundary component $S^\star(V_i)$ will have a d-dimensional volume $\mathrm{Vol}(H_i)$, with

$$\mathrm{Vol}(H_i) \leq \frac{52 \delta c_1}{c_0} (20 \sqrt{\delta c_1 M})^{d-1}.$$

Since $\delta = \delta_0$ this gives a d-volume of at most $1/d!$. Applying the minor arc case of Lemma 2.2 gives

$$K_i \leq (d+1)! \mathrm{Vol}(H_i),$$

where K_i is the number of integer points contained in $S^\star(V_i)$. However, the existence of a d-dimensional $S^\star(V_i)$ in $S'(V)$ requires that $K_i \geq d+1$, and so if we consider $\delta = \delta_0$, then K_i, the number of integer points in the boundary component, is exactly $d+1$. The number of vertices of the convex hull is

$$\leq 2(3\alpha_d d!)^{d/(d+1)}(2c_1 M)^{d(d-1)/(d+1)},$$

by (25) in Lemma 3.10 with $j = 1$. Hence, when $\delta = \delta_0$, the total number of integer points in the d-dimensional boundary components is estimated by

$$2(d+1)(3\alpha_d d!)^{d/(d+1)}(2c_1 M)^{d(d-1)/(d+1)}. \quad \blacksquare \tag{61}$$

5. Girdles and lattice determinants. We now recall Minkowski's Second Theorem [7].

LEMMA 5.1 (Minkowski's Second Theorem). *Let K be a convex body symmetrical in the origin. Let Λ be a lattice. Let the successive minima of K with respect to Λ be $\lambda_1, \ldots, \lambda_d$, defined by*

$$\lambda_i = \inf\{\lambda > 0 : \lambda K \text{ contains at least } i \text{ linearly independent vectors of } \Lambda\},$$

where

$$0 < \lambda_1 \leq \cdots \leq \lambda_d < +\infty.$$

Then they obey the inequality

$$\frac{2^d D(\Lambda)}{d!} \leq \lambda_1 \ldots \lambda_d V(K) \leq 2^d D(\Lambda) \tag{62}$$

where $V(K)$ is the volume of K and $D(\Lambda)$ is the determinant of the lattice.

COROLLARY. *Let Λ and $D(\Lambda)$ be defined as above, with $\lambda_1, \ldots, \lambda_d$ the ordinary Euclidean lengths of the lattice vectors. Let K be the open unit d-ball. Then the determinant or fundamental volume of the lattice satisfies*

$$\frac{\lambda_1 \ldots \lambda_d \alpha_d}{2^d} \leq D(\Lambda) \leq \lambda_1 \ldots \lambda_d. \tag{63}$$

Proof of Corollary. By construction, if $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the linearly independent vectors of Λ with respective Euclidean lengths $\lambda_1, \ldots, \lambda_d$, then the $\mathbf{e}_i$ are ordered by length. Let θ_i be the angle between $\mathbf{e}_{i+1}$ and the i-dimensional plane lattice defined by $\mathbf{e}_1, \ldots, \mathbf{e}_i$ with determinant $D(\Lambda_i)$. Then

$$\begin{aligned} D(\Lambda) &= \lambda_d \sin\theta_{d-1} D(\Lambda_{d-1}) = \lambda_d \lambda_{d-1} \sin\theta_{d-1} \sin\theta_{d-2} D(\Lambda_{d-2}) \\ &= \cdots = \lambda_1 \ldots \lambda_d \prod_{i=1}^{d} \sin\theta_i \leq \lambda_1 \ldots \lambda_d. \end{aligned}$$

The upper bound of (62) gives

$$\frac{\lambda_1 \ldots \lambda_d V(K)}{2^n} \leq D(\Lambda),$$

and taking $V(K) = \alpha_d$ gives the required result. $\blacksquare$

Here we introduce the idea of a j-dimensional girdle, $2 \leq j \leq d-2$, with fixed basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_j$. The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_j$ through the origin generate a j-dimensional lattice Λ in a j-plane Π_0. Each *j-girdle* is therefore defined to be a set of j-dimensional boundary components whose j-planes Π are all completely parallel to Π_0. The sets of integer points on each j-plane Π are cosets of Λ, congruent to Λ by translation, and the number of integer points lying on each j-girdle is related to the fundamental j-volume or determinant of the lattice Λ. Conversely, the lattice Λ determines the linearly independent vectors $\mathbf{e}_1, \ldots, \mathbf{e}_j$ in the Corollary to Lemma 5.1. We write $l(\Lambda)$ for the length λ_j of the longest basis vector $\mathbf{e}_j$ and introduce the following lemma to assist with our counting argument.

Lemma 5.2 (Sums of reciprocal lattice determinants). *For $k=1, \ldots, d-1$ we have*

$$\sum_{l(\Lambda)\leq E} \frac{1}{(D(\Lambda))^k} \leq \frac{(2^{2d+2k}E^{d-k})^j}{\alpha_j^k}, \tag{64}$$

where the sum ranges over all possible j-dimensional lattice determinants, $j \leq d-1$, whose basis vectors have length $\leq E$. When we take E to be the maximum possible length of a boundary component basis vector, then by (16), *$E = 10\sqrt{\delta c_1 M}$ and*

$$\sum_{l(\Lambda)\leq E} \frac{1}{(D(\Lambda))^k} \leq \frac{(2^{3d+k}(5\sqrt{\delta c_1 M})^{d-k})^j}{\alpha_j^k}. \tag{65}$$

Proof. By the Corollary to Lemma 5.1, there are linearly independent vectors $\mathbf{e}_i, 1 \leq i \leq j$, of the lattice Λ with

$$\frac{|\mathbf{e}_1|\ldots|\mathbf{e}_j|\alpha_j}{2^j} \leq D(\Lambda) \leq |\mathbf{e}_1|\ldots|\mathbf{e}_j|.$$

Hence by Lemmas 5.1 and 3.6,

$$\begin{aligned}\sum_{l(\Lambda)\leq E} \frac{1}{(D(\Lambda))^k} &\leq \left(\frac{2^j}{\alpha_j}\right)^k \sum_{|\mathbf{e}_1|\leq E} \cdots \sum_{|\mathbf{e}_j|\leq E} \frac{1}{|\mathbf{e}_1|^k \ldots |\mathbf{e}_j|^k} \\ &\leq \left(\frac{2^j}{\alpha_j}\right)^k (2^{2d+k}E^{d-k})^j = \frac{(2^{2d+2k}E^{d-k})^j}{\alpha_j^k}.\end{aligned}$$

By (16) the vectors $|\mathbf{e}_i|$ are non-zero integer vectors with

$$|\mathbf{e}_i| \leq l(\Lambda) \leq E = 10\sqrt{\delta c_1 M}, \tag{66}$$

so that

$$\sum_{l(\Lambda)\leq E} \frac{1}{(D(\Lambda))^k} \leq \frac{(2^{2d+2k}(10\sqrt{\delta c_1 M})^{d-k})^j}{\alpha_j^k} = \frac{(2^{3d+k}(5\sqrt{\delta c_1 M})^{d-k})^j}{\alpha_j^k},$$

which establishes the result. ∎

6. Summing the boundary components. When we consider a j-dimensional boundary component $S^\star(V)$, $2 \le j \le d-2$, there are geometrical considerations. The points of $S^\star(V)$ lie on some j-dimensional plane Π containing the vertex V. The lattice of integer points meets Π is some j-dimensional lattice Λ with a basis consisting of j integer vectors $\mathbf{e}_1, \ldots, \mathbf{e}_j$. The points of $S^\star(V)$ lie in the set E, the shell bounded by the surfaces C_1 and C_0. By the calculations of Lemma 3.3 the points of $S^\star(V)$ lie in a d-dimensional cylindrical slab G whose axis is the normal $\mathbf{n}$ to C_1 at R, the point of C_1 closest to the vertex V. The upper and lower faces of the d-cylinder G lie in the tangent hyperplane F at R and in a completely parallel hyperplane F', separated by a small distance

$$\eta = 52\delta c_1/c_0.$$

The upper and lower faces of the d-cylinder are $(d-1)$-spheres of radius $10\sqrt{\delta c_1 M}$ by (16) of Lemma 3.3.

As defined at the beginning of Section 4, in d-dimensional space, through a given point V on a j-plane Π, there exists a unique $(d-j)$-plane Ψ that is completely orthogonal to Π.

Let W_1 be a point of F' not in Π or Ψ and lying at a distance $10\sqrt{\delta c_1 M}$ from the axis of the d-cylinder. As $2 \le j, d-j \le d-2$, we can choose W_1 such that Y, the (two-dimensional) affine plane defined by $\mathbf{n}$ and W_1, contains at least one other point P of the j-plane Π in addition to the vertex V. Then $Y \cap G$ is a rectangle containing P, R and V, and W_1 is a corner of the rectangle. Hence the line segment VP is also contained in $Y \cap \Pi$. Let $\mathbf{k}$ be the line VP produced in $Y \cap \Pi$, cutting the hyperplanes of the upper and lower faces of the cylinder in W_3 and W_4. Let W_2 be the corner of the rectangle on F that is diametrically opposite W_1 as depicted in Figure 2.

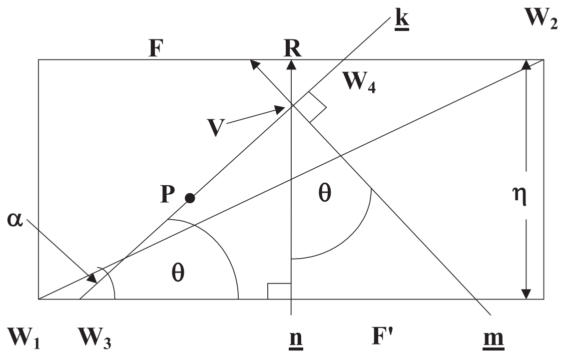

Fig. 2

We can construct in Y a line $\mathbf{m}$, through V, that is orthogonal to the line $\mathbf{k}$. By the definition of completely orthogonal planes, all lines perpendicular to $\mathbf{k}$ and not in Π must lie in Ψ. Therefore the line $\mathbf{m}$ lies in $Y \cap \Psi$ making an angle θ with $\mathbf{n}$, the normal to the tangent hyperplane to C_1 at R.

By construction, any vector lying wholly within the d-cylinder G has length $\le W_1W_2$, so that

$$W_3W_4 = \eta \operatorname{cosec} \theta \le W_1W_2 = \eta \operatorname{cosec} \alpha.$$

By equation (16), the distance of points of $S^\star(V)$ from V is at most

$$r = 10\sqrt{\delta c_1 M},$$

so that $S^\star(V)$ lies within a distance r of the line $\mathbf{k}$ in a j-dimensional plane Π. Hence $S^\star(V)$ must be contained in a j-cylinder, G', with axis k, whose upper and lower faces are $(j-1)$-spheres of radius r. The j-dimensional volume of G' is therefore

$$\alpha_{j-1} r^{j-1} W_3W_4 = \alpha_{j-1} r^{j-1} \eta \operatorname{cosec} \theta. \tag{67}$$

Suppose that the j-dimensional boundary component $S^\star(V_i)$ contains l points of S, where

$$L + 1 \le l \le 2L \tag{68}$$

for some L equal to a power of two. By Lemma 2.2 in dimension j, the convex hull of $S^\star(V)$ has j-dimensional volume

$$\operatorname{Vol}(S^\star(V)) \ge \frac{l-j}{j!} D(\Lambda) \ge \frac{L-j+1}{j!} D(\Lambda) \ge \frac{L}{(j+1)!} D(\Lambda), \tag{69}$$

where $|S^\star(V)|$ lies in the range of (68).

Comparing (67) and (69), we see that

$$\sin\theta \le \frac{(j+1)!\eta\alpha_{j-1}r^{j-1}}{D(\Lambda)L}, \tag{70}$$

and for acute angles we can write

$$\theta \le \frac{\pi}{2}\sin\theta \le \frac{\pi(j+1)!\eta\alpha_{j-1}r^{j-1}}{2D(\Lambda)L}. \tag{71}$$

As stated before, a j-girdle is a set of j-dimensional boundary components whose j-planes Π are all completely parallel. We want to count the number of components in the girdle for which (68) holds for each L equal to a power of two. Each boundary component $S^\star(V_i)$ gives rise to a set A_i along the surface of the sphere B, of radius c_1M, introduced in the proof of Lemma 4.1. The set A_i has a centre, the point W_i where the outward normal is parallel to the line VR normal to C_1. Corresponding to the unique pair of completely orthogonal j- and $(d-j)$-planes Π and Ψ through V, there are diametric planes of the sphere B, Π' parallel to Π, Ψ' parallel to Ψ, that form a unique completely orthogonal pair of planes through the centre of B. The distance of W_i from Ψ', measured along the surface of B, is $\theta c_1 M$. The distance of each point of A_i from W_i is

$$\le \sqrt{c_0\delta M/4},$$

so that the distance of each point of A_i from the $(d-j)$-plane Ψ' is

$$\le \theta c_1 M + \sqrt{c_0\delta M/4} \le 2\max(\theta c_1 M, \theta_0 c_1 M), \tag{72}$$

where

$$\theta_0 = \frac{1}{c_1}\sqrt{\frac{c_0\delta}{4M}}.$$

There are two cases according to which term gives the maximum in (72). In both cases we consider the maximum $(d-1)$-dimensional surface region available on the surface of the d-sphere B and relate this to the minimum surface requirement for each set A_i on the surface of B. We note that if more than one j-dimensional boundary component in a j-girdle of the convex hull H lies on the same j-plane, then the vertices V_i which label the boundary components $S^\star(V_i)$ must be different, so they are counted separately in this argument.

First we consider L so small that

$$\frac{\pi(j+1)!\eta\alpha_{j-1}r^{j-1}}{2D(\Lambda)L} \ge \frac{\pi}{2}\sin\theta \ge \theta \ge \theta_0 = \frac{1}{c_1}\sqrt{\frac{c_0\delta}{4M}}. \tag{73}$$

Then

$$\frac{\pi(j+1)!\eta\alpha_{j-1}r^{j-1}c_1 M}{D(\Lambda)L} \ge 2\max(\theta c_1 M, \theta_0 c_1 M).$$

The intersection of Ψ' with B is a $(d-j)$-dimensional sphere, B_1, with diameter $2c_1M$. The $(d-j-1)$-dimensional surface of B_1 is contained within the $(d-1)$-dimensional surface of B, and by (6) this is given by

$$(d-j)\alpha_{d-j}(c_1 M)^{d-j-1}. \tag{74}$$

The set A_i has distance at most $2\theta c_1 M$ from the $(d-j)$-plane Ψ' on the surface of B in j further perpendicular directions, and so has cross-section at most $4\theta c_1 M$ in these j dimensions. Hence the search region on the surface of B has $(d-1)$-dimensional volume at most

$$\begin{aligned}(d-j)\alpha_{d-j}(c_1M)^{d-j-1}(4\theta c_1 M)^j &\le (2\pi c_1 M)^{d-j-1}(4\theta c_1 M)^j\\ &\le (2\pi c_1 M)^{d-j-1}\left(\frac{2\pi(j+1)!\eta\alpha_{j-1}r^{j-1}c_1M}{D(\Lambda)L}\right)^j,\end{aligned}$$

where we have used (8). By (30), the number of such sets A is at most

$$\frac{1}{\alpha_{d-1}}\left(\sqrt{\frac{4}{c_0\delta M}}\right)^{d-1}(2\pi c_1 M)^{d-j-1}\left(\frac{2\pi(j+1)!\eta\alpha_{j-1}r^{j-1}c_1 M}{D(\Lambda)L}\right)^j$$
$$=\frac{2^{2(d-1)+j^2}5^{j(j-1)}13^j\alpha_{j-1}^j\pi^{d-1}((j+1)!)^j c_1^{\frac{2d+j^2+j-2}{2}}}{\alpha_{d-1}c_0^{\frac{d+2j-1}{2}}(D(\Lambda)L)^j}\delta^{\frac{j^2+j-d+1}{2}}M^{\frac{d+j^2-j-1}{2}}.$$

The corresponding boundary components $S^\star(V)$ have at most $2L$ points. We then sum over $L=2,4,8,\dots$ to get a contribution of at most

$$\frac{2^{2d+j^2}5^{j(j-1)}13^j\alpha_{j-1}^j\pi^{d-1}((j+1)!)^j c_1^{\frac{2d+j^2+j-2}{2}}}{\alpha_{d-1}c_0^{\frac{d+2j-1}{2}}(D(\Lambda))^j}\delta^{\frac{j^2+j-d+1}{2}}M^{\frac{d+j^2-j-1}{2}} \tag{75}$$

points to S from all the boundary components in the girdle in the cases (73).

For ranges of L for which (73) is false we have

$$\sin\theta\le\frac{(j+1)!\eta\alpha_{j-1}r^{j-1}}{D(\Lambda)L}<\frac{2\theta_0}{\pi}=\frac{1}{\pi c_1}\sqrt{\frac{c_0\delta}{M}},$$
$$\theta\le\frac{\pi}{2}\sin\theta<\theta_0=\frac{1}{2c_1}\sqrt{\frac{c_0\delta}{M}},$$
$$2\max(\theta c_1 M,\theta_0 c_1 M)<2\theta_0 c_1 M=\sqrt{c_0\delta M}.$$

The sets A_i corresponding to the extended components with all L for which (73) is false are disjoint, and they lie within a region of $(d-1)$-volume at most

$$(2\pi c_1 M)^{d-j-1}(4\theta_0 c_1 M)^j\le(2\pi c_1 M)^{d-j-1}(2\sqrt{c_0\delta M})^j$$
$$=2^{d-1}(\pi c_1)^{d-j-1}(c_0\delta)^{j/2}M^{(2d-j-2)/2},$$

using the same reasoning as that of the previous case.

By (30), the number of such sets A_i is at most

$$\frac{1}{\alpha_{d-1}}\left(\sqrt{\frac{4}{c_0\delta M}}\right)^{d-1}2^{d-1}(\pi c_1)^{d-j-1}c_0^{j/2}\delta^{j/2}M^{(2d-j-2)/2}$$
$$=\left(\frac{2^{2d-2}(\pi c_1)^{d-j-1}c_0^{(j+1-d)/2}}{\alpha_{d-1}}\right)\delta^{(j+1-d)/2}M^{(d-j-1)/2}.$$

However small θ is, the integer points of $S^\star(V)$ lie in a j-dimensional cube of j-volume

$$(20\sqrt{\delta c_1 M})^j,$$

so if there are $l\ge j+1$ integer points in $S^\star(V)$, by the minor arc case $d=j$

in Lemma 2.2,

$$\frac{l}{(j+1)!}\,D(\varLambda) \le \frac{l-j+1}{j!}\,D(\varLambda) \le (20\sqrt{\delta c_1 M})^j,$$

so that

$$l \le \frac{(j+1)!}{D(\varLambda)}\,(20\sqrt{\delta c_1 M})^j$$

and the boundary components $S^\star(V)$ in the girdle for which (73) is false contribute

$$\begin{aligned}(76)\qquad &\le \frac{(j+1)!}{D(\varLambda)}\,(20\sqrt{\delta c_1 M})^j\,\frac{2^{2d-2}(\pi c_1)^{d-j-1}c_0^{\frac{j+1-d}{2}}}{\alpha_{d-1}}\,\delta^{\frac{j+1-d}{2}}M^{\frac{d-j-1}{2}}\\ &= \frac{(j+1)!2^{2d+2j-2}5^j\pi^{d-j-1}c_0^{\frac{j+1-d}{2}}c_1^{\frac{2d-j-2}{2}}}{\alpha_{d-1}D(\varLambda)}\,\delta^{\frac{2j+1-d}{2}}M^{\frac{d-1}{2}}\end{aligned}$$

integer points to $S(H)$.

We use Lemma 5.2 with $j=k$ to estimate the contribution of all boundary components with L small in all j-girdles given by (75) as

$$(77)\qquad \frac{2^{3jd+2d+2j^2}5^{j(d-1)}13^j\alpha_{j-1}^j\pi^{d-1}((j+1)!)^j c_1^{(2d+jd+j-2)/2}}{\alpha_{d-1}\alpha_j^j c_0^{(d+2j-1)/2}} \times \delta^{(d+1)(j-1)/2+1}M^{(d-1)(j+1)/2}$$

integer points, and the contribution of all boundary components with L large from all j-girdles given by (76) as

$$(78)\qquad \frac{(j+1)!2^{3jd+3j+2d-2}5^{jd}\pi^{d-j-1}c_0^{(j+1-d)/2}c_1^{(jd+2d-2j-2)/2}}{\alpha_{d-1}\alpha_j} \times \delta^{(d+1)(j-1)/2+1}M^{(d-1)(j+1)/2}.$$

After some calculation we find that

$$\frac{c_1^{(2d+jd+j-2)/2}}{c_0^{(d+2j-1)/2}} \ge c_0^{(j+1-d)/2}c_1^{(jd+2d-2j-2)/2},$$

$$\frac{\alpha_{j-1}}{\alpha_j} \le j, \qquad (j(j+1)!)^j \ge \frac{(j+1)!}{\alpha_j},$$

for all $j\ge 0$, $d\ge 1$, where we have used (4), (6) and (7) to obtain the above inequalities. Hence we can write the sum of the two terms from (77) and (78) as

$$(79)\qquad \le \lambda_j\left(\frac{c_1^2}{c_0^2}\,\delta^{d+1}(c_1 M)^{d-1}\right)^{(j-1)/2}\left(\frac{c_1}{c_0}\right)^{(d+1)/2}\delta(c_1 M)^{d-1}$$

where we have written

$$\lambda_j = \frac{2^{3jd+2d+2j^2+2j}(5^j\pi)^{d-1}(9j(j+1)!)^j}{\alpha_{d-1}}.$$

We now consider the total number of integer points contributed by the j-girdles in all boundary components with $\delta \le \delta_0$, defined in (17). Hence

$$\delta^{d+1} < \delta_0^{d+1} = \left(\frac{c_0}{2^{2d}5^{d-1}13d!c_1}\right)^2 (c_1M)^{-(d-1)},$$

and

$$\left(\frac{c_1^2}{c_0^2}\,\delta^{d+1}(c_1M)^{d-1}\right)^{(j-1)/2} \le \left(\frac{1}{2^{2d}5^{d-1}13d!}\right)^{j-1} = \mu_j,$$

say, where μ_j is a constant depending only on d and j.

In this notation, the upper bound in (79) for the components with $\delta \le \delta_0$ is

$$\lambda_j\mu_j\left(\frac{c_1}{c_0}\right)^{(d+1)/2}\delta(c_1M)^{d-1}. \tag{80}$$

Using the inequalities

$$\frac{9^j}{13^{j-1}} \le 9, \quad j \ge 1,$$
$$\frac{j^j(j+1)!^j}{d!^{j-1}} \le d!, \quad j \le d-2,$$

we can write

$$\lambda_j\mu_j \le \frac{2^{8d+3jd+2j}d!}{\alpha_{d-1}}.$$

Now

$$\sum_{j=2}^{d-2} 2^{2j} = \frac{(2^d-8)(2^d+8)}{12} \le 2^{2d-3},$$

and

$$\sum_{j=2}^{d-2} 2^{3jd} = \frac{2^{3d^2}-2^{6d}}{2^{6d}-2^{3d}} \le 2^{3d^2-5d}.$$

Hence we estimate the contribution of integer points from all j-dimensional girdles, with $2 \le j \le d-2$, and $\delta \le \delta_0$ as

$$N_g \le \frac{2^{3d^2+5d-3}d!}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{(d+1)/2}\delta_0(c_1M)^{d-1}. \tag{81}$$

Next, for $\delta \le \delta_0$, we consider the integer points contributed by the boundary components of dimension $0, 1, d-1$ and d, along with the points lying strictly inside the convex hull H. These individual upper bounds correspond to (25), (27), (34), (60) and (23) respectively, and adding them we have

$$(82)\quad \leq 2^{\frac{d^2+10d+18}{2}} d!\left(\frac{c_1}{c_0}\right)^{\frac{d-1}{2}}\left((c_1M)^{\frac{d(d-1)}{d+1}} + \frac{2^4}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{\frac{d-1}{2}}\delta_0(c_1M)^{d-1}\right)$$

integer points. Combining (81) with (82) then gives the total number of integer points lying within a distance δ_0 from the convex hull H as at most

$$(83)\quad 2^{\frac{d^2+10d+18}{2}} d!\left(\frac{c_1}{c_0}\right)^{\frac{d-1}{2}}\left((c_1M)^{\frac{d(d-1)}{d+1}} + \frac{2^{\frac{5d^2-22}{2}}}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{\frac{d-1}{2}}\delta_0(c_1M)^{d-1}\right).$$

This result is valid for a shell of thickness $\delta = \delta_0$ and consists of terms independent of δ (degree zero), and those with a factor of δ (degree one).

We cover the shell E of all extended vertex components, bounded internally by C_0 and externally by C_1, by R thinner concentric shells $E_1, \ldots, E_R$ of thickness δ_0. The distance between C_1 and C_0 along any inward normal vector to these two surfaces is 2δ. Hence we choose R to be the smallest such integer with

$$R\delta_0 \geq 2\delta, \quad (R-1)\delta_0 < 2\delta,$$

so that

$$(84)\qquad R < 2\delta/\delta_0 + 1.$$

The shell E_r consists of the points on some inward normal whose distance l from the hypersurface C_1 lies in the range

$$(r-1)\delta_0 \leq l \leq r\delta_0.$$

Replacing δ with $r\delta_0$ in Lemma 2.1 implies that each shell E_r will satisfy the Curvature Condition, so that any two-dimensional plane sectional curve of E_r will lie in the range

$$c_0M \leq \varrho \leq c_1M.$$

Therefore, (83) gives a uniform upper bound for the number of integer points contributed by any shell E_r. Now let

$$(85)\qquad \eta = \delta_0(c_1M)^{(d-1)/(d+1)} = \left(\frac{c_0}{2^{2d}5^{d-1}13d!c_1}\right)^{2/(d+1)} \leq \frac{1}{2^6}.$$

Then

$$(86)\qquad \frac{1}{\eta} = \left(\frac{2^{2d}5^{d-1}13d!c_1}{c_0}\right)^{2/(d+1)} \leq 2^{d+8}\,\frac{c_1}{c_0}$$

and

$$(87)\qquad \delta_0 = \frac{\eta}{(c_1M)^{(d-1)/(d+1)}} \leq \frac{1}{2^6(c_1M)^{(d-1)/(d+1)}}.$$

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We multiply the upper bound (83) by the maximum number of shells allowed by (84). For the degree zero terms this yields

$$
\begin{aligned}
(88)\qquad &\le \left(\frac{2\delta}{\delta_0}+1\right)2^{\frac{d^2+10d+18}{2}}d!\left(\frac{c_1}{c_0}\right)^{\frac{d-1}{2}}(c_1M)^{\frac{d(d-1)}{d+1}}\\
&\le \frac{2^{\frac{d^2+10d+20}{2}}d!}{\eta}\left(\frac{c_1}{c_0}\right)^{\frac{d-1}{2}}\delta(c_1M)^{d-1}+2^{\frac{d^2+10d+18}{2}}d!\left(\frac{c_1}{c_0}\right)^{\frac{d-1}{2}}(c_1M)^{\frac{d(d-1)}{d+1}}\\
&\le 2^{\frac{d^2+12d+36}{2}}d!\left(\frac{c_1}{c_0}\right)^{\frac{d+1}{2}}\delta(c_1M)^{d-1}+2^{\frac{d^2+10d+18}{2}}d!\left(\frac{c_1}{c_0}\right)^{\frac{d-1}{2}}(c_1M)^{\frac{d(d-1)}{d+1}}.
\end{aligned}
$$

For the degree one terms we have

$$
\begin{aligned}
(89)\qquad &\le \left(\frac{2\delta}{\delta_0}+1\right)\frac{2^{3d^2+5d-2}d!}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{d-1}\delta_0(c_1M)^{d-1}\\
&\le \frac{2^{3d^2+5d-1}d!}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{d-1}\delta(c_1M)^{d-1}+\frac{2^{3d^2+5d-2}d!}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{d-1}\eta(c_1M)^{\frac{d(d-1)}{d+1}}\\
&\le \frac{2^{3d^2+5d-1}d!}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{d-1}\delta(c_1M)^{d-1}+\frac{2^{3d^2+5d-8}d!}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{d-1}(c_1M)^{\frac{d(d-1)}{d+1}}.
\end{aligned}
$$

Finally, we combine the terms from (88) and (89) to estimate the total number of integer points by

$$
\le \frac{2^{3d^2+5d-7}d!}{\alpha_{d-1}}\left(\frac{c_1}{c_0}\right)^{d-1}\left((c_1M)^{d(d-1)/(d+1)}+2^9\delta(c_1M)^{d-1}\right),
$$

as required. ∎

Proof of Theorem 1.2. In the proof of Theorem 1.1, we consider an enlarged component $S'(V)$, where all the calculations for distances between points on the outer surface C_1 take place within the reach $U(V)$ of $S'(V)$, with respect to V.

By Lemma 3.7, the Local Curvature Condition holds at all points in $U(V)$, so the calculations which establish Theorem 1.1 are valid under the weaker hypothesis of the Local Curvature Condition. ∎

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