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Zero-sums of length kq in $\mathbb{Z}_q^d$

by

Silke Kubertin (Uetze)

1. Introduction. Let n and d be positive integers. A sequence $\mathcal{A}$ in $\mathbb{Z}_n^d$ is called a *zero-sum* if the sum of all elements of $\mathcal{A}$ is zero in $\mathbb{Z}_n^d$. By $s_k(\mathbb{Z}_n^d)$ we denote the smallest integer t such that any sequence of length t in $\mathbb{Z}_n^d$ contains a zero-sum of length kn. The case $k = 1$, $s_1(\mathbb{Z}_n^d)$ then denoted by $f(n, d)$, was first studied by Harborth ([7]) and generated a lot of research. Already in 1961 the one-dimensional case had been solved by Erdős, Ginzburg and Ziv, which initiated a whole new branch in combinatorial number theory.

Theorem (P. Erdős, A. Ginzburg, A. Ziv, 1961 [3]). *For any positive integer n we have $f(n, 1) = s_1(\mathbb{Z}_n) = 2n - 1$.*

Kemnitz' Conjecture $f(n, 2) = s_1(\mathbb{Z}_n^2) = 4n - 3$ (see [8]) was open for about twenty years and was recently proved by Reiher in [10]. The best result until then was the following:

Theorem (W. D. Gao, 2001 [4]). *Let q be a prime power. Then we have $f(q, 2) = s_1(\mathbb{Z}_q^2) \le 4q - 2$ and $s_2(\mathbb{Z}_q^2) \le 4q - 2$.*

This improves a result of Rónyai ([11]) who showed this only a little earlier for primes p instead of prime powers q. Up to now the best general bounds for odd primes p and higher dimensions d are

$$f(p, d) \ge 1.125^{\lfloor d/3 \rfloor} 2^d (p - 1) + 1,$$

by Elsholtz ([2]), where $2^d(p - 1) + 1$ is the trivial lower bound, and

$$f(p, d) \le (cd \log d)^d p$$

by Alon and Dubiner ([1]). They conjectured that $f(p, d) \le c^d p$.

For $k \ne 1$ the constant $s_k(\mathbb{Z}_n^d)$ was first studied by Gao and Thangadurai. They verified that $s_k(\mathbb{Z}_p^3) = (k + 3)p - 3$ for $k \ge 4$ (see [6]) and in higher dimensions $s_k(\mathbb{Z}_q^d) = (k + d)q - d$ for $k \ge q^{d-1}$ (see [5]).

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The sequence consisting of $kn-1$ copies of the zero-vector and $n-1$ copies of each of the d basis vectors obviously does not contain a zero-sum of length kn. Therefore we have

$$s_k(\mathbb{Z}_n^d) \geq kn-1+(n-1)d+1=(k+d)n-d.$$

For $k<d$ the above example can be extended by $\left\lfloor \frac{d-k}{d-1} n \right\rfloor - 1$ copies of the vector $(1,\ldots,1)$. So we get

$$s_k(\mathbb{Z}_n^d) \geq (k+d)n-d+\left\lfloor \frac{d-k}{d-1}\, n-1 \right\rfloor.$$

Again this example can be improved by using vectors with exactly l $(>k)$ entries 1 and the other entries 0 instead of the all-one vector. But as opposed to the case $k=1$, where a simple example shows that $s_1(\mathbb{Z}_n^d) > 2^d(n-1)$, it is not obvious that for $2 \leq k < d$ the growth of $s_k(\mathbb{Z}_n^d)$ is not linear in d.

In this paper we suggest the following conjecture:

Conjecture. *For positive integers $k \geq d$ and n we have*

$$s_k(\mathbb{Z}_n^d) = (k+d)n-d.$$

This has been proved by Gao ([5]) for prime powers $n=q$ and $k \geq q^{d-1}$ using Olson's result about Davenport's Constant ([9]). Here the Conjecture will be verified for a large class of smaller values of k in the case of general d (Theorems 2 and 4) as well as for $d \leq 4$ (Theorem 1).

These are our main results:

Theorem 1. *Let p be a prime and q be a power of p. For any positive integer k we have*

(1) $s_k(\mathbb{Z}_q) = (k+1)q-1$ (Gao, Thangadurai, 2003 [6]),
(2) $s_k(\mathbb{Z}_q^2) = (k+2)q-2$ *for* $k \geq 2$ (Gao, Thangadurai, 2003 [6]),
(3) $s_k(\mathbb{Z}_q^3) = (k+3)q-3$ *for* $k \geq 3$ *and* $s_2(\mathbb{Z}_q^3) \leq 6q-3$, *both for* $p>3$,
(4) $s_k(\mathbb{Z}_q^4) = (k+4)q-4$ *for* $k \geq 4$ *and* $p \geq 7$ (*actually for* $p \geq 5$, *if* k *is even*), *and* $s_2(\mathbb{Z}_q^4) \leq 8q-4$ *and* $s_3(\mathbb{Z}_q^4) \leq 8q-4$, *both for* $p \geq 5$.

Theorem 2. *Let p be a prime and q be a power of p. Then the Conjecture holds for $s_{np}(\mathbb{Z}_q^d)$, where n and d are any positive integers*:

$$s_{np}(\mathbb{Z}_q^d) = (np+d)q-d.$$

Our next result is a general upper bound for $s_k(\mathbb{Z}_q^d)$ with $k \geq d$.

Theorem 3. *Let d and $k \geq d$ be positive integers, $p > \min(2k, 2d)$ be a prime and q be a power of p. Then*

$$s_k(\mathbb{Z}_q^d) \leq \left(\frac{3}{8}\, d^2+\frac{3}{2}\, d-\frac{3}{8}+k\right)q-d.$$

Certainly this could be improved with some additional effort but we do not see how to obtain an upper bound for $s_k(\mathbb{Z}_q^d)$ with linear growth in d.

As a corollary of Theorems 2 and 3 we prove the Conjecture for sufficiently large k.

THEOREM 4. *Let d and k be positive integers, $p > 2d$ be a prime and q be a power of p. If $\lfloor \frac{k-d}{p} \rfloor p \geq \frac{3}{8}d^2 + \frac{d}{2} - \frac{3}{8}$, then the Conjecture holds for $s_k(\mathbb{Z}_q^d)$.*

The cases $d = 1$ and $d = 2$ are simple consequences of the Erdős–Ginzburg–Ziv Theorem and of the above theorem of Gao ($s_2(\mathbb{Z}_q^2) \leq 4q - 2$). To handle the other cases in the following sections we will generalize the method that Rónyai ([11]) developed to prove $f(p, 2) \leq 4p - 2$.

2. Rónyai's method. In order to prove $f(p, 2) \leq 4p - 2$, Rónyai ([11]) used special polynomial functions $P : \{0, 1\}^m \to \mathbb{F}_p$, depending on the given sequence $\mathcal{A}$. For sufficiently large $m = |\mathcal{A}|$ there is an $x \in \{0, 1\}^m$, $x \neq \mathbf{0}$, such that $P(x) \neq 0$. This x is related to a zero-sum with length p within $\mathcal{A}$.

In order to adapt these polynomials to prime powers q instead of primes p we had to change them a bit. Furthermore, in higher dimensions $d > 2$ this method can be generalized to prove that, for a given set $\mathcal{L} = \{l_1, \ldots, l_{\lceil d/2 \rceil}\}$, any sufficiently long sequence in $\mathbb{Z}_q^d$ contains a zero-sum of length lq for at least one $l \in \mathcal{L}$ and, iterating this, the existence of zero-sums of length kq for any given $k \geq d$.

We use the following easy fact, proved e.g. by Rónyai ([11]).

LEMMA 2.1. *Let $\mathbb{F}$ be a field and m be a positive integer. Then the monomials $\prod_{i \in I} x_i$, $I \subseteq \{1, \ldots, m\}$, constitute a base of the $\mathbb{F}$-linear space of all functions $f : \{0, 1\}^m \to \mathbb{F}$. (Here 0 and 1 are viewed as elements of $\mathbb{F}$.)*

Therefore any polynomial function $P : \{0, 1\}^m \to \mathbb{F}_p$, $p > 2$, has a unique representation of the form $\sum_{I \subseteq \{1, \ldots, m\}} a_I \prod_{i \in I} x_i$. With respect to this representation we define the *degree* of P as

$$\deg P = \max_{\substack{I \subseteq \{1, \ldots, m\} \\ a_I \neq 0}} \deg\Big(\prod_{i \in I} x_i\Big) = \max_{\substack{I \subseteq \{1, \ldots, m\} \\ a_I \neq 0}} |I|.$$

DEFINITION 1. Let $d > 1$ be an integer and $\mathcal{L} \subset \mathbb{N}$ be a set at least of cardinality $\lceil \frac{d}{2} \rceil$. An integer K is said to have *Property* (1) if

$$|\mathcal{L} \cup (K - \mathcal{L})| \geq d. \tag{1}$$

Here $K - \mathcal{L}$ denotes the set $\{K - l \mid l \in \mathcal{L}\}$.

Note that all $K > 2 \max_{l \in \mathcal{L}} l$ have Property (1).

Now we can prove the following theorem.

THEOREM 2.1. *Let $d > 1$ be an integer, p be a prime and q be a power of p. Let $\mathcal{L}$ be a set of at least $\lceil \frac{d}{2} \rceil$ positive integers and K be an integer with*

Property (1). *If* $p > \max(\{1, \dots, K-1\} \setminus (\mathcal{L} \cup (K - \mathcal{L})))$, *then any sequence* $(a_1, \dots, a_{Kq})$ *in* $\mathbb{Z}_q^d$ *with* $\sum_{i=1}^{Kq} a_i = \mathbf{0}$ (*in* $\mathbb{Z}_q^d$) *contains a zero-sum of length* lq *for at least one* $l \in \mathcal{L}$.

Proof. Assume to the contrary that for no $l \in \mathcal{L}$ there is a zero-sum of length lq within $(a_1, \dots, a_{Kq})$. Then there is no zero-sum of length $(K-l)q$, $l \in \mathcal{L}$, either. So if there are zero-sums of length $kq > 0$ apart from the whole sequence, k has to be in $J = \{1, \dots, K-1\} \setminus (\mathcal{L} \cup (K - \mathcal{L}))$, $|J| \le K - 1 - d$.

We define the polynomial function $P : \{0,1\}^{Kq} \to \mathbb{F}_p$ as

$$P(x) = Q(x) \prod_{\delta=1}^{d} R_\delta(x) \prod_{j \in J} S_j(x),$$

where

$$Q(x) = \binom{g(x) - 1}{q - 1}, \qquad R_\delta(x) = \binom{\sum_{i=1}^{Kq} a_{i,\delta} x_i - 1}{q - 1}, \qquad S_j(x) = \binom{g(x)}{q} - j.$$

Here $g(x)$ is the Hamming weight of $x \in \{0,1\}^{Kq}$,

$$g(x) = \sum_{i=1}^{Kq} x_i.$$

Any vector $x \in \{0,1\}^{Kq}$ corresponds to a subsequence $\mathcal{B}_x = (a_i)_{x_i = 1}$ of length $g(x)$. Note that $P(x)$ vanishes in each of the following three cases:

(1) $g(x)$ is not divisible by q (because of Q),
(2) the corresponding subsequence $\mathcal{B}_x$ is not a zero-sum (because of the R_δ),
(3) $\mathcal{B}_x$ is of length jq with $j \in J + p\mathbb{N}$ (because of S_j).

Therefore we get

$$P(x) = P(\mathbf{0})\chi_{\mathbf{0}}(x) + P(\mathbf{1})\chi_{\mathbf{1}}(x)$$

where $\chi_{\mathbf{0}}(x) = \prod_{i=1}^{Kq}(1 - x_i)$ and $\chi_{\mathbf{1}}(x) = \prod_{i=1}^{Kq} x_i$ are the characteristic functions of the all-zero resp. the all-one vector and

$$P(\mathbf{0}) = \prod_{j \in J}(-j) = (-1)^{|J|} P(\mathbf{1}).$$

So the degree of P is at least $\deg P \ge Kq - 1$.

On the other hand the degree of P can be determined via the representation as a linear combination of monomials one gets using the relations $x_i^2 = x_i$ ($x_i \in \{0,1\}$). Since this reduction cannot increase the degree, we have

$$\deg P \le \deg Q + \sum_{\delta=1}^{d} \deg R_\delta + \sum_{j \in J} \deg S_j \le (d+1)(q-1) + |J|q \le Kq - d,$$

a contradiction to $\deg P \ge Kq - 1$. ∎

In a second step we will start with sequences which are not necessarily zero-sums of length Kq.

THEOREM 2.2. *Let p be a prime, q be a power of p and $d > 1$ be an integer. Given a set $\mathcal{L} = \{l_1, \dots, l_{\lceil d/2\rceil}\} \subset \mathbb{N}$ let $K_1 < \dots < K_{\lfloor d/2\rfloor}$ be the $\lfloor \frac{d}{2} \rfloor$ smallest positive integers with Property* (1). *Define the set $J := \{1, \dots, K_{\lfloor d/2\rfloor}\} \setminus (\mathcal{L} \cup \{K_1, \dots, K_{\lfloor d/2\rfloor}\})$. Then for $m \geq (K_{\lfloor d/2\rfloor} + 1)q - d$ and $p > \max_{j\in J} j$ any sequence $(a_i)_{i=1,\dots,m}$ in $\mathbb{Z}_q^d$ has a zero-sum of length lq for at least one $l \in \mathcal{L}$.*

Proof. Assume to the contrary that a sequence $(a_i)_{i=1,\dots,m}$ contains no zero-sums of length lq for any $l \in \mathcal{L}$. Then by Theorem 2.1 for any K with Property (1) there are no zero-sums of length Kq either. Now look at $P : \{0, 1\}^m \to \mathbb{F}_p$,

$$P(x) = Q(x) \prod_{\delta=1}^{d} R_\delta(x) \prod_{j\in J} S_j(x),$$

and proceed as above. ∎

Theorem 2.2 has the following immediate consequences:

COROLLARY 2.1. *For q a power of the prime p and an integer $d \geq 2$ let $(a_i)_{i=1,\dots,m}$ be a sequence in $\mathbb{Z}_q^d$.*

(1) *If $m \geq \left(2d - \lceil \frac{d}{2} \rceil + 1\right)q - d$ and $p > d$, then (a_i) contains a zero-sum of length lq for at least one $l \in \left\{1, \dots, \lceil \frac{d}{2} \rceil\right\}$.*
(2) *If $m \geq \left(2d - \lceil \frac{d}{2} \rceil\right)q - d$ and $p \geq d$, then (a_i) contains a zero-sum of length lq for at least one $l \in \left\{1, \dots, \lceil \frac{d}{2} \rceil, d\right\}$.*
(3) *If $m \geq 2dq - d$ and $p \geq d + \lceil \frac{d}{2} \rceil$, then (a_i) contains a zero-sum of length lq for at least one $l \in \left\{1, \dots, \lceil \frac{d}{2} \rceil - 1, d\right\}$.*
(4) *If $m \geq \left(2d - \lceil \frac{d}{2} \rceil\right)q - d$ and $p > d$, then (a_i) contains a zero-sum of length lq for at least one $l \in \left\{1, \dots, \lceil \frac{d}{2} \rceil + 1\right\}$.*
(5) *If $m \geq 2dq - d$ and $p \geq 2d - 1$, then (a_i) contains a zero-sum of length lq for at least one odd $l \leq d$.*
(6) *If $m \geq 2dq - d$ and $p \geq 2d - 1$, then (a_i) contains a zero-sum of length lq for at least one even $l \leq d + 1$.*

Proof. This directly follows by an application of Theorem 2.2 with the following choice of the sets J:

(1) $J = \left\{\lceil \frac{d}{2} \rceil + 1, \dots, d\right\}$,
(2) $J = \left\{\lceil \frac{d}{2} \rceil + 1, \dots, d - 1\right\}$,
(3) $J = \left\{\lceil \frac{d}{2} \rceil, \dots, d + \lceil \frac{d}{2} \rceil - 1\right\} \setminus \{d\}$,
(4) $J = \left\{\lceil \frac{d}{2} \rceil + 2, \dots, d\right\}$,
(5) $J = \{2, 4, \dots, 2(d - 1)\}$,
(6) $J = \left\{1, 3, \dots, 2\lceil \frac{d}{2} \rceil - 1\right\} \cup \left\{2\lceil \frac{d}{2} \rceil + 2, 2\lceil \frac{d}{2} \rceil + 4, \dots, 2(d - 1)\right\}$. ∎

A slightly weaker result than item (5) in the above corollary is due to Gao and Thangadurai ([6]) who showed that for primes $p > 2$ any sequence of length $2(d+1)(p-1)+1$ in $\mathbb{Z}_p^d$ has a zero subsequence of length lp for some odd l.

3. Proofs of our main results. To handle the higher-dimensional problem we combine the parts of Corollary 2.1 in order to ensure the existence of a zero-sum of length kq for a fixed $k \geq d$ within a sufficiently large sequence in $\mathbb{Z}_q^d$. We point out that a slightly weaker version of part (3) of Theorem 1 (for primes $p > 3$ and $k \geq 4$) has been proved by Gao and Thangadurai in [6], using different methods.

Proof of Theorem 1. Since $s_k(\mathbb{Z}_q^d) \geq (k+d)q - d$ (see introduction) we only have to show that the claimed constants are upper bounds.

(3) Let $p > 3$ and $\mathcal{A}$ be a sequence in $\mathbb{Z}_q^3$ of length $6q-3$. First we search for zero-sums of length $2q$ and $3q$. By Corollary 2.1(1), (3), $\mathcal{A}$ contains a zero-sum of length q or two zero-sums of length $2q$ and of length $3q$. In the first case there are $5q-3$ elements left, which provide by Corollary 2.1(1) another zero-sum of length $2q$ (then we are done) or of length q. In this last case, to find a zero-sum of length $3q$, we apply Corollary 2.1(2) to the remaining $4q-3$ elements. So we have $s_2(\mathbb{Z}_q^3), s_3(\mathbb{Z}_q^3) \leq 6q-3$. Therefore any sequence in $\mathbb{Z}_q^3$ of cardinality $(k+3)q-3$ ($k \equiv 2, 3$ modulo 3, $k \geq 3$) contains disjoint zero-sums, one of length $2q$ resp. $3q$ and $\left\lfloor \frac{k-2}{3} \right\rfloor$ of length $3q$. We get $s_k(\mathbb{Z}_q^d) \leq (k+3)q-3$ for all $k \equiv 2, 3$ modulo 3, $k \geq 3$.

To show the upper bound for $k = 4$ take a sequence of length $7q-3$. Repeated application of Corollary 2.1(1) proves the existence either of a zero-sum of length $4q$ or of two zero-sums of lengths q and $2q$. In this second case $4q-3$ elements are left which by Corollary 2.1(2) contain a zero-sum of length q, $2q$ or $3q$. Therefore we have $s_k(\mathbb{Z}_q^d) \leq (k+3)q-3$ for all $k \equiv 1$ modulo 3, $k \geq 4$.

(4) We get $s_2(\mathbb{Z}_q^4) \leq 8q-4$ from Corollary 2.1(1), and Corollary 2.1(2) tells us $s_4(\mathbb{Z}_q^4) \leq 8q-4$, both for $p \geq 5$.

Let now $p \geq 7$. To show $s_3(\mathbb{Z}_q^4) \leq 8q-4$ let $\mathcal{A}$ be a sequence in $\mathbb{Z}_q^4$ of length $8q-4$. By Corollary 2.1(5) it contains a zero-sum of length q or $3q$. In the first case within the $7q-4$ remaining elements we find by Corollary 2.1(1) a zero-sum of length q or $2q$. If this again is a zero-sum of length q, then the last $6q-4$ elements contain a zero-sum of length q, $2q$ or $3q$ and so we are done.

Now we search for a zero-sum of length $5q$ within an arbitrary sequence in $\mathbb{Z}_q^4$ of length $9q-4$. We already know that there must be a zero-sum of length $3q$. By Corollary 2.1(2) the $6q-4$ remaining elements contain a zero-sum of length q, $2q$ or $4q$. In the case of length $2q$ we are done. If there is a zero-sum of length q, we delete these q elements from the original sequence and because of $s_4(\mathbb{Z}_q^4) \leq 8q-4$ we find a zero-sum of length $4q$ and so have a zero-sum of length $5q$. In the last case (i.e. of disjoint zero-sums of lengths $3q$ and $4q$) we apply Theorem 2.1 to the zero-sum of length $7q$ and $\mathcal{L} = \{1, 5\}$. So either we directly get a zero-sum of length $5q$ or in the case of length q we proceed as above. So we have shown $s_5(\mathbb{Z}_q^4) \leq 9q-4$. Combining the results in this part we get $s_k(\mathbb{Z}_q^4) = (k+4)q - 4$ for all $k \geq 4$. ∎

Proof of Theorem 2. The proof of $s_p(\mathbb{Z}_q^d) = (p+d)q - d$ is analogous to that of Theorem 2.2 with $m = (p+d)q - d$ and $P : \{0,1\}^m \to \mathbb{F}_p$ defined as

$$P = Q \prod_{\delta=1}^{d} R_\delta \prod_{j=1}^{p-1} S_j$$

where Q, R_δ and S_j are as above. So, within a sequence of length $(np+d)q-d$ there are n disjoint zero-sums of length pq. ∎

Proof of Theorem 3. First let k be in $\{d, \ldots, 2d-1\}$. The idea is to use Theorem 2.2 in order to extract $\lceil \frac{d}{2} \rceil - 1$ pairwise disjoint zero-sums of different lengths $l_j q$ $(\neq kq)$ first and then to find a zero-sum of length $(k - l_j)q$ or kq.

So let $\mathcal{A}$ be a sequence in $\mathbb{Z}_q^d$ of length

$$\left(\frac{3}{8}\,d^2 + \frac{3}{2}\,d - \frac{3}{8} + k\right)q - d \geq \left(\frac{3}{8}\,d^2 - \frac{d}{2} + \frac{5}{8} + 2k\right)q - d.$$

By Theorem 2.2 the sequence $\mathcal{A}_1 := \mathcal{A}$ contains a zero-sum of length $l_1 q$ for at least one $l_1 \in \mathcal{L}_1 := \left\{1, 2, \ldots, \lceil \frac{d}{2} \rceil - 1, k\right\}$. Let $\mathcal{A}_2$ be the sequence $\mathcal{A}_1$ without this zero-sum. So either $\mathcal{A}_2$ has length at least $|\mathcal{A}_1| - \left(\lceil \frac{d}{2} \rceil - 1\right)q$ or we have already obtained a zero-sum of length kq.

We use Theorem 2.2 with $\mathcal{L}_j := \left\{1, 2, \ldots, \lceil \frac{d}{2} \rceil - 2 + j, k\right\} \setminus \{l_1, \ldots, l_{j-1}\}$ iteratively where $\mathcal{A}_j$ is the sequence $\mathcal{A}_{j-1}$ without the zero-sum of length $l_{j-1}q$ until we have found a zero-sum of length kq or $\lceil \frac{d}{2} \rceil - 1$ pairwise disjoint zero-sums of lengths $l_j q$. In both cases $\mathcal{A}' := \mathcal{A}_{\lceil d/2 \rceil}$ has length at least

$$|\mathcal{A}| - \sum_{j=1}^{\lceil d/2 \rceil - 1} \left(\left\lceil \frac{d}{2} \right\rceil - 2 + j\right)q \geq \left(d + 2k - \left\lceil \frac{d}{2} \right\rceil\right)q - d.$$

Therefore by Theorem 2.2 the sequence $\mathcal{A}'$ contains a zero-sum of length $l'q$ for at least one $l' \in \mathcal{L}' = \{k - l_1, \ldots, k - l_{\lceil d/2 \rceil - 1}, k\}$.

Note that in all these steps $\max J$ does not exceed $2k$, so $p > 2k$ guarantees $p > \max J$.

Now if $k \geq 2d$, then $\mathcal{A}$ contains $\lfloor \frac{k}{d} \rfloor - 1$ disjoint zero-sums of length dq and because of

$$\frac{3}{8}d^2 + \frac{3}{2}d - \frac{3}{8} + k - \left(\left\lfloor \frac{k}{d} \right\rfloor - 1 \right) d \geq \frac{3}{8}d^2 - \frac{d}{2} + \frac{5}{8} + 2 \underbrace{\left(k - \left(\left\lfloor \frac{k}{d} \right\rfloor - 1 \right) d \right)}_{\leq 2d-1}$$

there is a zero-sum of length $\big(k - \big(\lfloor \frac{k}{d} \rfloor - 1\big)d\big)q$ within the remaining sequence. ■

Proof of Theorem 4. Let $\mathcal{A}$ be a sequence in $\mathbb{Z}_q^d$ of length $(d+k)q - d$ where k is of the form $np + r$ with $d \leq r \leq p + d - 1$ and $np \geq \frac{3}{8}d^2 + \frac{d}{2} - \frac{3}{8}$. Since $d + k = d + np + r \geq \frac{3}{8}d^2 + \frac{3}{2}d - \frac{3}{8} + r$ the given sequence $\mathcal{A}$ contains by Theorem 3 a zero-sum of length rq. Within the remaining $(d + np)q - d$ elements there is by Theorem 2 a zero-sum of length npq. ■

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Zur Seebecke 6
D-31311 Uetze, Germany
E-mail: silke.kubertin@web.de

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