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Products of special values of modular L-functions and their applications

by

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1. Introduction. Let $f(z)=\sum_{n=1}^{\infty} a_f(n)q^n$ be a cusp form of weight $2k$ for $\Gamma_0(N)$ with trivial character. We denote by $L(f,s)=\sum_{n=1}^{\infty} a_f(n)/n^s$ the L-function of f. For a square-free integer D, let $L\big(f,\big(\frac{D}{\cdot}\big),s\big)$ be the *D-quadratic twist of* $L(f,s)$ given by

$$L\left(f,\left(\frac{D}{\cdot}\right),s\right)=\sum_{n=1}^{\infty}\frac{a_f(n)\big(\frac{D}{n}\big)}{n^s}.$$

Recently there have been a number of investigations regarding the distribution of analytic ranks of the families of quadratic twists of L-functions (see [7], [16], [17]). Goldfeld [8] conjectured that for newforms f of weight 2,

$$\#\left\{-X\le m\le X\;\middle|\; m \text{ is square-free and } L\left(f,\left(\frac{m}{\cdot}\right),1\right)\neq 0\right\}\sim X/2.$$

Given an elliptic curve $E: y^2=x^3+ax+b$ $(a,b\in\mathbb{Z})$ and a square-free integer D, we define D-quadratic twist of E to be the curve $E_D: y^2=x^3+aD^2x+bD^3$. A weaker version of Goldfeld's conjecture, which is still unproved, is

$$\#\{-X\le m\le X \mid m \text{ is square-free and } \operatorname{rank} E_m(\mathbb{Q})=0\}\gg X.$$

Heath-Brown [9] confirmed this conjecture for the congruent number elliptic curve. Moreover, this assertion has been proved for a variety of elliptic curves with rational torsion points of order 3 by the works of James, Vatsal and Wong [10], [22], [25]. For general elliptic curves over $\mathbb{Q}$, Ono and Skinner [17] proved that

$$\#\{-X\le m\le X \mid m \text{ is square-free and } \operatorname{rank} E_m(\mathbb{Q})=0\}\gg X/\log X.$$

Also, it is conjectured that there are infinitely many primes p for which $\operatorname{rank} E_{-p}(\mathbb{Q})=0$. Ono [15] confirmed this conjecture for some special elliptic

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curves. Ono and Skinner [17] checked that if $E/\mathbb{Q}$ is an elliptic curve with conductor $N \le 100$, then either $\operatorname{rank} E_{-p}(\mathbb{Q}) = 0$ or $\operatorname{rank} E_p(\mathbb{Q}) = 0$ for infinitely many primes p.

In this paper we examine the following question.

QUESTION 1.1. Let r be a positive integer. If f_i are newforms of weight $2k_i$ for $\Gamma_0(N_i)$ with trivial character for each $i = 1, \dots, r$, then are there infinitely many square-free integers m such that

$$L\biggl(f_1, \biggl(\frac{(-1)^{k_1} m}{\cdot}\biggr), k_1\biggr) \cdots L\biggl(f_r, \biggl(\frac{(-1)^{k_r} m}{\cdot}\biggr), k_r\biggr) \ne 0?$$

Using the idea of Ono–Skinner's result ([17, Fundamental Lemma]), we have some non-vanishing lemmas on Fourier coefficients of half-integral weight modular forms and their applications to special values of L-functions of modular forms. Let $S_k(N, \chi)$ be the space of cusp forms of weight k for $\Gamma_0(N)$ with character χ, and let $S_k^{\mathrm{new}}(N, \chi)$ be the set of newforms of weight k for $\Gamma_0(N)$ with character χ. Our main result is the following theorem.

THEOREM 1.2. *Let $g_i(z) = \sum_{n=1}^{\infty} b_i(n) q^n \in S_{k_i+1/2}(M_i, 1)$ be eigenforms of the Hecke operators T_{p^2} for all $p \nmid M_i$ such that the image of $g_i(z)$ under Shimura correspondence is $f_i(z) = \sum_{n=1}^{\infty} a_i(n) q^n \in S_{2k_i}^{\mathrm{new}}(N_i, 1)$. Suppose that the coefficients $b_i(m)$ are algebraic integers contained in a number field K. Let v be a place of K over 2 and put*

$$s_i = \min\{\operatorname{ord}_v(b_i(m)) \mid m > 1 \text{ is square-free and } (m, M_1 \cdots M_r) = 1\}.$$

If $s_i < \infty$ $(i = 1, \dots, r)$ and there exists a square-free integer m_0 with exactly l prime factors $(m_0 = p_1 \cdots p_l)$ such that $(m_0, M_1 \cdots M_r) = 1$ and for $i = 1, \dots, r$,

$$\operatorname{ord}_v(b_i(m_0)) = s_i, \qquad L\biggl(f_i, \biggl(\frac{(-1)^{k_i} m_0}{\cdot}\biggr), k_i\biggr) \ne 0,$$

then there are infinitely many square-free integers m with exactly l prime factors for which

$$\prod_{i=1}^{r} L\biggl(f_i, \biggl(\frac{(-1)^{k_i} m}{\cdot}\biggr), k_i\biggr) \ne 0.$$

Moreover,

$$\#\biggl\{0 < m \le X \;\bigg|\; m \text{ square-free}, \prod_{i=1}^{r} L\biggl(f_i, \biggl(\frac{(-1)^{k_i} m}{\cdot}\biggr), k_i\biggr) \ne 0\biggr\} \gg \frac{X}{\log X}.$$

2. Shimura correspondence. We briefly review the theory of Shimura correspondence. For a positive integer N divisible by 4, let $S_{k+1/2}(N, \chi)$ be the space of cusp forms of half-integral weight $k+1/2$ for $\Gamma_0(N)$ with character χ. It is known that there is a close connection between Fourier coefficients of half-integral weight modular forms and critical values of twisted modular L-functions. Let

$$g(z)=\sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N,\chi),$$

where $q=e^{2\pi iz}$. Let t be a positive square-free integer. Now define $A_t(n)$ by the formal product of Dirichlet series

$$\sum_{n=1}^{\infty}\frac{A_t(n)}{n^s}=\Bigg(\sum_{n=1}^{\infty}\frac{\chi(n)\big(\frac{-1}{n}\big)^k\big(\frac{t}{n}\big)}{n^{s-k+1}}\Bigg)\Bigg(\sum_{n=1}^{\infty}\frac{b(tn^2)}{n^s}\Bigg).$$

Then Shimura [18] proved that there is a positive integer M such that

$$\mathrm{SH}_t(g(z))=f_t(z)=\sum_{n=1}^{\infty}A_t(n)q^n \in M_{2k}(M,\chi^2),$$

where $M_{2k}(M,\chi)$ is the space of cusp forms of weight $2k$ for $\Gamma_0(M)$ with character χ. In fact, we can take $M=N/2$. If $f(z)=\sum_{n=1}^{\infty}a(n)q^n$ is a modular form, then its L-function $L(f,s)$ is

$$L(f,s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}.$$

Furthermore, for a Dirichlet character ψ we put

$$L(f,\psi,s)=\sum_{n=1}^{\infty}\frac{a(n)\psi(n)}{n^s}.$$

In [23] Waldspurger proved formulae connecting Fourier coefficients of half-integral weight modular forms and critical values of twisted modular L-functions. The following theorem is a special case of his results.

THEOREM 2.1 ([23, Corollary 2]). *Let*

$$g(z)=\sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}\bigg(N,\Big(\frac{d}{\cdot}\Big)\bigg)$$

be an eigenform of the Hecke operators T_{p^2} for all $p\nmid M$ such that $\mathrm{SH}_1(g(z)) = f(z)=\sum_{n=1}^{\infty}a(n)q^n\in S_{2k}^{\mathrm{new}}(M,1)$ for an appropriate positive integer M. Let n_1 and n_2 be two positive square-free integers such that $n_1/n_2\in\mathbb{Q}_p^{\times 2}$ for all $p\mid N$. Then

$$b(n_1)^2L\bigg(f,\Big(\frac{(-1)^kdn_2}{\cdot}\Big),k\bigg)n_2^{k-1/2}=b(n_2)^2L\bigg(f,\Big(\frac{(-1)^kdn_1}{\cdot}\Big),k\bigg)n_1^{k-1/2}.$$

Now we define some notation that is used in the next theorem. Let E be an elliptic curve over $\mathbb{Q}$ with conductor N, and let

$$L(E,s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

be the Hasse–Weil L-function of E. Then, by the works of Wiles, Wiles and Taylor, Diamond, Conrad, Diamond and Taylor, and Breuil, Conrad, Diamond and Taylor, [24], [20], [6], [3], [2], it is now known that $L(E,s)$ is the Mellin transform of a weight 2 newform $f(z)=\sum_{n=1}^{\infty}a(n)q^n \in S_2^{\mathrm{new}}(N,1)$. Therefore by Kolyvagin's result [13] if $L(E,1)\neq 0$, then $\operatorname{rank} E(\mathbb{Q})=0$. Now suppose that for some positive integer M there exists a cusp form $g(z)=\sum_{n=1}^{\infty}b(n)q^n \in S_{3/2}\big(M,\big(\frac{d}{\cdot}\big)\big)$ that is an eigenform of the Hecke operators T_{p^2} for all $p\nmid M$ such that the image of $g(z)$ under Shimura correspondence is $f(z)$.

Remark 2.2 (cf. [12]). It is known that if the conductor of E is a square-free odd integer, then there exists a weight $3/2$ eigenform for which its image under Shimura correspondence is $f(z)$.

By using Waldspurger's result, K. Ono proved the following theorem.

Theorem 2.3 ([14, Theorem 2]). *The notation being as above, let n_1 be a positive square-free integer such that $b(n_1)\neq 0$ and $L(E_{-dn_1},1)\neq 0$. Suppose that n_2 is a positive square-free integer such that $n_1/n_2\in\mathbb{Q}_p^{\times 2}$ for all $p\,|\,N$. If $b(n_2)\neq 0$, then* $\operatorname{rank} E_{-dn_2}(\mathbb{Q})=0$.

Remark 2.4. Let E be an elliptic curve over $\mathbb{Q}$, and let

$$L(E,s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

be the Hasse–Weil L-function of E. Then the Hasse–Weil L-function of E_D is

$$L(E_D,s)=\sum_{n=1}^{\infty}\frac{a(n)\big(\frac{D}{n}\big)}{n^s}.$$

3. Fourier coefficients of half-integral weight modular forms. In this section we prove some non-vanishing lemmas for the Fourier coefficients of half-integral weight modular forms. From now on, we assume that the Fourier coefficients of half-integral weight modular forms are algebraic integers contained in a number field K. Let v be a place of K over 2 and let e be the ramification index of v over 2.

Definition 3.1. Let r be a positive integer. Then we put

$$\begin{aligned} P_r &= \{D\in\mathbb{Z}\mid D \text{ is square-free with exactly } r \text{ prime factors}\},\\ P(X) &= \{p\mid p \text{ is prime and } p\leq X\},\\ P_r(X) &= \{D\in P_r\mid |D|\leq X\}. \end{aligned}$$

The following lemma is an application of Ono–Skinner's result [17].

LEMMA 3.2. *For $i = 1, \dots, r$, let*

$$g_i(z) = \sum_{n=1}^{\infty} b_i(n) q^n \in S_{k_i+1/2}(\Gamma_1(N_i))$$

be non-zero half-integral weight modular forms. Put

$$N = N_1 \cdots N_r, \quad s_i = \min\{\mathrm{ord}_v(b_i(m)) \mid m \text{ is positive integer}\}.$$

Assume that there exists a square-free integer m_0 with $m_0 = p_1 \cdots p_l \in P_l$ such that $s_i = \mathrm{ord}_v(b_i(m_0))$ and $(m_0, N) = 1$. Then there is a finite Galois extension k of $\mathbb{Q}$ having the following property: If a square-free integer m with $m = q_1 \cdots q_l \in P_l$ satisfies $\mathrm{Frob}_{q_j}(k/\mathbb{Q}) = \mathrm{Frob}_{p_j}(k/\mathbb{Q})$ *for $j = 1, \dots, l$, then* $\mathrm{ord}_v(b_i(m)) = s_i$ *for all i.*

Proof. The proof of this lemma is similar to the proof of the Fundamental Lemma in [17]. Let

$$G_i(z) = \sum_{n=1}^{\infty} c_i(n) q^n = g_i(z)\Big(1 + 2\sum_{n=1}^{\infty} q^{n^2}\Big).$$

Then G_i is in $S_{k_i+1}(\Gamma_1(N_i))$. Since

$$c_i(n) = b_i(n) + 2 \sum_{m+y^2=n,\, y>0} b_i(m),$$

it follows that $\mathrm{ord}_v(c_i(m_0)) = \mathrm{ord}_v(b_i(m_0))$. By the theory of newforms, $G_i(z)$ can be uniquely expressed as a linear combination

$$G_i(z) = \sum_{u=1}^{a} \alpha_{i,u} f_{i,u}(z) + \sum_{v=1}^{b} \beta_{i,v} h_v(l_{i,v} z),$$

where $f_{i,u}(z) = \sum_{n=1}^{\infty} a_{i,u}(n) q^n$ and $h_{i,v}(z)$ are newforms of weight $k_i + 1$ and level a divisor of N_i, and where each $l_{i,v}$ is a non-trivial divisor of N_i. Therefore, if $(n, N_i) = 1$, then

$$c_i(n) = \sum_{u=1}^{a} \alpha_{i,u} a_{i,u}(n).$$

Let L be a finite extension of $\mathbb{Q}$ containing K, the Fourier coefficients of each $f_{i,u}$, and the $\alpha_{i,u}$'s. Let w be a place of L over v, let e be the ramification index of w over v, let O_w be the completion of the ring of integers of L at the place w, and let λ be a uniformizer. Moreover, put

$$E_i = \max_{1 \le u \le a} |\mathrm{ord}_w(\alpha_{i,u})|.$$

Then, by the theory of Galois representations (cf. [5]) there are representations

$$\varrho_{i,u} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(O_w)$$

such that trace $\varrho_{i,u}(\mathrm{Frob}_p) = a_{i,u}(p)$. Now, let

$$\varrho_i = \bigoplus_{u=1}^{a} \varrho_{i,u} \bmod \lambda^{E_i+es_i+1}.$$

Since its image is finite, there is a finite Galois extension k_i of $\mathbb{Q}$ such that the restriction of ϱ_i to k_i is an isomorphism. Therefore, for a prime q_t with $\mathrm{Frob}_{q_t}(k_i/\mathbb{Q}) = \mathrm{Frob}_{p_t}(k_i/\mathbb{Q})$, we see that $a_{i,u}(q_t) \equiv a_{i,u}(p_t) \bmod \lambda^{E_i+es_i+1}$ for $u = 1, \ldots, a$. By the multiplicativity of the Fourier coefficients of newforms, we have $a_{i,u}(m) \equiv a_{i,u}(m_0) \bmod \lambda^{E_i+es_i+1}$, where $m = q_1 \cdots q_l$. It follows that $c_i(m) \equiv c_i(m_0) \bmod \lambda^{es_i+1}$, so $\mathrm{ord}_w(c_i(m)) = \mathrm{ord}_w(c_i(m_0)) = es_i$. Hence, we find that $\mathrm{ord}_v(b_i(m)) = \mathrm{ord}_v(b_i(m_0)) = s_i$. Since

$$\mathrm{Frob}_p(k_1 \cdots k_r/\mathbb{Q})|_{k_i} = \mathrm{Frob}_p(k_i/\mathbb{Q}),$$

this lemma follows by putting $k = k_1 \cdots k_r$. ■

The following lemma is easy.

LEMMA 3.3. *Let $L_1, \ldots, L_r$ be finite Galois extensions of $\mathbb{Q}$. Let A_i be a conjugate class in* $\mathrm{Gal}(L_i/\mathbb{Q})$. *Assume that there exists a prime p_0 such that* $\mathrm{Frob}_{p_0}(L_i/\mathbb{Q}) = A_i$ *for $i = 1, \ldots, r$. Then there are infinitely many primes p for which* $\mathrm{Frob}_p(L_i/\mathbb{Q}) = A_i$ *for $i = 1, \ldots, r$. Moreover,*

$$\#\{p \in P(X) \mid \mathrm{Frob}_p(L_i/\mathbb{Q}) = A_i \text{ for } i = 1, \ldots, r\} \gg X/\log X.$$

Proof. Put $L = L_1 \cdots L_r$. Since $\mathrm{Frob}_p(L/\mathbb{Q})|_{L_i} = \mathrm{Frob}_p(L_i/\mathbb{Q})$, we infer that if $\mathrm{Frob}_p(L/\mathbb{Q}) = \mathrm{Frob}_{p_0}(L/\mathbb{Q})$, then $\mathrm{Frob}_p(L_i/\mathbb{Q}) = A_i$. By the Chebotarev Density Theorem, the set of primes for which $\mathrm{Frob}_p(L_i/\mathbb{Q}) = A_i$ for $i = 1, \ldots, r$ has positive density. ■

LEMMA 3.4. *Let r, a, t be positive integers and for $i = 1, \ldots, r$ let*

$$g_i(z) = \sum_{n=1}^{\infty} b_i(n) q^n \in S_{k_i+1/2}(N_i, \chi_i)$$

be eigenforms. Let

$$s_i = \min\{\mathrm{ord}_v(b_i(n)) \mid n \text{ is positive integer}\}.$$

Assume that there exists a square-free integer m_0 with $m_0 = p_1 \cdots p_l$ such that $m_0 \equiv a \bmod t$, $(m_0, N_1 \cdots N_r) = 1$, and $\mathrm{ord}_v(b_i(m_0)) = s_i$ for all i. Then there are infinitely many square-free integers $m = q_1 \cdots q_l$ for which $\mathrm{ord}_v(b_i(m)) = s_i$ $(i = 1, \ldots, r)$ *and $m \equiv a \bmod t$. Moreover,*

$$\#\{m \in P_l(X) \mid \mathrm{ord}_v(b_i(m)) = s_i \ (\forall i),\ m \equiv a \bmod t\} \gg \frac{X(\log\log X)^{l-1}}{\log X}.$$

Proof. We see that the g_i satisfy the condition of Lemma 3.2. By applying Lemma 3.2 for $g_i(z)$, we find that there is a finite Galois extension k_1 of $\mathbb{Q}$ having the following property: if a square-free integer m with

$m = q_1 \cdots q_l$ satisfies $\mathrm{Frob}_{q_j}(k_1/\mathbb{Q}) = \mathrm{Frob}_{p_j}(k_1/\mathbb{Q})$ for $j = 1, \ldots, l$, then $\mathrm{ord}_v(b_i(m)) = s_i$ for all i. Let $k_2 = \mathbb{Q}(\zeta_t)$, where $\zeta_t = \exp(2\pi i/t)$. If $\mathrm{Frob}_{q_j}(k_2/\mathbb{Q}) = \mathrm{Frob}_{p_j}(k_2/\mathbb{Q})$ for $j = 1, \ldots, l$, then we see that $m \equiv m_0 \bmod t$. Therefore by Lemma 3.3 there are infinitely many primes q_j for which $\mathrm{Frob}_{q_j}(k_n/\mathbb{Q}) = \mathrm{Frob}_{p_j}(k_n/\mathbb{Q})$ for $n = 1, 2$ and $j = 1, \ldots, l$. Hence,

$$\#\{m \in P_l(X) \mid \mathrm{ord}_v(b_i(m)) = s_i \ (\forall i),\ m \equiv a \bmod t\} \gg X(\log\log X)^{l-1}/\log X. \ \blacksquare$$

In this lemma, we dealt with the case of

$$s_i = \min\{\mathrm{ord}_v(b_i(n)) \mid n \text{ is a positive integer}\}.$$

The assumption of this lemma is not enough to compute examples in Section 5, but we can extend this lemma to the case of some weaker conditions. For example, by taking twists of cuspforms, we can prove the case of

$$s_i = \min\{\mathrm{ord}_v(b_i(n)) \mid n \text{ is a positive integer and } (n, N_i) = 1\}.$$

Moreover, if g_i are eigenforms, it follows that $\mathrm{ord}_v(b_i(n)) \le \mathrm{ord}_v(b_i(nm^2))$ for a square-free integer n and a positive integer m. Therefore we have

$$\min\{\mathrm{ord}_v(b_i(n)) \mid n \text{ is positive}\} = \min\{\mathrm{ord}_v(b_i(n)) \mid n \text{ is positive square-free}\}.$$

The following lemma is a basic fact about taking twists of cuspforms.

LEMMA 3.5 ([18, Lemma 3.6]). *Let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi),$$

and let ψ be a primitive character modulo r. Let s be the conductor of χ and let M be the least common multiple of N, r^2 and rs. Put

$$g'(z) = \sum_{n=1}^{\infty} \psi(n)b(n)q^n.$$

Then $g'(z)$ belongs to $S_{k+1/2}(M, \psi^2\chi)$.

We shall require the following lemma.

LEMMA 3.6. *Let r, a, t be positive integers and for $i = 1, \ldots, r$ let*

$$g_i(z) = \sum_{n=1}^{\infty} b_i(n)q^n \in S_{k_i+1/2}(N_i, \chi_i)$$

be eigenforms. Let

$$s_i = \min\{\mathrm{ord}_v(b_i(n)) \mid n > 1 \text{ is square-free and } (n, N_1 \cdots N_r) = 1\}.$$

Assume that there exists a square-free integer m_0 with $m_0 = p_1 \cdots p_l$ such that $m_0 \equiv a \bmod t$, $(m_0, N_1 \cdots N_r) = 1$, and $\mathrm{ord}_v(b_i(m_0)) = s_i$ for all i.

Then there are infinitely many square-free integers $m = q_1 \cdots q_l$ *for which* $\mathrm{ord}_v(b_i(m)) = s_i$ $(i = 1, \ldots, r)$ *and* $m \equiv a \bmod t$. *Moreover*,

$$\#\{m \in P_l(X) \mid \mathrm{ord}_v(b_i(m)) = s_i\ (\forall i),\ m \equiv a \bmod t\} \gg \frac{X(\log\log X)^{l-1}}{\log X}.$$

Proof. We choose a prime p satisfying $\left(\frac{m_0}{p}\right) = -1$. For the prime p, let

$$\delta(n) = \begin{cases} 1 & \text{if } (n, N_1 \cdots N_r) = 1,\ \left(\frac{n}{p}\right) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Lemma 3.5, we have

$$g_i'(z) = \sum_{n=1}^{\infty} b_i'(n) q^n = \sum_{n=1}^{\infty} b_i(n)\delta(n) q^n \in S_{k_i+1/2}(\Gamma_1(p^2(N_1 \cdots N_r)^2)).$$

Since $g_i(z)$ are eigenforms, it follows that $\mathrm{ord}_v(b_i(n)) \le \mathrm{ord}_v(b_i(nm^2))$. Therefore

$$\begin{aligned} s_i &= \min\left\{ \mathrm{ord}_v(b_i(n)) \,\middle|\, 0 < n \in \mathbb{Z},\ (n, N_1 \cdots N_r) = 1 \text{ and } \left(\frac{n}{p}\right) = -1 \right\} \\ &= \min\{\mathrm{ord}_v(b_i'(n)) \mid 0 < n \in \mathbb{Z}\}. \end{aligned}$$

We find that g_i' satisfy the assumption of Lemma 3.4. This completes the proof. ∎

For the applications to elliptic curves given in the next section, we introduce the following lemma.

LEMMA 3.7 ([18, Section 1]). *Let*

$$g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{k+1/2}(N, \chi),$$

and let p be a prime. Then

$$f(z) = \sum_{n=1}^{\infty} b(pn) q^n \in S_{k+1/2}\left(pN, \left(\frac{4p}{\cdot}\right)\chi\right).$$

COROLLARY 3.8. *Let a, t be positive integers and let*

$$g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{k+1/2}(N, \chi)$$

be an eigenform. Let l be a positive square-free integer such that $(l, M) = 1$ *and put*

$$\begin{aligned} s &= \min\{\mathrm{ord}_v(b(n)) \mid n > 1 \text{ is square-free and } (n, N) = 1\}, \\ s' &= \min\{\mathrm{ord}_v(b(ln)) \mid n > 1 \text{ is square-free and } (n, lN) = 1\}. \end{aligned}$$

Assume that there exists a square-free integer m_0 *with* $m_0 = p_1 \cdots p_r$ *such that* $m_0 \equiv a \bmod t$, $(m_0, lN) = 1$, $\mathrm{ord}_v(b(m_0)) = s$ *and* $\mathrm{ord}_v(b(lm_0)) = s'$. *Then there are infinitely many square-free integers* $m = q_1 \cdots q_r$ *for which* $\mathrm{ord}_v(b(m)) = s$, $\mathrm{ord}_v(b(lm)) = s'$ *and* $m \equiv a \bmod t$. *Moreover*,

$$\#\{m \in P_r(X) \mid \mathrm{ord}_v(b(m)) = s,\ \mathrm{ord}_v(b(lm)) = s',\ m \equiv a \bmod t\} \gg X(\log\log X)^{r-1}/\log X.$$

Proof. Let

$$g'(z) = \sum_{n=1}^{\infty} b'(n) q^n = \sum_{n=1}^{\infty} b(ln) q^n.$$

By Lemma 3.7, we have $g'(z) \in S_{k+1/2}(\Gamma_1(l^2N^2))$. Now, $g'(z)$ is not always an eigenform, but it has the property that $\mathrm{ord}_v(b'(n)) \le \mathrm{ord}_v(b'(nm^2))$. Hence we can prove this assertion in the same way as Lemma 3.6. ∎

Proof of Theorem 1.2. Put $M = N_1 \cdots N_r$ and let $t = 8\prod_{q|M} q$, where the product is over the odd prime divisors of M. If m is square-free integer with $m \equiv m_0 \bmod t$, then $m/m_0 \in \mathbb{Q}_p^{\times 2}$ for all $p \mid M$. By assumption there exists a square-free integer m_0 with $(m_0, M) = 1$ such that $\mathrm{ord}_v(b_i(m_0)) = s_i$ for $i = 1, \ldots, r$. By applying Lemma 3.6 to $g_i(z)$'s, we find that there are infinitely many square-free integers m for which $b_i(m) \neq 0$ for $i = 1, \ldots, r$ and $m \equiv m_0 \bmod t$. Therefore, by Theorem 2.1 we see that $L\big(f_i, \big(\frac{(-1)^{k_i} m}{\cdot}\big), k_i\big) \neq 0$. Hence there are infinitely many square-free integers m for which

$$L\bigg(f_1, \bigg(\frac{(-1)^{k_1} m}{\cdot}\bigg), k_1\bigg) \cdots L\bigg(f_r, \bigg(\frac{(-1)^{k_r} m}{\cdot}\bigg), k_r\bigg) \neq 0. \ ∎$$

4. Applications to elliptic curves. Let E be an elliptic curve over $\mathbb{Q}$ with $L(E, s) = \sum_{n=1}^{\infty} a(n)/n^s$ and let $g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{3/2}\big(M, \big(\frac{d}{\cdot}\big)\big)$ be an eigenform of the Hecke operators T_{p^2} for all $p \nmid M$ such that

$$\mathrm{SH}_1(g(z)) = \sum_{n=1}^{\infty} a(n) q^n.$$

Suppose that the coefficients $b(m)$ are algebraic integers contained in a number field K. Let l be a positive square-free integer such that $(l, M) = 1$ and let v be a place of K over 2. Put

$$s = \min\{\mathrm{ord}_v(b(m)) \mid m > 1 \text{ is square-free and } (m, M) = 1\},$$
$$s' = \min\{\mathrm{ord}_v(b(lm)) \mid m > 1 \text{ is square-free and } (m, lM) = 1\}.$$

THEOREM 4.1. *Let the notation be as above. Assume that* $s < \infty$, $s' < \infty$ *and there exists a prime* p_0 *with* $p_0 \nmid Ml$ *such that* $\mathrm{ord}_v(b(p_0)) = s$,

$\mathrm{ord}_v(b(lp_0)) = s'$, $L(E_{-dp_0}, 1) \neq 0$ *and* $L(E_{-dlp_0}, 1) \neq 0$. *Then there are infinitely many primes* p *for which* $\mathrm{rank}\, E_{-dp}(\mathbb{Q}(\sqrt{l})) = 0$. *Moreover*,

$$\#\{p \in P(X) \mid \mathrm{rank}\, E_{-dp}(\mathbb{Q}(\sqrt{l})) = 0\} \gg X/\log X.$$

Proof. Let $t = 8\prod_{q|M} q$, where the product is over the odd prime divisors of M. If p_1 is a prime with $p_1 \equiv p_0 \bmod t$, then $p_1/p_0 \in \mathbb{Q}_p^{\times 2}$ for all $p \mid M$. By assumption there exists a prime p_0 with $p_0 \nmid M$ such that $\mathrm{ord}_v(b(p_0)) = s$ and $\mathrm{ord}_v(b(lp_0)) = s'$. Hence, by applying Corollary 3.8 to $g(z)$ and $r = 1$, we deduce that there are infinitely many primes p for which $b(p) \neq 0$, $b(lp) \neq 0$ and $p \equiv p_0 \bmod t$. Therefore, by Theorem 2.3 we see that $\mathrm{rank}\, E_{-dp}(\mathbb{Q}) = \mathrm{rank}\, E_{-dlp}(\mathbb{Q}) = 0$. Since $\mathrm{rank}\, E_{-dp}(\mathbb{Q}(\sqrt{l})) = \mathrm{rank}\, E_{-dp}(\mathbb{Q}) + \mathrm{rank}\, E_{-dlp}(\mathbb{Q}) = 0$, there are infinitely many primes p for which $\mathrm{rank}\, E_{-dp}(\mathbb{Q}(\sqrt{l})) = 0$. ∎

Let E be an elliptic curve over $\mathbb{Q}$ and let $k = \mathbb{Q}(\sqrt{l})$ be a quadratic field. Then it is known that the L-function of E over k is given by

$$L(E, k, s) = L(E, s)L(E_l, s).$$

Corollary 4.2. *Let the notation be as above. Put*

$$s_1 = \min\{\mathrm{ord}_v(b(m)) \mid m > 1 \text{ is square-free}, (m, M) = 1\},$$

$$s_2 = \min\{\mathrm{ord}_v(b(lm)) \mid m > 1 \text{ is square-free}, (m, lM) = 1\}.$$

If there exists a prime p_0 *with* $p_0 \nmid lM$ *such that* $\mathrm{ord}_v(b(p_0)) = s < \infty$ *and* $\mathrm{ord}_v(b(lp_0)) = s' < \infty$, *then there are infinitely many primes* p *for which*

$$\mathrm{ord}_v\left(\frac{L(E_{-dp}, \mathbb{Q}(\sqrt{l}), 1)}{\Omega(E_{-dp}, \mathbb{Q}(\sqrt{l}))}\right) = \mathrm{ord}_v\left(\frac{L(E_{-dp_0}, \mathbb{Q}(\sqrt{l}), 1)}{\Omega(E_{-dp_0}, \mathbb{Q}(\sqrt{l}))}\right),$$

where $\Omega(E, k)$ *is the period of* E *over* k.

Proof. This follows from Theorem 2.1 and the proof of Theorem 4.1. ∎

Corollary 4.3. *The notation being as above, assume that* E *has no* $\mathbb{Q}$*-rational* 2*-torsion points and there exists a prime* p_0 *with* $p_0 \nmid Ml$ *such that*

$$\mathrm{ord}_v(b(p_0)) = \mathrm{ord}_v(L(E_{-dp_0}, 1)/\Omega(E_{-dp}, \mathbb{Q})) = 0,$$

$$\mathrm{ord}_v(b(lp_0)) = \mathrm{ord}_v(L(E_{-dlp_0}, 1)/\Omega(E_{-dlp}, \mathbb{Q})) = 0.$$

Then assuming the Birch and Swinnerton-Dyer conjecture there are infinitely many primes p *for which* $\mathrm{rank}\, E_{-dp}(\mathbb{Q}(\sqrt{l})) = 0$ *and* $\#\mathrm{Sel}_2(E_{-dp}, \mathbb{Q}(\sqrt{l})) = 1$, *where* $\mathrm{Sel}_2(E, K)$ *is the* 2*-Selmer group of* E *over* K.

Proof. Let $K = \mathbb{Q}(\sqrt{l})$. The following exact sequence is known:

$$0 \to E(K)/2E(K) \to \mathrm{Sel}_2(E, K) \to Ш_2(E, K) \to 0.$$

By assumption that E is an elliptic curve without $\mathbb{Q}$-rational 2-torsion points, if $E(K)$ is a finite group, then $E(K)/2E(K)$ is trivial. Therefore if

$\#Ш_2(E,K)=1$, then $\#\mathrm{Sel}_2(E,K)=1$. By Corollary 4.2 and the Birch and Swinnerton-Dyer conjecture, there are infinitely many primes p for which

$$\mathrm{ord}_v \#Ш_2(E,K) = \mathrm{ord}_v\left(\frac{L(E_{-dp},\mathbb{Q}(\sqrt{l}),1)}{\Omega(E_{-dp},\mathbb{Q}(\sqrt{l}))}\right)=0.$$

This completes the proof. ∎

In [4], Coogan and Jiménez–Urroz prove the following theorem.

THEOREM 4.4 ([4, Theorem 3]). *Let E^1 and E^2 be two elliptic curves over $\mathbb{Q}$ without $\mathbb{Q}$-rational 2-torsion points. Then there exist fundamental discriminants D_1, D_2 and a set of primes T of positive density such that*

$$\mathrm{rank}\, E^1_{dD_1}(\mathbb{Q}) = \mathrm{rank}\, E^2_{dD_2}(\mathbb{Q}) = 0,$$

where d is any product of an even number of distinct primes in T.

We consider general cases in the result of Coogan and Jiménez–Urroz.

THEOREM 4.5. *For $i=1,\dots,r$, let E^i be an elliptic curve over $\mathbb{Q}$ with $L(E^i,s)=\sum_{n=1}^{\infty}a_i(n)/n^s$ and let $g_i(z)=\sum_{n=1}^{\infty}b_i(n)q^n\in S_{3/2}(M_i,1)$ be an eigenform of the Hecke operators T_{p^2} for all $p\nmid M_i$ such that the image of $g_i(z)$ under Shimura correspondence is $f_i(z)=\sum_{n=1}^{\infty}a_i(n)q^n$. Suppose that the coefficients $b_i(m)$ are algebraic integers contained in a number field K. Let v be a place of K over 2 and put*

$$s_i = \min\{\mathrm{ord}_v(b_i(m)) \mid m>1 \text{ is square-free and } (m, M_1\cdots M_r)=1\}.$$

If $s_i<\infty$ and there exists a square-free integer m_0 with $(m_0,M_1\cdots M_r)=1$ such that for $i=1,\dots,r$,

$$\mathrm{ord}_v(b_i(m_0))=s_i,\qquad L(E^i_{-m_0},1)\neq 0,$$

then there are infinitely many square-free integers m for which $\mathrm{rank}\,E^1_{-m}(\mathbb{Q})=\cdots=\mathrm{rank}\,E^r_{-m}(\mathbb{Q})=0$. *Moreover,*

$$\#\{0<m\le X \mid m \text{ is square-free and } \mathrm{rank}\,E^i_{-m}(\mathbb{Q})=0\ (i=1,\dots,r)\} \gg X/\log X.$$

Proof. This follows from Theorems 1.2 and 2.3. ∎

5. Examples. In this section, we give some examples pertaining to our results.

EXAMPLE 5.1 (cf. [1, Examples 3.6.1]). Let E be the elliptic curve given by

$$E:\ y^2+y=x^3-x^2-10x-20.$$

Then the conductor of E is 11, in fact E is the modular curve $X_0(11)$. In this case the weight 3/2 eigenform

$$g(z) = \frac{1}{2}\Big(\sum_{x,y,z\in\mathbb{Z}} q^{x^2+11y^2+11z^2} - q^{3x^2+2xy+4y^2+11z^2} \Big) = \sum_{n=1}^{\infty} b(n)q^n$$
$$= q - q^3 - q^5 + q^{11} + 2q^{12} - 2q^{14} + q^{15} - \cdots$$

is in $S_{3/2}(44,1)$. Its image $\mathrm{SH}_1(g(z)) = \sum_{n=1}^{\infty} a(n)q^n$ is the weight 2 newform whose Mellin transform is $L(E,s)$. We can find that $\mathrm{ord}_2(b(3)) = 0$, $\mathrm{ord}_2(b(5)) = 0$ and $\mathrm{ord}_2(b(15)) = 0$, and

$$\min\{\mathrm{ord}_v(b(m)) \mid m > 1 \text{ is square-free and } (m,44) = 1\} = 0.$$

Moreover, we can verify that

$$\mathrm{ord}_2(L(E_{-3},1)/\Omega(E_{-3},\mathbb{Q})) = 0, \quad \mathrm{ord}_2(L(E_{-5},1)/\Omega(E_{-5},\mathbb{Q})) = 0$$

and

$$\mathrm{ord}_2(L(E_{-15},1)/\Omega(E_{-15},\mathbb{Q})) = 0.$$

Therefore by Theorem 4.1, for $l = 3,5$ there are infinitely many primes p for which $\mathrm{rank}\, E_{-p}(\mathbb{Q}(\sqrt{l})) = 0$.

Furthermore by Corollary 4.3, assuming the Birch and Swinnerton-Dyer conjecture, for $l = 3,5$ there are infinitely many primes p for which

$$\mathrm{rank}\, E_{-p}(\mathbb{Q}(\sqrt{l})) = 0, \quad \#\mathrm{Sel}_2(E_{-p},\mathbb{Q}(\sqrt{l})) = 1.$$

When $s \geq \mathrm{ord}_v 2$, to determine

$$s = \min\{\mathrm{ord}_v(b(m)) \mid m > 1 \text{ is square-free and } (m,lM) = 1\}$$

we use the following theorem.

Theorem 5.2 ([19, Theorem 1]). *Let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in M_k(N,\chi)$$

be a half-integral or integral weight modular form whose coefficients $b(m)$ *are algebraic integers contained in a number field* K. *Let* v *be a finite place of* K *and*

$$\lambda = \frac{k}{12}\,[\Gamma_0(1):\Gamma_0(N)] + 1 = \frac{kN}{12}\prod_{p\mid N}\frac{p+1}{p} + 1.$$

Assume that $b(n) \equiv 0 \bmod v$ *for* $n = 1,\ldots,\lambda$. *Then* $b(n) \equiv 0 \bmod v$ *for all* n.

Remark 5.3 (cf. [11, Proposition 5]). In [19], Sturm proved this theorem for integral weight modular forms and trivial character, but the general case follows by taking an appropriate power of f.

EXAMPLE 5.4 (cf. [21]). Let E be the elliptic curve given by

$$E:\ y^2 = x^3 - x.$$

Then the conductor of E is 27 and E has $\mathbb{Q}$-rational torsion points of order 2. In this case the weight 3/2 eigenform

$$\begin{aligned} g(z) &= \sum_{x,y,z\in\mathbb{Z}} (-1)^{x+y} q^{(4x+1)^2+16y^2+2z^2} = \sum_{n=1}^{\infty} b(n)q^n \\ &= q + 2q^3 + q^9 - 2q^{11} - 4q^{17} - 2q^{19} - 3q^{25} + \cdots \end{aligned}$$

is in $S_{3/2}(128,1)$. Its image $\mathrm{SH}_1(g(z)) = \sum_{n=1}^{\infty} a(n)q^n$ is the weight 2 newform whose Mellin transform is $L(E,s)$. Tunnell [21] proved that for a square-free integer d,

$$L(E_d) = \frac{b(d)^2 \Omega}{4\sqrt{d}},$$

where Ω is the real period of E. Let

$$\delta(n) = \begin{cases} 1 & \text{if } (n,128)=1, \\ 0 & \text{otherwise}, \end{cases} \qquad \delta'(n) = \begin{cases} 1 & \text{if } (n,1408)=1, \\ 0 & \text{otherwise}. \end{cases}$$

Then, by Lemmas 3.5 and 3.7, we have

$$\begin{aligned} g_1(z) = \sum_{n=1}^{\infty} b_1(n)q^n &= \sum_{n=1}^{\infty} b(n)\delta(n)q^n - \Big(1 + 2\sum_{n=1}^{\infty} q^{n^2}\Big)^2 \Big(\sum_{n=1}^{\infty} \delta(n)q^{n^2}\Big) \\ &\in M_{3/2}(128,\chi_1) \end{aligned}$$

and

$$\begin{aligned} g_2(z) = \sum_{n=1}^{\infty} b_2(n)q^n &= \sum_{n=1}^{\infty} b(11n)\delta'(n)q^n - 2\Big(1 + 2\sum_{n=1}^{\infty} q^{n^2}\Big)^2 \Big(\sum_{n=1}^{\infty} \delta'(n)q^{n^2}\Big) \\ &\in M_{3/2}(15488,\chi_2). \end{aligned}$$

Now, let $\lambda_1 = 25$ and $\lambda_2 = 3169$. Then we can verify that $b_1(n) \equiv 0 \bmod 2$ for $n = 1,\ldots,\lambda_1$ and $b_2(n) \equiv 0 \bmod 4$ for $n = 1,\ldots,\lambda_2$. Hence, by Theorem 5.2,

$$\min\{\mathrm{ord}_2(b(m)) \mid m > 1 \text{ is square-free}, (m,128)=1\} = 1,$$

$$\min\{\mathrm{ord}_2(b(11m)) \mid m > 1 \text{ is square-free}, (m,15488)=1\} = 2.$$

Also we can verify that $\mathrm{ord}_2(b(59)) = 1$ and $\mathrm{ord}_2(b(59\cdot 11)) = 2$. By Theorem 4.1, there are infinitely many primes p for which $\operatorname{rank} E_{-p}(\mathbb{Q}(\sqrt{11})) = \operatorname{rank} E_{-p}(\mathbb{Q}(\sqrt{-11})) = 0$. In fact, by computing the 2-Selmer groups of E_{-p}, it is known that if p and q are primes with $p \equiv q \equiv 3 \bmod 8$, then $\operatorname{rank} E_{-p}(\mathbb{Q}(\sqrt{q})) = 0$.

EXAMPLE 5.5 (cf. [1, Examples 3.6.2]). Let E be the elliptic curve given by

$$E:\ y^2 + xy + y = x^3 + 4x - 6.$$

Then the conductor of E is 14 and E has a $\mathbb{Q}$-rational torsion point of order 2. In this case the weight 3/2 eigenform

$$\begin{aligned} g(z) &= \frac{1}{2}\Big(\sum_{x,y,z\in\mathbb{Z}} q^{x^2+14y^2+14z^2} - q^{2x^2+7y^2+14z^2}\Big) = \sum_{n=1}^{\infty} b(n)q^n \\ &= q - q^2 + q^4 - q^7 - q^8 - q^9 + \cdots \end{aligned}$$

is in $S_{3/2}(56,1)$. Its image $\mathrm{SH}_1(g(z)) = \sum_{n=1}^{\infty} a(n)q^n$ is the weight 2 newform whose Mellin transform is $L(E,s)$. Let

$$\delta(n) = \begin{cases} 1 & \text{if } (n,56)=1, \\ 0 & \text{otherwise}, \end{cases} \qquad \delta'(n) = \begin{cases} 1 & \text{if } (n,88200)=1, \\ 0 & \text{otherwise}. \end{cases}$$

Then, by Lemmas 3.5 and 3.7, we have

$$\begin{aligned} g_1(z) &= \sum_{n=1}^{\infty} b_1(n)q^n = \sum_{n=1}^{\infty} b(n)\delta(n)q^n - \Big(1+2\sum_{n=1}^{\infty} q^{n^2}\Big)^2\Big(\sum_{n=1}^{\infty}\delta(n)q^{n^2}\Big) \\ &\in M_{3/2}(504,\chi_1) \end{aligned}$$

and

$$\begin{aligned} g_2(z) &= \sum_{n=1}^{\infty} b_2(n)q^n = \sum_{n=1}^{\infty} b(15n)\delta'(n)q^n - 2\Big(1+2\sum_{n=1}^{\infty} q^{n^2}\Big)^2\Big(\sum_{n=1}^{\infty}\delta'(n)q^{n^2}\Big) \\ &\in M_{3/2}(88200,\chi_2). \end{aligned}$$

Now, let $\lambda_1 = 145$ and $\lambda_2 = 30241$. Then we can verify that $b_1(n) \equiv 0 \bmod 2$ for $n = 1,\dots,\lambda_1$ and $b_2(n) \equiv 0 \bmod 4$ for $n = 1,\dots,\lambda_2$. Hence, by Theorem 5.2,

$$\min\{\mathrm{ord}_2(b(m)) \mid m>1 \text{ is square-free, } (m,56)=1\} = 1,$$
$$\min\{\mathrm{ord}_2(b(15m)) \mid m>1 \text{ is square-free, } (m,88200)=1\} = 2.$$

Also, we can see that $\mathrm{ord}_2(b(71)) = 1$, $\mathrm{ord}_2(b(71\cdot 15)) = 2$, $L(E_{-71},1) \neq 0$ and $L(E_{-71\cdot 15},1) \neq 0$. By Theorem 4.1, there are infinitely many primes p for which $\mathrm{rank}\, E_{-p}(\mathbb{Q}(\sqrt{15})) = 0$.

Example 5.6. Let $E^1 = X_0(11)$ and $E^2 = X_0(14)$. Moreover, we put

$$\begin{aligned} g_1(z) &= \frac{1}{2}\Big(\sum_{x,y,z\in\mathbb{Z}} q^{x^2+11y^2+11z^2} - q^{3x^2+2xy+4y^2+11z^2}\Big) = \sum_{n=1}^{\infty} b_1(n)q^n \\ &= q - q^3 - q^5 + q^{11} + 2q^{12} - 2q^{14} + q^{15} - \cdots \end{aligned}$$

and

$$\begin{aligned} g_2(z) &= \frac{1}{2}\Big(\sum_{x,y,z\in\mathbb{Z}} q^{x^2+14y^2+14z^2} - q^{2x^2+7y^2+14z^2}\Big) = \sum_{n=1}^{\infty} b_2(n)q^n \\ &= q - q^2 + q^4 - q^7 - q^8 - q^9 + q^{14} + 2q^{15} + \cdots. \end{aligned}$$

Then

$$\min\{\mathrm{ord}_2(b_1(m)) \mid m > 1 \text{ is square-free}, (m,44)=1\} = 0,$$
$$\min\{\mathrm{ord}_2(b_2(m)) \mid m > 1 \text{ is square-free}, (m,56)=1\} = 1,$$

$\mathrm{ord}_2(b_1(15)) = 0$, $\mathrm{ord}_2(b_2(15)) = 1$, $L(E^1_{-15},1) \neq 0$ and $L(E^2_{-15},1) \neq 0$. Therefore by Theorem 4.5 there are infinitely many square-free integers m for which $\operatorname{rank} E^1_{-m}(\mathbb{Q}) = \operatorname{rank} E^2_{-m}(\mathbb{Q}) = 0$. Moreover,

$$\#\{0 < m \le X \mid m \text{ is square-free and } \operatorname{rank} E^i_{-m}(\mathbb{Q}) = 0 \ (i=1,2)\} \gg X/\log X.$$

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