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Mahler measure of the Horie unit and Weber's class number problem in the cyclotomic $\mathbb{Z}_3$-extension of $\mathbb{Q}$

by

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1. Introduction. Let p be a prime number and $\mathbb{Q}(\mu_{p^\infty})$ the cyclotomic field of all p-power roots of unity. Let $\mathbb{B}_{p,n}$ be the unique real subfield of $\mathbb{Q}(\mu_{p^\infty})$ which is cyclic of degree p^n over $\mathbb{Q}$. Then we call $\mathbb{B}_{p,\infty} = \bigcup_{n\geq 1} \mathbb{B}_{p,n}$ the *cyclotomic* $\mathbb{Z}_p$-*extension* of $\mathbb{Q}$, and $\mathbb{B}_{p,n}$ the nth layer of this extension. We denote the class number of $\mathbb{B}_{p,n}$ by $h_{p,n}$. We consider the following problem:

WEBER'S CLASS NUMBER PROBLEM. Is $h_{p,n}$ equal to one for every positive integer n?

Direct calculation only gives information on $h_{p,n}$ with small n. To obtain information for large n, we study the ℓ-part of $h_{p,n}$ for each prime number ℓ.

PROBLEM. Does a prime number ℓ divide $h_{p,n}$ for some positive integer n?

In the case $\ell = p$, Iwasawa [I] proved that p does not divide $h_{p,n}$. In the case $\ell \neq p$, Washington [Was] proved that the ℓ-part of $h_{p,n}$ is bounded as n tends to ∞. Recently Horie [H1]–[H3] developed an ingenious method to attack $h_{p,n}$ by studying the Galois action on the Horie unit (cf. (2.1)).

In the case $p = 2$, Fukuda–Komatsu [FK1], [FK2] proved several results by applying Washington's method and Horie's method. Moreover, Okazaki [O] developed a theory for this problem making use of lower bounds for the trace of the square of relative units and the Mahler measure of relative units, where a *relative unit* is a unit ϵ of $\mathbb{B}_{2,n}$ which satisfies $\mathrm{Nr}_{\mathbb{B}_{2,n}/\mathbb{B}_{2,n-1}}(\epsilon) = \pm 1$.

In this paper, we investigate the case of $p = 3$. We will employ one new idea which enables us to parallel the above works.

To ease notation, put $\zeta_n = \exp(2\pi\sqrt{-1}/3^n)$, $\mathbb{B}_n = \mathbb{B}_{3,n}$ and $h_n = h_{3,n}$. Then $\mathbb{B}_n = \mathbb{Q}(2\cos(2\pi/3^{n+1}))$. Masley [Ma] and van der Linden [Li] proved the following:

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Theorem 1.1 (Masley). *We have* $h_1 = h_2 = h_3 = 1$.

Theorem 1.2 (van der Linden). *If the Generalized Riemann Hypothesis is valid, then* $h_4 = 1$.

Horie [H2] proved the following:

Theorem 1.3 (Horie). *Let* $\ell \geq 5$ *be a prime number and* 3^s *the exact power of* 3 *dividing* $\ell^2 - 1$. *Put* $c = 2 \cdot 3^{s-1}$ *and*

$$H(s) = \left(\frac{(3^{s-1}\log(3/2) + (6s+4)\log 3)c^3(c-1)^{(c-1)/2}}{(\log 2) \cdot 3^{(2s-1)(c-1)/4}} \right)^c.$$

If ℓ *does not divide* h_{2s-2} *and* $\ell \geq H(s)$, *then* ℓ *does not divide* h_n *for any positive integer* n.

Horie then treated small prime numbers ℓ to obtain the following:

Theorem 1.4 (Horie). *Let* ℓ *be a prime number. If* $\ell \not\equiv \pm 1 \pmod 9$, *then* ℓ *does not divide* h_n *for any positive integer* n.

In this paper, we prove the following results by using the methods of Fukuda–Komatsu, Horie and Okazaki:

Theorem A. *Let* $\ell \geq 5$ *be a prime number and* 3^s *the exact power of* 3 *dividing* $\ell^2 - 1$. *Put*

$$m_\ell = 2s - 1 + [\log_3 \ell],$$

where $[x]$ *denotes the greatest integer not exceeding* x. *Then* ℓ *does not divide* h_n/h_{m_ℓ} *for any* $n \geq m_\ell$. *Therefore, if* ℓ *does not divide* h_{m_ℓ}, *then* ℓ *does not divide* h_n *for any positive integer* n.

Remark 1.5. Friedman–Sands [FSW] give an explicit bound of the stabilization on the ℓ-part of the minus part of class groups in the cyclotomic $\mathbb{Z}_3$-extension over imaginary abelian fields.

Theorem B. *Let* $\ell \geq 5$ *be a prime number,* n *a positive integer and* 3^s *the exact power of* 3 *dividing* $\ell^2 - 1$. *Put* $r = \min\{n, s\}$ *and* $c = 2 \cdot 3^{r-1}$, *and denote by* f *the inertia degree of* ℓ *in* $\mathbb{Q}(\zeta_r)/\mathbb{Q}$. *If* $\ell^f > 2^{c/2} \cdot c!$, *then* ℓ *does not divide* h_n/h_{n-1}.

This is analogous to a result of [O], which states that no prime number ℓ with $\ell^g > (2^{t-1})!$ divides $h_{2,n}$ for any n, where $g = 1$ or 2 according as $\ell \equiv 1$ or $-1 \pmod 4$ and 2^t is the exact power of 2 dividing $\ell^g - 1$.

In our case $p = 3$, we put $L(s) = 2^{3^{s-1}} \cdot (2 \cdot 3^{s-1})!$. Then Theorem B implies that ℓ does not divide h_n for any n if $\ell \geq L(s)$. This lower bound $L(s)$ is smaller than Horie's lower bound $H(s)$ in Theorem 1.3. For example, if $\ell \equiv 8, 10, 17, 19 \pmod{27}$, that is, $s = 2$, then

$$H(2) = 226586234472011388846 81.21742\ldots$$

and

$$L(2) = 5760. \tag{1.1}$$

We obtain the following result using Theorem A, an algorithm in [Mo] and numerical calculation.

THEOREM C. *Let ℓ be a prime number less than $4.0 \cdot 10^5$. Then ℓ does not divide h_n for any positive integer n.*

This corresponds to a result of [FK1], [FK2] that no prime number $\ell < 1.2 \cdot 10^8$ divides $h_{2,n}$ for any n.

By (1.1) and Theorem C, we obtain the following improvement upon Theorem 1.4 of Horie:

COROLLARY D. *Let ℓ be a prime number. If $\ell \not\equiv \pm 1 \pmod{27}$, then ℓ does not divide h_n for any positive integer n.*

2. Proof of Theorem A. We prove Theorem A by using Horie's method [H3]. We put $\zeta_n = \exp(2\pi\sqrt{-1}/3^n)$ and

$$\eta_n = \frac{\zeta_{n+1} - \zeta_{n+1}^{-1}}{\zeta_1\zeta_{n+1} - \zeta_1^{-1}\zeta_{n+1}^{-1}} \tag{2.1}$$

for any positive integer n. Then η_n is a unit and contained in $\mathbb{B}_n$. We call η_n the nth *Horie unit.*

Every element α in $\mathbb{Z}[\zeta_n]$ is uniquely expressed in the form

$$\alpha = \sum_{i=0}^{2\cdot 3^{n-1}-1} a_i \zeta_n^i \quad (a_i \in \mathbb{Z}).$$

For each such α and each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n+1})/\mathbb{Q}(\zeta_1))$, we define the element α_σ in the group ring $\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{n+1})/\mathbb{Q}(\zeta_1))]$ by

$$\alpha_\sigma = \sum_{i=0}^{2\cdot 3^{n-1}-1} a_i \sigma^i.$$

Horie proved the following lemmas:

LEMMA 2.1 (cf. [H2]). *Let $\ell \geq 5$ be a prime number, σ a generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_{n+1})/\mathbb{Q}(\zeta_1))$ and F a subfield of $\mathbb{Q}(\zeta_n)$ containing the decomposition field of ℓ in $\mathbb{Q}(\zeta_n)/\mathbb{Q}$. Then ℓ divides the integer h_n/h_{n-1} if and only if there exists a prime ideal $\mathfrak{L}$ of F dividing ℓ such that $\eta_n^{\alpha_\sigma}$ is an ℓth power of a unit in $\mathbb{B}_n$ for any α in the ideal $\ell\mathfrak{L}^{-1}$ of F.*

LEMMA 2.2 (cf. [H1]). *Let $\ell \geq 5$ be a prime number and φ the Frobenius automorphism of ℓ in $\mathbb{Q}(\zeta_{n+1})/\mathbb{Q}$. If an element β in $\mathbb{Z}[\zeta_{n+1}]$ is an ℓth power in $\mathbb{Z}[\zeta_{n+1}]$, then $\beta^\varphi - \beta^\ell \in \ell^2\mathbb{Z}[\zeta_{n+1}]$.*

Moreover, we use the following lemma:

Lemma 2.3. *Let a_i be elements in $\mathbb{Z}$ and ζ a primitive 3^{n+1}th root of unity. If*

$$\sum_{i=0}^{2\cdot 3^{n-1}-1} a_i\zeta^i \equiv 0 \pmod{\ell},$$

then $a_j \in \ell\mathbb{Z}$ for $0 \leq j \leq 2\cdot 3^{n-1}-1$.

Let ℓ and φ be as in Lemma 2.2, $\zeta = \zeta_{n+1}^2$ a primitive 3^{n+1}th root of unity, $\omega = \zeta_1^2$, σ a generator of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}(\omega))$ and recall

$$\eta = \frac{1}{\omega}\cdot\frac{\zeta-1}{\omega\zeta-1}$$

is the nth Horie unit. Let s be as in Theorem A and choose $F = \mathbb{Q}(\zeta_s)$. We assume that $n \geq s$ and ℓ divides h_n/h_{n-1}. Then, by Lemma 2.1, there exists a prime ideal $\mathfrak{L}$ in $\mathbb{Q}(\zeta_s)$ dividing ℓ such that η^{α_σ} is an ℓth power of a unit in $\mathbb{B}_n$ for any α in the ideal $\ell\mathfrak{L}^{-1}$ of $\mathbb{Q}(\zeta_s)$. Let

$$\alpha = \sum_{i=0}^{2\cdot 3^{s-1}-1} a_i(\zeta_n^{3^{n-s}})^i$$

be an element of $\ell\mathfrak{L}^{-1}$ with $a_i \in \mathbb{Z}$ and put $\tau = \sigma^{3^{n-s}}$. Then $\alpha_\sigma = \sum_{i=0}^{2\cdot 3^{s-1}-1} a_i\tau^i$. Noting that

$$\begin{aligned}(\beta+\gamma)^{a\ell} &= \left(\beta^\ell+\gamma^\ell+\sum_{k=1}^{\ell-1}\binom{\ell}{k}\beta^k\gamma^{\ell-k}\right)^a \\ &\equiv (\beta^\ell+\gamma^\ell)^a + a(\beta^\ell+\gamma^\ell)^{a-1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\beta^k\gamma^{\ell-k} \pmod{\ell^2}\end{aligned}$$

for $\beta,\gamma \in \mathbb{Z}[\zeta]$ and $a \in \mathbb{Z}$, we obtain, mod ℓ^2,

$$(\zeta^{\tau^i}-1)^{\ell a_i} \equiv (\zeta^{\ell\tau^i}-1)^{a_i} + a_i(\zeta^{\ell\tau^i}-1)^{a_i-1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\zeta^{k\tau^i}(-1)^{\ell-k},$$

$$(\omega\zeta^{\tau^i}-1)^{-\ell a_i} \equiv (\omega^\ell\zeta^{\ell\tau^i}-1)^{-a_i} - a_i(\omega^\ell\zeta^{\ell\tau^i}-1)^{-a_i-1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\omega^k\zeta^{k\tau^i}(-1)^{\ell-k}.$$

From these congruences and from the consequence

$$\begin{aligned}\frac{(\eta^{\alpha_\sigma})^\ell-(\eta^{\alpha_\sigma})^\varphi}{\omega^{-\ell\alpha_\sigma}} &= \prod_{i=0}^{2\cdot 3^{s-1}-1}\frac{(\zeta^{\tau^i}-1)^{\ell a_i}}{(\omega\zeta^{\tau^i}-1)^{\ell a_i}} - \prod_{i=0}^{2\cdot 3^{s-1}-1}\frac{(\zeta^{\ell\tau^i}-1)^{a_i}}{(\omega^\ell\zeta^{\ell\tau^i}-1)^{a_i}} \\ &\equiv 0 \pmod{\ell^2}\end{aligned}$$

of Lemmas 2.1 and 2.2, we obtain

$$\sum_{i=0}^{2\cdot 3^{s-1}-1}\left(\frac{a_i}{\zeta^{\ell\tau^i}-1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\zeta^{k\tau^i}(-1)^{\ell-k}\right.$$
$$\left.-\frac{a_i}{\omega^\ell\zeta^{\ell\tau^i}-1}\sum_{k=1}^{\ell-1}\binom{\ell}{k}\omega^k\zeta^{k\tau^i}(-1)^{\ell-k}\right)\equiv 0 \pmod{\ell^2},$$

since $\zeta^{\ell\tau^i}-1$ are prime to ℓ. By the congruence

$$\binom{\ell}{k}\equiv\frac{\ell(-1)^{k-1}}{k}\pmod{\ell^2}\qquad(1\le k\le \ell-1),$$

we have

$$\sum_{i=0}^{2\cdot 3^{s-1}-1}\left(\frac{a_i}{\zeta^{\ell\tau^i}-1}\sum_{k=1}^{\ell-1}\frac{\ell}{k}\zeta^{k\tau^i}-\frac{a_i}{\omega^\ell\zeta^{\ell\tau^i}-1}\sum_{k=1}^{\ell-1}\frac{\ell}{k}\omega^k\zeta^{k\tau^i}\right)\equiv 0\pmod{\ell^2}.$$

Hence

$$\sum_{i=0}^{2\cdot 3^{s-1}-1}a_i\sum_{k=1}^{\ell-1}\frac{1}{k}\left(\frac{1}{\zeta^{\ell\tau^i}-1}-\frac{\omega^k}{\omega^\ell\zeta^{\ell\tau^i}-1}\right)\zeta^{k\tau^i}\equiv 0\pmod{\ell}.$$

By substituting $(\zeta^{3^s})^{\tau^i}=\zeta^{3^s}$, we obtain

$$\begin{aligned}0&\equiv\sum_{i=0}^{2\cdot 3^{s-1}-1}a_i\sum_{k=1}^{\ell-1}\frac{1}{k}\Bigl(\sum_{j=0}^{3^s-1}(\zeta^{\ell\tau^i})^j-\omega^k\sum_{j=0}^{3^s-1}(\omega^\ell\zeta^{\ell\tau^i})^j\Bigr)\zeta^{k\tau^i}\\&\equiv\sum_{i=0}^{2\cdot 3^{s-1}-1}a_i\sum_{k=1}^{\ell-1}\sum_{j=0}^{3^s-1}\frac{1-\omega^{\ell j+k}}{k}\zeta^{(\ell j+k)\tau^i}\pmod{\ell}.\end{aligned}$$

Now we have the following:

Lemma 2.4. *Let α be as in Lemma 2.1 and*

$$\alpha=\sum_{i=0}^{2\cdot 3^{s-1}-1}a_i(\zeta_n^{3^{n-s}})^i \tag{2.2}$$

with $a_i\in\mathbb{Z}$. If ℓ divides h_n/h_{n-1}, then

$$\sum_{i=0}^{2\cdot 3^{s-1}-1}a_i\sum_{k=1}^{\ell-1}\sum_{j=0}^{3^s-1}\frac{1-\omega^{\ell j+k}}{k}\zeta^{(\ell j+k)\tau^i}\equiv 0\pmod{\ell}.$$

We put

$$S=\{b_0 3^{n-s+1}+b_1 3^{n-s+2}+\cdots+b_{s-1}3^n\mid b_j=0,1,2\text{ for }0\le j\le s-1\}$$

and define

$$S' = \bigcup_{i=0}^{2\cdot 3^{s-1}-1} \{r \in S \mid \zeta^{\tau^i - 1} = \zeta^r\}.$$

LEMMA 2.5. *Let j and k be rational integers with $0 \le j \le 3^s - 1$, $1 \le k \le \ell - 1$ and $r \in S'$. Let ℓ be a prime number with $5 \le \ell < 3^{n-2s+1}$. If $(r+1)(\ell j + k) \equiv 2 \cdot 3^{s-1}\ell - 1 \pmod{3^n}$, then $j = 2 \cdot 3^{s-1} - 1$, $k = \ell - 1$ and $r = 0$ or 3^n.*

Proof. We have

$$-3^{n-s+1} < (2 \cdot 3^{s-1} - j)\ell - k - 1 < 3^{n-s+1}$$

because $0 \le j \le 3^s - 1$, $1 \le k \le \ell - 1$ and $\ell < 3^{n-2s+1}$.

Since $(2 \cdot 3^{s-1} - j)\ell - k - 1 \equiv 0 \pmod{3^{n-s+1}}$, we have

$$(2 \cdot 3^{s-1} - j)\ell - k - 1 = 0.$$

Since $2 \le k + 1 = (2 \cdot 3^{s-1} - j)\ell \le \ell$, we have $k = \ell - 1$ and $j = 2 \cdot 3^{s-1} - 1$, which implies $r \equiv 0 \pmod{3^n}$. Hence either $r = 0$, $r = 3^n$ or $r = 2 \cdot 3^n$. Since $r \in S'$, we have $r = 0$ or $r = 3^n$. ∎

Proof of Theorem A. The assertion is trivial when $n = m_\ell$. So we assume that there exists an integer $n > m_\ell$ such that ℓ divides h_n/h_{n-1}. Then $\ell < 3^{n-2s+1}$ and Lemma 2.4 yields

$$\sum_{i=0}^{2\cdot 3^{s-1}-1} a_i \sum_{k=1}^{\ell-1} \sum_{j=0}^{3^s-1} \frac{1 - \omega^{\ell j + k}}{k} \zeta^{(\ell j + k)\tau^i} \equiv 0 \pmod{\ell} \tag{2.3}$$

where a_i is the rational integer defined by (2.2). Since $\zeta_n^{3^{n-s}}$ is a unit, we may assume $a_0 \not\equiv 0 \pmod{\ell}$. From Lemmas 2.3 and 2.5, and (2.3), we have

$$a_0 \frac{1 - \omega^2}{\ell - 1} \zeta^{2\cdot 3^{s-1}\ell - 1} + a_{3^{s-1}} \frac{1 - \omega^2}{\ell - 1} \zeta^{(2\cdot 3^{s-1}\ell - 1)(3^n + 1)} \equiv 0 \pmod{\ell}.$$

Hence $a_0 \equiv 0 \pmod{\ell}$, which is a contradiction. ∎

3. Lower bound of Mahler measure of relative units. Let α be an algebraic number. Denote by $d = \deg \alpha$ its degree. Let

$$a(X - \alpha_1) \cdots (X - \alpha_d)$$

be its minimal polynomial in $\mathbb{Z}[X]$. We define the *Mahler measure* of α by

$$M(\alpha) = |a| \prod_{j=1}^{d} \max\{1, |\alpha_j|\}$$

(cf. [EW], [Wal]). From the definition we have:

Proposition 1. *Let α, β be algebraic numbers.*

(1) *Let r be a positive integer. If $\deg \alpha^r = \deg \alpha$, then $M(\alpha^r) = M(\alpha)^r$.*
(2) *If α and β are algebraic integers with $\deg \alpha\beta \le \deg \alpha$ and $\deg \alpha\beta \le \deg \beta$, then $M(\alpha\beta) \le M(\alpha)M(\beta)$.*
(3) *If σ is an automorphism of $\mathbb{Q}(\alpha)$, then $M(\alpha^\sigma) = M(\alpha)$.*
(4) *If α is a unit, then $M(\alpha^{-1}) = M(\alpha)$.*

Schinzel showed that

$$M(\alpha) \ge \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}$$

whenever $\alpha \ne \pm 1$ is a totally real algebraic number of degree d (cf. [S] and [EW, Theorem 1.14]).

Let $F(x)$ be the minimal polynomial of a totally real unit ϵ. We point out Remark 1.16 in [EW] and notice that $F(1)F(-1)$ has an exponential lower bound in some important cases as we will see in Lemma 3.3 below. Now we can show the following inequality by tracing the proof of Theorem 1.14 in [EW].

Theorem 3.1. *Let ε be a unit other than ± 1 and $\mathcal{O}$ the ring of integers of $\mathbb{Q}(\varepsilon)$. Assume $\varepsilon - 1 \in \mathfrak{M}$ and $\varepsilon + 1 \in \mathfrak{N}$ for some ideals $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathcal{O}$. Then*

$$M(\varepsilon) \ge \left(\frac{C^{1/d} + \sqrt{C^{2/d}+4}}{2}\right)^{d/2}$$

where $d = \deg \varepsilon$ and $C = (\mathcal{O} : \mathfrak{M}\mathfrak{N})$, the absolute norm of $\mathfrak{M}\mathfrak{N}$.

We come back to the field $\mathbb{B}_n$ and we put $\zeta = \zeta_{n+1}$. Let $\mathfrak{P}$ be a prime ideal in $\mathbb{Q}(\zeta)$ dividing 3, and $w(x)$ the normalized additive $\mathfrak{P}$-adic valuation of x. Moreover, we let $\mathfrak{p}$ be a prime ideal in $\mathbb{B}_n$ dividing 3, and $v(x)$ the normalized additive $\mathfrak{p}$-adic valuation of x. Then $v(x) = 2 \cdot w(x)$ for x in $\mathbb{B}_n$. We denote by τ a generator of $\mathrm{Gal}(\mathbb{B}_n/\mathbb{B}_{n-1})$ which satisfies $\zeta^\tau = \zeta^{3^n+1}$.

Lemma 3.2. *Let ε be a unit in $\mathbb{B}_n$. If $\mathrm{Nr}_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\varepsilon) = 1$, then*

$$v(\varepsilon - 1) \ge \frac{3^n - 1}{2}.$$

Proof. There exists x in $\mathbb{Z}[\zeta]$ such that $\varepsilon = x^{1-\tau}$, by Hilbert's Theorem 90. Since $\mathfrak{P}^3 = (1-\zeta^3)$ and $(1-\zeta^3)^\tau = 1-\zeta^3$, we may assume $w(x) = 0, 1, 2$. Note that if $\alpha \in \mathbb{Z}[\zeta]$ then $w(\alpha - \alpha^\tau) \ge 3^n$. Hence

$$w(\varepsilon - 1) = w\left(\frac{x - x^\tau}{x^\tau}\right) \ge 3^n - 2,$$

that is, $2 \cdot v(\varepsilon - 1) \ge 3^n - 2$. Since $v(\varepsilon - 1)$ is a rational integer, we obtain the assertion. ∎

REMARK. We put $\omega = \zeta^{3^n}$ and recall the nth Horie unit

$$\eta_n = \frac{\zeta - \zeta^{-1}}{\omega\zeta - \omega^{-1}\zeta^{-1}}.$$

We have $\mathrm{Nr}_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\eta_n) = 1$ and

$$\begin{aligned}\eta_n^{-1} - 1 &= \frac{\omega\zeta - \omega^{-1}\zeta^{-1} - \zeta + \zeta^{-1}}{\zeta - \zeta^{-1}} = \frac{\omega\zeta^2 - \omega^2 - \zeta^2 + 1}{\zeta^2 - 1} \\ &= \frac{(\omega - 1)\zeta^2 - (\omega - 1)(\omega + 1)}{\zeta^2 - 1} = \frac{\omega - 1}{\zeta^2 - 1}(\zeta^2 + \omega^2).\end{aligned}$$

Since $(\zeta^2 - 1) = \mathfrak{P}$, $(\omega - 1) = \mathfrak{P}^{3^n}$ and $\zeta^2 + \omega^2$ is a unit in $\mathbb{Q}(\zeta)$, we have $(\eta_n - 1) = \mathfrak{P}^{3^n - 1}$ in $\mathbb{Q}(\zeta)$. Hence $(\eta_n - 1) = \mathfrak{p}^{(3^n-1)/2}$, that is, the inequality in Lemma 3.2 is best possible.

On the other hand,

$$\begin{aligned}\eta_n^{-1} + 1 &= \frac{\omega\zeta - \omega^{-1}\zeta^{-1} + \zeta - \zeta^{-1}}{\zeta - \zeta^{-1}} = \frac{\omega\zeta^2 - \omega^2 + \zeta^2 - 1}{\zeta^2 - 1} \\ &= \frac{(\omega + 1)\zeta^2 - (\omega^2 + 1)}{\zeta^2 - 1} = \frac{-\omega^2\zeta^2 + \omega}{\zeta^2 - 1} = -\omega\frac{\omega\zeta^2 - 1}{\zeta^2 - 1}.\end{aligned}$$

Hence $\eta_n^{-1} + 1$ is a unit, that is, $\eta_n + 1$ is a unit.

Note that the absolute norm of $\mathfrak{p}$ is 3. From Theorem 3.1 and Lemma 3.2, we conclude the following:

LEMMA 3.3. *Let ε be a unit in $\mathbb{B}_n$ with $\mathrm{Nr}_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\varepsilon) = 1$ and put $N = 3^n$. Then*

$$M(\varepsilon) \geq \left(\frac{3^{(N-1)/2N} + \sqrt{3^{(N-1)/N} + 4}}{2}\right)^{N/2}.$$

In particular, if $n \geq 4$, then

$$M(\varepsilon) \geq \left(\frac{3^{40/81} + \sqrt{3^{80/81} + 4}}{2}\right)^{N/2}.$$

4. Upper bound of Mahler measure of the Horie unit. We put $N = 3^n$ and $\Theta = \pi/6N$. Note the nth Horie unit can be written in terms of real trigonometric functions as follows:

$$\eta_n = \frac{\zeta_{n+1} - \zeta_{n+1}^{-1}}{\zeta_1\zeta_{n+1} - \zeta_1^{-1}\zeta_{n+1}^{-1}} = \frac{\sin(4\Theta)}{\sin(4(1+N)\Theta)}.$$

Let σ be a generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_{n+1})/\mathbb{Q}(\zeta_1))$ with $\zeta_{n+1}^{\sigma} = \zeta_{n+1}^{4}$. We have

$$\begin{aligned} M(\eta_n) &= \prod_{i=0}^{N-1} \max\{1, |\eta_n^{\sigma^i}|\} = \prod_{\substack{0\le j<3N \\ j\equiv 1 \ (\mathrm{mod}\ 3)}} \max\left\{1, \left|\frac{\sin(4j\Theta)}{\sin(4(j+N)\Theta)}\right|\right\} \\ &= \prod_{\substack{0\le j<3N \\ j\equiv 1 \ (\mathrm{mod}\ 3)}} \max\left\{1, \left|\frac{\sin(4((2N-j)+N)\Theta)}{\sin(4(2N-j)\Theta)}\right|\right\} \\ &= \prod_{\substack{0\le j<3N \\ j\equiv -1 \ (\mathrm{mod}\ 3)}} \max\left\{1, \left|\frac{\sin(4(j+N)\Theta)}{\sin(4j\Theta)}\right|\right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} M(\eta_n) = M(\eta_n^{-1}) &= \prod_{i=0}^{N-1} \max\{1, |(\eta_n^{-1})^{\sigma^i}|\} \\ &= \prod_{\substack{0\le j<3N \\ j\equiv 1 \ (\mathrm{mod}\ 3)}} \max\left\{1, \left|\frac{\sin(4(j+N)\Theta)}{\sin(4j\Theta)}\right|\right\}. \end{aligned}$$

Hence

$$\begin{aligned} M(\eta_n)^2 &= \prod_{0\le j<3N}^{*} \max\left\{1, \left|\frac{\sin(4(j+N)\Theta)}{\sin(4j\Theta)}\right|\right\} \\ &= \prod_{0\le j<N}^{*} \prod_{0\le i<3} \max\left\{1, \left|\frac{\sin(4(j+(i+1)N)\Theta)}{\sin(4(j+iN)\Theta)}\right|\right\}, \end{aligned}$$

where $\prod^{*}$ denotes the product over indices coprime with 3. Write the set $\{|\sin(4(j+iN)\Theta)| \mid i = 0,1,2\}$ as $\{s_0, s_1, s_2\}$ with $s_0 < s_1, s_2$. Then

$$\begin{aligned} \prod_{0\le i<3} \max\left\{1, \left|\frac{\sin(4(j+(i+1)N)\Theta)}{\sin(4(j+iN)\Theta)}\right|\right\} &= \begin{cases} s_2/s_0 & \text{if } s_1 < s_2, \\ s_1/s_0 & \text{if } s_1 > s_2, \end{cases} \\ &= \frac{\max\{s_0, s_1, s_2\}}{\min\{s_0, s_1, s_2\}}. \end{aligned}$$

The maximum on the right hand side can be found by considering the inequality

$$|\sin(4j\Theta + 4Ni\Theta)| \ge |\sin(4j\Theta + 4N(i+1)\Theta)|, |\sin(4j\Theta + 4N(i-1)\Theta)|.$$

Hence, the maximum is attained at i with

$$4j\Theta + 4Ni\Theta \in (\pi/3, 2\pi/3) \cup (4\pi/3, 5\pi/3),$$

or equivalently

$$j + iN \in (N/2, N) \cup (2N, 5N/2).$$

Similarly, the minimum is attained at i with

$$4j\Theta + 4Ni\Theta \in (-\pi/6, \pi/6) \cup (5\pi/6, 7\pi/6),$$

or equivalently

$$j + iN \in (-N/4, N/4) \cup (5N/4, 7N/4).$$

Therefore,

$$M(\eta_n)^2 = \frac{\prod^*_{N/2<j<N} |\sin(4j\Theta)| \cdot \prod^*_{2N<j<5N/2} |\sin(4j\Theta)|}{\prod^*_{-N/4<j<N/4} |\sin(4j\Theta)| \cdot \prod^*_{5N/4<j<7N/4} |\sin(4j\Theta)|}.$$

Thus, we get

$$\begin{aligned}
M(\eta_n) &= \frac{\prod^*_{N/2<j<N} \sin(4j\Theta)}{\prod^*_{0<j<N/4} \sin(4j\Theta) \cdot \prod^*_{5N/4<j<3N/2} \sin(4j\Theta)} \\
&= \frac{\prod^*_{N/2<j<N} \cos((4j-3N)\Theta)}{\prod^*_{0<j<N/4} \sin(4j\Theta) \cdot \prod^*_{5N/4<j<3N/2} \sin((6N-4j)\Theta)} \\
&= \frac{\prod^*_{-N<4j-3N<N} \cos((4j-3N)\Theta)}{\prod^*_{0<j<N/4} \sin(4j\Theta) \cdot \prod^*_{0<3N-2j<N/2} \sin((6N-4j)\Theta)} \\
&= \frac{\prod^*_{0<3N-4j<N} \cos((3N-4j)\Theta) \cdot \prod^*_{0<4j-3N<N} \cos((4j-3N)\Theta)}{\prod^*_{0<4j<N} \sin(4j\Theta) \cdot \prod^*_{0<6N-4j<N} \sin((6N-4j)\Theta)}.
\end{aligned}$$

Noting that the ranges of the products are

$$\begin{aligned}
&\{k \in \mathbb{Z} \mid k \equiv +3N \pmod 4,\ k \not\equiv 0 \pmod 3\} \cap (0, \pi/3), \\
&\{k \in \mathbb{Z} \mid k \equiv -3N \pmod 4,\ k \not\equiv 0 \pmod 3\} \cap (0, \pi/3), \\
&\{k \in \mathbb{Z} \mid k \equiv 0 \pmod 4,\ k \not\equiv 0 \pmod 3\} \cap (0, \pi/3), \\
&\{k \in \mathbb{Z} \mid k \equiv 2 \pmod 4,\ k \not\equiv 0 \pmod 3\} \cap (0, \pi/3),
\end{aligned}$$

we get

$$M(\eta_n) = \frac{\prod^*_{0<k<N,\, 2\nmid k} \cos(k\Theta)}{\prod^*_{0<k<N,\, 2\mid k} \sin(k\Theta)}.$$

Thus we have

(4.1)
$$M(\eta_n) = \frac{\cos((N-2)\Theta)}{\sin((N-1)\Theta)} \cdot \prod_{0<3K<N,\, 2\nmid K} \frac{\cos((3K-2)\Theta) \cdot \cos((3K+2)\Theta)}{\sin((3K-1)\Theta) \cdot \sin((3K+1)\Theta)}.$$

For $(t, v) \in \mathbb{R}^2$ such that $0 < t - v \le t \le t + v < \pi/4$, we have

$$\frac{\partial}{\partial v} \log \frac{\cos(t-v)}{\sin(t+v)} = \tan(t-v) - \cot(t+v) < 0.$$

Thus

$$\cot\frac{(2N-3)\Theta}{2} > \frac{\cos((N-2)\Theta)}{\sin((N-1)\Theta)}.$$

For $(t,u,v)\in\mathbb{R}^3$ such that $0 < t-u-v \le t-u+v \le t \le t+u-v \le t+u+v < \pi/4$, put

$$g(t,u,v) = \log\frac{\cos(t-u-v)\cos(t+u+v)}{\sin(t-u+v)\sin(t+u-v)}.$$

Then

$$\frac{\partial}{\partial v}g(t,u,v) = \tan(t-u-v) - \tan(t+u+v) - \cot(t-u+v) + \cot(t+u-v).$$

Since

$$\tan(t-u-v) \le \tan t \le \tan(t+u+v), \quad \cot(t-u+v) \ge \cot t \ge \cot(t+u-v),$$

it follows that

$$\frac{\partial}{\partial v}g(t,u,v) \le 0.$$

Hence

$$g(t,u,v) \le g(t,u,0).$$

Therefore, the factors in the product of (4.1) are estimated as follows:

$$\frac{\cos((3K-2)\Theta)\cdot\cos((3K+2)\Theta)}{\sin((3K-1)\Theta)\cdot\sin((3K+1)\Theta)} \le \cot\frac{(6K-3)\Theta}{2}\cdot\cot\frac{(6K+3)\Theta}{2}.$$

Summing up, we get

$$\begin{aligned} M(\eta_n) &\le \cot\frac{(2N-3)\Theta}{2}\cdot\prod_{0<3K<N,\,2\nmid K}\left(\cot\frac{(6K-3)\Theta}{2}\cdot\cot\frac{(6K+3)\Theta}{2}\right) \\ &= \prod_{0<J<2N/3,\,2\nmid J}\cot\frac{J\pi}{4N}. \end{aligned}$$

Since

$$\frac{d^2}{dt^2}\log\cot t = -\frac{d}{dt}\frac{1}{\sin t\cos t} = -\frac{d}{dt}\frac{2}{\sin 2t} = \frac{4\cos t}{(\sin 2t)^2} > 0$$

for $0 < t < \pi/4$, we have

$$\sum_{0<J<2N/3,\,2\nmid J}\log\cot\frac{J\pi}{4N} \le \frac{2N}{\pi}\int_0^{\pi/6}\log\cot t\,dt.$$

Recall the Lobachevskiĭ function ([GR], [Lo])

$$-\int_0^{\theta}\log\cos t\,dt = \theta\log 2 + \sum_{m=1}^{\infty}\frac{(-1)^m}{2m^2}\sin 2m\theta$$

and its companion function

$$-\int_0^\theta \log\sin t\,dt = \theta\log 2 + \sum_{m=1}^{\infty}\frac{1}{2m^2}\sin 2m\theta$$

for $0 \le \theta < \pi/2$. Subtracting, we get

$$\int_0^\theta \log\cot t\,dt = \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\sin(2(2m+1)\theta).$$

Substituting $\theta = \pi/6$ yields

$$\begin{aligned}\int_0^{\pi/6} \log\cot t\,dt &= \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\sin\frac{(2m+1)\pi}{3}\\ &= \frac{\sqrt{3}}{2}\biggl(1-\sum_{m=1}^{\infty}\biggl(\frac{1}{(6m-1)^2}-\frac{1}{(6m+1)^2}\biggr)\biggr).\end{aligned}$$

Since $\frac{1}{(6m-1)^2} - \frac{1}{(6m+1)^2} > 0$, the right hand side above is smaller than

$$\frac{\sqrt{3}}{2}\biggl(1-\sum_{m=1}^{1000}\biggl(\frac{1}{(6m-1)^2}-\frac{1}{(6m+1)^2}\biggr)\biggr) < 0.845785.$$

As $\frac{2}{\pi}0.845785 < 0.53845$, we deduce

Lemma 4.1. *Let* $N = 3^n$. *Then*

$$M(\eta_n) \le \exp(0.53845\cdot N).$$

5. Minkowski's Convex Body Theorem. Let $\ell \ge 5$ be a prime number, n a positive integer and 3^s the exact power of 3 dividing ℓ^2-1. We put $r = \min\{n,s\}$ and $c = 2\cdot 3^{r-1}$. In this section, we consider the mapping

$$\mu : \mathbb{Q}(\zeta_r) \to \mathbb{C}^c, \qquad \alpha \mapsto \overrightarrow{\alpha} := (\alpha^\rho)_{\rho\in\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})},$$

and the $\mathbb{R}$-vector space

$$V = \mathbb{R}\overrightarrow{1} + \mathbb{R}\overrightarrow{\zeta_r} + \cdots + \mathbb{R}\overrightarrow{\zeta_r^{c-1}} \cong \mathbb{R}^c. \tag{5.1}$$

We put

$$X = \biggl\{\sum_{i=0}^{c-1} a_i\overrightarrow{\zeta_r^i} \in V \biggm| |a_0| + |a_1| + \cdots + |a_{c-1}| \le \frac{\ell}{\sqrt{2}}\biggr\}$$

and define $|\cdot|_1$ on $\mathbb{Z}[\zeta_r]$ by

$$|a_0 + a_1\zeta^r + \cdots + a_{c-1}\zeta_r^{c-1}|_1 = |a_0| + |a_1| + \cdots + |a_{c-1}|.$$

We consider the volume $\mathrm{vol}(\cdot)$ on V induced by the standard volume on $\mathbb{R}^c$ via (5.1). For an ideal $\mathfrak{a}$ of $\mathbb{Q}(\zeta_r)$, we also denote by $\mathrm{vol}(\mathfrak{a})$ the volume of the fundamental domain of the lattice $\mu(\mathfrak{a})$. Then

$$\mathrm{vol}(X) = \frac{(\sqrt{2}\ell)^c}{c!} \quad \text{and} \quad \mathrm{vol}(\ell\mathfrak{L}^{-1}) = \ell^{c-f}$$

where $\mathfrak{L}$ is a prime ideal of $\mathbb{Q}(\zeta_r)$ dividing ℓ, and f is the inertia degree of $\mathfrak{L}$ in $\mathbb{Q}(\zeta_r)/\mathbb{Q}$. Now we apply the Minkowski Convex Body Theorem to get:

LEMMA 5.1. *Let ℓ, n, s, r, c and X be as above and $\mathfrak{L}$ a prime ideal of $\mathbb{Q}(\zeta_r)$ dividing ℓ. Denote by f the inertia degree of $\mathfrak{L}$ in $\mathbb{Q}(\zeta_r)/\mathbb{Q}$. If $\ell^f > 2^{c/2} \cdot c!$, then there exists a non-zero α in $X \cap \mu(\ell\mathfrak{L}^{-1})$. Therefore, if $\ell^f > 2^{c/2} \cdot c!$, then there exists a non-zero α in $\ell\mathfrak{L}^{-1}$ such that $|\alpha|_1 \le \ell/\sqrt{2}$.*

Proof. Since $\ell^f > 2^{c/2} \cdot c!$, we have $\mathrm{vol}(X) > 2^c \mathrm{vol}(\ell\mathfrak{L}^{-1})$. ■

6. Proof of Theorem B. Let $\ell \ge 5$ be a prime number, 3^s the exact power of 3 dividing $\ell^2 - 1$, and n a positive integer. By Theorem 1.1, we may assume $n \ge 4$. We put $N = 3^n$, $r = \min\{n, s\}$ and $c = 2 \cdot 3^{r-1}$. We denote by f the inertia degree of ℓ in $\mathbb{Q}(\zeta_r)/\mathbb{Q}$. Assume that $\ell^f > 2^{c/2} \cdot c!$ and ℓ divides h_n/h_{n-1}. By Lemmas 2.1 and 5.1, there exist α in $\ell\mathfrak{L}^{-1}$ and a unit ε in $\mathbb{B}_n$ such that

$$\eta_n^{\alpha\sigma} = \varepsilon^\ell \tag{6.1}$$

and

$$|\alpha|_1 < \frac{\ell}{\sqrt{2}}, \tag{6.2}$$

where $\mathfrak{L}$ is a prime ideal in $\mathbb{Q}(\zeta_r)$ dividing ℓ. Since $\mathrm{Nr}_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\eta_n) = 1$, we have $\mathrm{Nr}_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\varepsilon) = 1$. By Lemma 3.3,

$$M(\varepsilon) \ge \left(\frac{3^{40/81} + \sqrt{3^{80/81} + 4}}{2}\right)^{N/2}. \tag{6.3}$$

By taking the logarithm, we have

$$\log\left(\frac{3^{40/81} + \sqrt{3^{80/81} + 4}}{2}\right) > \log\left(\frac{\sqrt{3} + \sqrt{7}}{2}\right) - \frac{1}{162}\log 3 > 0.77661.$$

Hence

$$M(\varepsilon) > \exp(0.77661 \cdot N/2). \tag{6.4}$$

Since $\deg \varepsilon^\ell = \deg \varepsilon$ and $\deg \eta_n^{\alpha\sigma} \le \deg \eta_n$, we have

$$M(\varepsilon^\ell) = M(\varepsilon)^\ell \tag{6.5}$$

and

$$M(\eta_n^{\alpha\sigma}) \le M(\eta_n)^{|\alpha|_1}. \tag{6.6}$$

By (6.1)–(6.6) and Lemma 4.1, we obtain

$$\exp(0.77661 \cdot \ell \cdot N/2) < M(\varepsilon^{\ell}) = M(\eta_n^{\alpha_\sigma}) \leq M(\eta_n)^{|\alpha|_1} \leq \exp(0.53845 \cdot N \cdot \ell/\sqrt{2}).$$

Hence $0.77661 \leq 0.53845 \cdot \sqrt{2} = 0.761483\ldots$, a contradiction.

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