ACTA ARITHMETICA
116.2 (2005)

On the arithmetic mean of Dedekind sums

by

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Introduction and main result. In what follows, $\mathbb{N}$ denotes the set of positive integers. We consider a number $N \in \mathbb{N}$ (which will often tend to infinity) and integers m with $(m,N)=1$. The classical Dedekind sum $s(m,N)$ is defined by

$$s(m,N)=\sum_{k=1}^{N}((k/N))((mk/N)),$$

where $((\ldots))$ is the usual sawtooth function (see [2]). In the present setting it is more natural to work with

$$S(m,N)=12s(m,N).$$

Since $S(m+N,N)=S(m,N)$, it suffices to study $S(m,N)$ for numbers m in the range $0\le m<N$, $(m,N)=1$.

The *values* of $S(m,N)$ lie between $-N$ and N; their distribution has attracted considerable interest (see [2] for a survey). For instance, the limiting distribution of these sums shows that, on average, $|S(m,N)|\le 12\log N$ for about 90% of all possible $m\in[0,N[$ when N tends to infinity (see [12]). On the other hand, in the neighbourhood of *Farey points* $N\cdot c/d$, $0\le c\le d$, $(c,d)=1$, $d\le\sqrt{N}$, there is quite a number of integers m with relatively large values of $|S(m,N)|$. This phenomenon was studied (among other things) in [1], [3], and also in [5], [6]. It is responsible for the fact that the *quadratic mean value* of the sums $S(m,N)$ is relatively large. Indeed,

$$\Bigg(\frac{1}{\varphi(N)}\sum_{\substack{0\le m<N\\(m,N)=1}}|S(m,N)|^2\Bigg)^{1/2}\asymp N^{1/2}$$

for $N\to\infty$ (here $\varphi(N)$ is the Euler function; see [4], [13] for details).

2000 *Mathematics Subject Classification*: Primary 11F20.

Whereas higher power mean values and related moments of Dedekind sums have been studied thoroughly (see [8], [7]), it seems that not much is known about the asymptotic behaviour of the *arithmetic* mean

$$\frac{1}{\varphi(N)} \sum_{\substack{0\le m<N\\(m,N)=1}} |S(m,N)|.$$

The only result we know of is the upper bound

$$\frac{1}{\varphi(N)} \sum_{\substack{0\le m<N\\(m,N)=1}} |S(m,N)| \le \frac{6}{\pi^2}\log^2 N + O(\log N) \tag{1}$$

(for $N\to\infty$), which is an easy corollary to a result about continued fractions, as we shall point out below. In this paper we show

$$\frac{1}{\varphi(N)} \sum_{\substack{0\le m<N\\(m,N)=1}} |S(m,N)| \ge \frac{3}{\pi^2}\log^2 N + O(\log^2 N/\log\log N)$$

for $N\to\infty$. In fact, we present a more precise statement. For $N\ge 2$ put

$$x = \min\{\sqrt{N}/\log N, \sqrt{N}/\tau(N)\}, \tag{2}$$

where $\tau(N)$ denotes the number of divisors of N. Let c,d be integers such that $0\le c\le d\le x$ and $(c,d)=1$. For these we define

$$I_{c/d} = [\,0,N\,]\cap\{z\in\mathbb{R} : |z-N\cdot c/d|\le x/d\}.$$

So $I_{c/d}$ is a certain interval around the Farey point $N\cdot c/d$ (but it is in general larger than the Farey neighbourhood of [6] denoted in the same way). Further put

$$\mathcal{F} = \bigcup_{1\le d\le x}\ \bigcup_{\substack{0\le c\le d\\(c,d)=1}} I_{c/d}.$$

It is not hard to see that this union is *disjoint* if N is large (see Section 1 below). The set $\mathcal{F}$ contains only relatively *few* of all integers m, $0\le m<N$, $(m,N)=1$; indeed, $|\mathcal{F}\cap\mathbb{Z}| = O(\varphi(N)/\log N)$ for large values of N (see Section 2). We show

THEOREM 1. *Let N tend to infinity. Then*

$$\frac{1}{\varphi(N)} \sum_{\substack{m\in\mathcal{F}\\(m,N)=1}} |S(m,N)| = \frac{3}{\pi^2}\log^2 N + O(\log^2 N/\log\log N).$$

We should say some words about the upper bound (1). Fix N for a moment. Let $a_1,\ldots,a_n\in\mathbb{N}$ be the continued fraction expansion of m/N, i.e.,

$$m/N = \frac{1|}{|a_1} + \cdots + \frac{1|}{|a_n}$$

(so n depends on m and $a_n \geq 2$). Put $T(m,N) = a_1 + \cdots + a_n$. Then

$$\frac{1}{\varphi(N)} \sum_{\substack{0 \leq m < N \\ (m,N)=1}} T(m,N) = \frac{6}{\pi^2} \log^2 N + O(\log N) \tag{3}$$

(see [9] and [10]). The Dedekind sum $S(m,N)$ is nearly the same as the *alternating* sum $a_1 - a_2 + \cdots + (-1)^{n-1} a_n$. More precisely,

$$|S(m,N) - (a_1 - a_2 + \cdots + (-1)^{n-1} a_n)| \leq 5$$

(see [4, Lemma 4]). Together with (3) this clearly implies (1).

So this upper bound is in some sense trivial since it is just based on the estimate $|a_1 - a_2 + \cdots \pm a_n| \leq a_1 + \cdots + a_n$. Nevertheless, numerical computations suggest that (1) is basically *sharp*. We have the following explanation for this somewhat strange observation: Apparently the sign changes in $a_1 - a_2 + \cdots \pm a_n$ have no influence on the *main term* of (3) but only on the *error term*. One can see from the original papers that the error term of (3) is $\geq C \log N$ for some *positive* constant C. It seems likely that a lower bound of the following kind is nearly optimal: There are constants $C' < 0$ and $k \geq 1$ such that, for $N \to \infty$,

$$\frac{1}{\varphi(N)} \sum_{\substack{0 \leq m < N \\ (m,N)=1}} |S(m,N)| \geq \frac{6}{\pi^2} \log^2 N + C' \log N \log^k \log N.$$

In this sense the contribution of the relatively few values $|S(m,N)|$, $m \in \mathcal{F}$, is just *half* of the conjectured asymptotic arithmetic mean of *all* Dedekind sums. It seems that our method does not yield more.

1. Plan of the proof. For the time being, let $N \geq 5$ and x be as in (2). In addition, assume $0 \leq m < N$, $0 \leq c \leq d \leq x$, and $(m,N) = (c,d) = 1$. Our first observation concerns the *disjointness* of the intervals $I_{c/d}$ as mentioned above. Indeed, suppose $I_{c/d} \cap I_{c'/d'}$ is nonempty for $0 \leq c' \leq d' \leq x$, $(c',d') = 1$. Then $|N \cdot c/d - N \cdot c'/d'| \leq x/d + x/d'$. Together with (2) this gives $|d'c - c'd| \leq 2x^2/N \leq 2/\log^2 N < 1$.

A basic tool for our proof of Theorem 1 is the *generalized reciprocity law* for Dedekind sums, which we are going to state now. Choose $k, j \in \mathbb{Z}$ such that $-cj + dk = 1$ and define $r, q \in \mathbb{Z}$ by

$$\begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} j & -k \\ d & -c \end{pmatrix} \begin{pmatrix} m \\ N \end{pmatrix}. \tag{4}$$

So the 2×2-matrix of (4) has determinant 1 and $md - Nc = q$. As $(m,N) = 1$, we have $(r,q)=1$, and the conditions $(c,d)=1$, $d<N$, imply $q\neq 0$. Moreover, r is uniquely determined mod q; indeed, the substitutions $j\mapsto j+td$, $k\mapsto k+tc$, $t\in\mathbb{Z}$, entail $r\mapsto r+tq$. Accordingly, the Dedekind sum $S(r,|q|)$ is *uniquely determined* by m,N,c,d. The generalized reciprocity law says

$$S(m,N)=S(c,d)\pm S(r,|q|)+\frac{N^2+d^2+q^2}{Ndq}\pm 3,$$

where the $\pm$ sign is the sign of q in both cases (see, e.g., [5, Lemma 1]). This gives

$$S(m,N)=\frac{N}{dq}+S(c,d)\pm S(r,|q|)+O(1), \tag{5}$$

the O-term standing for an error of absolute value ≤ 5.

Let $I^+_{c/d}=[\,0,N\,]\cap\{z:0<z-N\cdot c/d\leq x/d\}$ be the *right half* of the interval $I_{c/d}$ and $\mathcal{F}^+$ the union of all $I^+_{c/d}$, $1\leq d\leq x$, $0\leq c\leq d$, $(c,d)=1$. Since $I^+_{1/1}=\emptyset$, it suffices that the union is taken over $c<d$ only. We shall show that

$$\frac{1}{\varphi(N)}\sum_{\substack{m\in\mathcal{F}^+\\(m,N)=1}}S(m,N)=\frac{3}{2\pi^2}\log^2 N+O(\log^2 N/\log\log N). \tag{6}$$

The analogue for the *left* halves $I^-_{c/d}$ and their respective union $\mathcal{F}^-$ reads

$$\frac{1}{\varphi(N)}\sum_{\substack{m\in\mathcal{F}^-\\(m,N)=1}}S(m,N)=-\frac{3}{2\pi^2}\log^2 N+O(\log^2 N/\log\log N) \tag{7}$$

(the union may be taken over $c>0$).

Theorem 1 is an immediate consequence of (6) and (7). These assertions are proved as follows: By (5), we have

$$\sum_{\substack{m\in\mathcal{F}^+\\(m,N)=1}}S(m,N)=\sum_{1\leq d\leq x}\sum_{\substack{0\leq c<d\\(c,d)=1}}\sum_{\substack{m\in I^+_{c/d}\\(m,N)=1}}\frac{N}{dq}+O(E_1+E_2+E_3), \tag{8}$$

where $q=md-Nc>0$ is as above and E_1, E_2, E_3 are the error terms

$$E_1=\sum_{1\leq d\leq x}\sum_{\substack{0\leq c<d\\(c,d)=1}}\sum_{\substack{m\in I^+_{c/d}\\(m,N)=1}}|S(c,d)|,\qquad E_2=\sum_{1\leq d\leq x}\sum_{\substack{0\leq c<d\\(c,d)=1}}\sum_{\substack{m\in I^+_{c/d}\\(m,N)=1}}|S(r,q)|,$$

$$E_3=\sum_{1\leq d\leq x}\sum_{\substack{0\leq c<d\\(c,d)=1}}\sum_{\substack{m\in I^+_{c/d}\\(m,N)=1}}1;$$

here r has the properties implied by (4). In the next section we show that the total contribution of E_1, E_2, E_3 is $O(\varphi(N)\log N\log\log N)$. The asymptotic expansion of the main term of (8), namely,

$$\sum_{1\le d\le x}\ \sum_{\substack{0\le c<d\\(c,d)=1}}\ \sum_{\substack{m\in I^+_{c/d}\\(m,N)=1}} \frac{N}{dq} = \frac{6}{\pi^2}\,\varphi(N)\log^2 x + O(\varphi(N)\log N\log^3\log N), \tag{9}$$

is more laborious. It is contained in Proposition 1 below, whose proof fills Section 3. Our choice (2) of x implies $\log x=(1/2)\log N+O(\log N/\log\log N)$ (see [11, p. 82]). So all these results together yield (6). Item (7) is treated in the same way: Equation (5) shows that (8) remains valid if $\mathcal{F}^+$ is replaced by $\mathcal{F}^-$ and $I^+_{c/d}$ by $I^-_{c/d}$ in each error term (here $q=-|q|$).

2. The error terms. We start with the above error terms. In order to treat E_1 and E_3 we use $|\mathbb{Z}\cap I^+_{c/d}|\le x/d+1\le 2x/d$ for $1\le d\le x$. Thereby,

$$E_3 \ll \sum_{1\le d\le x}\ \sum_{\substack{0\le c<d\\(c,d)=1}} \frac{x}{d} \ll x^2 \ll \frac{N}{\log^2 N} \ll \frac{\varphi(N)}{\log N}.$$

This estimate also shows $|\mathcal{F}^+\cap\mathbb{Z}|\ll\varphi(N)/\log N$ and, thus, the aforementioned bound $|\mathcal{F}\cap\mathbb{Z}|=O(\varphi(N)/\log N)$.

Further,

$$E_1 \le \sum_{1\le d\le x} \frac{2x}{d} \sum_{\substack{0\le c<d\\(c,d)=1}} |S(c,d)|.$$

From (1) we have

$$\sum_{\substack{0\le c<d\\(c,d)=1}} |S(c,d)| \ll d\log^2 d. \tag{10}$$

Accordingly,

$$E_1 \ll \sum_{1\le d\le x} x\log^2 d \ll x^2\log^2 N \ll N \ll \varphi(N)\log\log N$$

(for the last estimate see [11, p. 84]).

The most critical item is E_2; in particular, E_2 requires choosing x relatively small when N has many prime divisors (see (2)). By (4), $q=md-Nc$ for each $m\in I^+_{c/d}$, $(m,N)=(c,d)=1$. Therefore,

$$E_2 = \sum_{1\le q\le x}\ \sum_{\substack{0\le r<q\\(r,q)=1}} |S(r,q)|\cdot b_{r,q}, \tag{11}$$

where $b_{r,q}$ is the number of pairs (d,m), $1\le d\le x$, $1\le m\le N$, $(m,N)=1$ such that (4) holds for some c, $0\le c<d$, $(c,d)=1$, and suitable integers

j, k. The said equation shows, first, that $md \equiv q \bmod N$, whence because of $(m, N) = 1$ the condition $(d, N) = (q, N) = (d, q)$ follows; second, it gives an expression for N in terms of r and q, namely $N = -dr + jq$, so $dr \equiv -N \bmod q$. Accordingly,

$$b_{r,q} \leq \sum_{\substack{1 \leq d \leq x \\ (d,N)=(q,N)=(d,q) \\ dr \equiv -N \bmod q}} |\{1 \leq m < N : (m, N) = 1, md \equiv q \bmod N\}|.$$

If $\delta = (d, N) = (q, N) = (d, q)$, then the congruence $md \equiv q \bmod N$ has exactly δ solutions m, $0 \leq m < N$. Therefore,

$$b_{r,q} \leq \sum_{\delta|(q,N)} \delta \cdot |\{1 \leq d \leq x : dr \equiv -N \bmod q\}|.$$

Because $(r, q) = 1$, the congruence $dr \equiv -N \bmod q$ has exactly one solution d in each of the intervals $[1, q]$, $[q + 1, 2q]$, $[2q + 1, 3q]$, $\ldots$, and so it has at most $x/q + 1 \leq 2x/q$ solutions in the interval $[1, x]$ (in view of (11), only numbers $q \leq x$ are of interest). Thus,

$$b_{r,q} \leq \sum_{\delta|(q,N)} \frac{2\delta x}{q},$$

and, because of (11) and (10),

$$E_2 \ll \sum_{1 \leq q \leq x} \sum_{\substack{0 \leq r < q \\ (r,q)=1}} |S(r, q)| \sum_{\delta|(q,N)} \frac{\delta x}{q} \ll \sum_{1 \leq q \leq x} q \log^2 q \sum_{\delta|(q,N)} \frac{\delta x}{q}$$
$$\ll x \log^2 x \sum_{\delta|N} \delta \sum_{1 \leq q \leq x,\, \delta|q} 1 \ll x^2 \log^2 x \cdot \tau(N).$$

Our choice (2) of x shows $E_2 = O(N \log N) = O(\varphi(N) \log N \log\log N)$, which is also the contribution of all three error terms together.

3. The main term. In the following, d, q, and k are positive integers. The left side of (9) has the form

$$H(x) = \sum_{d \leq x} \sum_{\substack{0 \leq c < d \\ (c,d)=1}} \sum_{\substack{0 \leq m < N,\, (m,N)=1 \\ 1 \leq md - Nc \leq x}} \frac{N}{d(md - Nc)}. \tag{12}$$

The following proposition clearly contains (9).

PROPOSITION 1. *Let $\alpha > 0$, N tend to infinity, and $N^\alpha \leq x \leq N$. Then*

$$H(x) = \frac{6}{\pi^2} \varphi(N) \log^2 x + O(\varphi(N) \log N \log^3 \log N). \tag{13}$$

Proof. We use the standard sieving technique based on the Möbius function in order to remove the condition $(c,d)=1$ from (12). This gives

$$H(x)=\sum_{k\le x}\frac{\mu(k)}{k^2}\sum_{d\le x/k}\sum_{0\le c<d}\sum_{\substack{0\le m<N,\,(m,N)=1\\ 1\le md-Nc\le x/k}}\frac{N}{d(md-Nc)}.$$

With $q=md-Nc\in\mathbb{N}$ this reads

$$H(x)=N\sum_{k\le x}\frac{\mu(k)}{k^2}\sum_{d\le x/k}\sum_{q\le x/k}\frac{a_{d,q}}{dq},$$

where $a_{d,q}$ is the number of solutions m, $0\le m<N$, $(m,N)=1$, of the congruence $md\equiv q \bmod N$. Suppose $a_{d,q}\ne 0$. Because $(m,N)=1$, this can only happen if $(d,N)=(q,N)$.

Therefore, put $\delta=(d,N)=(q,N)$. Determining the exact value of $a_{d,q}$ is now an exercise in the Chinese remainder theorem; one obtains

$$a_{d,q}=\delta\prod_{\substack{p\mid\delta\\ p\nmid N/\delta}}(1-1/p)=\varphi(N)/\varphi(N/\delta)$$

(p runs through the respective primes). Accordingly,

$$H(x)=N\sum_{k\le x}\frac{\mu(k)}{k^2}\sum_{\delta\mid N}\frac{\varphi(N)}{\varphi(N/\delta)}\sum_{\substack{d,q\le x/k\\ (d,N)=(q,N)=\delta}}\frac{1}{dq}.$$

Here the innermost sum equals

$$\frac{1}{\delta^2}\sum_{\substack{d,q\le x/(k\delta)\\ (d,N/\delta)=(q,N/\delta)=1}}\frac{1}{dq}.$$

If we replace this sum by the same sum over $d,q\le x$, we obtain $H(x)=H_1(x)+R_1(x)$ with

$$H_1(x)=N\sum_{k\le x}\frac{\mu(k)}{k^2}\sum_{\delta\mid N}\frac{\varphi(N)}{\varphi(N/\delta)\delta^2}\sum_{\substack{d,q\le x\\ (d,N/\delta)=(q,N/\delta)=1}}\frac{1}{dq},$$

$$R_1(x)\ll N\sum_{k\le x}\frac{1}{k^2}\sum_{\delta\mid N}\frac{1}{\delta}\sum_{\substack{x/(k\delta)\le d\le x\\ 1\le q\le x}}\frac{1}{dq}$$

(where we have used $\varphi(N)\le\varphi(N/\delta)\delta$). Since

$$(14)\qquad \sum_{x/(k\delta)\le d\le x}\frac{1}{d}\ll 1+\log k\delta,\quad \sum_{\delta\mid N}\frac{1}{\delta}\ll\log\log N,\quad \sum_{\delta\mid N}\frac{\log\delta}{\delta}\ll\log^2\log N,$$

(see [11, p. 86], for the second item; the proof of the third one will be given in Lemma 1) we have

$$R_1(x) \ll N \log x \sum_{k \le x} \frac{1}{k^2} \sum_{\delta | N} \frac{1 + \log k + \log \delta}{\delta} \ll N \log N \log^2 \log N.$$

Because $N \ll \varphi(N) \log\log N$, the size of $R_1(x)$ is compatible with (13), so it suffices to consider $H_1(x)$.

Using $\sum_{k \le x} \mu(k)/k^2 = 6/\pi^2 + O(1/x)$ we obtain $H_1(x) = H_2(x) + R_2(x)$ with

$$H_2(x) = \frac{6N\varphi(N)}{\pi^2} \sum_{\delta | N} \frac{1}{\varphi(N/\delta)\delta^2} \sum_{\substack{d,q \le x \\ (d,N/\delta)=(q,N/\delta)=1}} \frac{1}{dq},$$

$$R_2(x) \ll \frac{N}{x} \sum_{\delta | N} \frac{1}{\delta} \sum_{d,q \le x} \frac{1}{dq} \ll \frac{N}{x} \log^3 N \ll \varphi(N).$$

Accordingly, we study $H_2(x)$ now. To this end we use

$$\sum_{d \le x,\, (d,N/\delta)=1} \frac{1}{d} = \frac{\delta\varphi(N/\delta)}{N} \log x + O(\log^2 \log N) \tag{15}$$

(see Lemma 1 below), which yields

$$H_2(x) = \frac{6N\varphi(N)}{\pi^2} \sum_{\delta | N} \frac{1}{\varphi(N/\delta)\delta^2} \left(\frac{\delta\varphi(N/\delta)}{N} \log x + O(\log^2 \log N) \right)^2.$$

Hence,

$$H_2(x) = \frac{6\varphi(N)}{\pi^2} \log^2 x + R_3(x)$$

with

$$R_3(x) \ll \varphi(N) \log x \log^2 \log N \sum_{\delta | N} \frac{1}{\delta} \ll \varphi(N) \log N \log^3 \log N.$$

This completes the proof. ■

The justification of the last entry of (14) and of (15) is afforded by

LEMMA 1. *Let N tend to infinity. Then*

$$\sum_{d | N} \frac{\log d}{d} \ll \log^2 \log N.$$

Let $\alpha > 0$, $x \ge N^\alpha$, and $n \le N$. If N tends to infinity, then

$$\sum_{d \le x,\, (d,n)=1} \frac{1}{d} = \frac{\varphi(n)}{n} \log x + O(\log^2 \log N)$$

with an O-constant independent of n and x.

Proof. We start with the first assertion. Let $N = \prod_p p^{e_p}$ be the decomposition of N into prime factors. It is not hard to check that

$$\sum_{d|N} \frac{\log d}{d} \le \sum_{p|N} \sum_{k=1}^{e_p} \frac{k \log p}{p^k} \sum_{d|N} \frac{1}{d} \ll \sum_{p|N} \frac{\log p}{p} \log\log N.$$

Combining this with

$$\sum_{p|N} \frac{\log p}{p} \ll \log\log N$$

we obtain the assertion. The proof of the last-mentioned estimate follows a pattern that can be found in [11, p. 14]. As to the second statement,

$$\sum_{d\le x,\,(d,n)=1} \frac{1}{d} = \sum_{k|n} \frac{\mu(k)}{k} \sum_{d\le x/k} \frac{1}{d} = \sum_{k|n,\,k\le x} \frac{\mu(k)}{k}\Big(\log\frac{x}{k} + O(1)\Big).$$

By a straightforward computation we see that this equals

$$\frac{\varphi(n)}{n}\log x + O\Big(\frac{\tau(n)\log x}{x} + \sum_{k|n} \frac{\log k}{k} + \log\log N\Big).$$

Since $x \ge N^\alpha$, the first assertion of the lemma yields the desired result. ∎

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Received on 29.6.2004
and in revised form on 21.7.2004 (4795)