ACTA ARITHMETICA
127.1 (2007)

# The torsion elements in $K_2$ of some local fields

by

Xuejun Guo (Nanjing)

**1. Introduction.** Let $F$ be a field. Then $K_2(F)$ can be generated by the Steinberg symbols. It is always an interesting problem to express elements in $K_2(F)$ in certain special forms. John Tate proved [7] that if a global field $F$ contains an $n$th primitive root of unity $\zeta_n$, where $n$ is a prime number, then every element of order $n$ in $K_2(F)$ is of the form $\{\zeta_n, a\}$ for some $a \in F^*$. Later Merkurjev and Suslin proved [4] that Tate's theorem holds for any field $F$ and any $n$, i.e., if a field $F$ contains an $n$th primitive root of unity $\zeta_n$, then every element of order $n$ in $K_2(F)$ is of the form $\{\zeta_n, a\}$ for some $a \in F^*$.

For a field $F \neq \mathbb{F}_2$, Jerzy Browkin proved [1] that if $n = 1,\ 2,\ 3,\ 4$ or $6$, then every element of order $n$ in $K_2(F)$ is of the form $\{a, X_n(a)\}$ for some $a \in F^*$ satisfying $X_n(a) \neq 0$. He conjectured that the same does not hold for other values of $n$. Later Hourong Qin proved [5] that Browkin's conjecture holds for $n = 5$ and $F = \mathbb{Q}$.

Let $F$ be a local field with a $q$-element residue field of characteristic $p$. In [9], Xu and Qin proved that if $n \mid (q-1)$, then every element of order $n$ is of the form $\{a, X_n(a)\}$ for some $a \in F^*$ satisfying $X_n(a) \neq 0$. Their proof is quite long. We will give a simple proof of their result in this paper. In [8], Xu and Qin proved that every element of order $p$ in $K_2(\mathbb{Q}_p(\zeta_p))$ is of the form $\{a, X_p(a)\}$ for some $a \in F^*$ satisfying $X_p(a) \neq 0$. Let $F$ be an unramified extension of $\mathbb{Q}_p$ and $E = F(\zeta_{p^m})$. We will prove that every element of order $p^k$ $(1 \leq k \leq m)$ in $K_2(E)$ is of the form $\{a, X_{p^k}(a)\}$ for some $a \in F^*$ satisfying $X_{p^k}(a) \neq 0$.

**2. Main results.** The following theorem is proved in [9]. Here we give a much simpler proof.

2000 *Mathematics Subject Classification*: 11R70, 19F15.

*Key words and phrases*: explicit reciprocity laws of Artin and Hasse.

Supported by NSFC 10401014, and partially supported by the Natural Science Foundation of Jiangsu province of China (BK 2005207).

THEOREM 2.1 ([9]). *Let $F$ be a local field, $\pi$ a uniformizer of $F$, $R$ the ring of integers in $F$ and $\mathbb{F}_q$ the residue field with characteristic $p$. If $n \mid (q-1)$, then every element of order $n$ in $K_2(F)$ is of the form $\{a, X_n(a)\}$ for some $a \in F^*$ satisfying $X_n(a) \neq 0$.*

*Proof.* Let $S = \{1 \le i \le n \mid (i,n) = 1\}$. Let $\zeta$ be a primitive $n$th root of unity in $F$. For every $i \in S$, let $a_i = \pi + \zeta^i \in F$. Let $T = \{\{a_i, X_n(a_i)\} \mid i \in S\}$.

Browkin proved [1] that $\{a_i, X_n(a_i)\}^n = 1$. And we know [3] that there are exactly $\varphi(n)$ elements of order $n$ in $K_2(F)$. Hence it is sufficient to prove that every element in $T$ is of order $n$ and the cardinality of $T$ is $\varphi(n)$. Let $\partial$ be the tame symbol:

$$\partial : K_2(\mathbb{Q}_p) \to k, \qquad \{a, b\} \mapsto (-1)^{v(a)v(b)} a^{v(b)}/b^{v(a)} \pmod{\pi},$$

where $v$ is the standard discrete valuation of $F$.

Since $X_n(a_i) = \prod_{j\in S}(\pi + \zeta^i - \zeta^j)$, we have $v(X_n(a_i)) = 1$. Hence $\partial(\{a_i, X_n(a_i)\}) = \zeta^i \pmod{\pi}$ for every $i \in S$. So the images of $\{a_i, X_n(a_i)\} \in T$ under the map $\partial$ are different and of order $n$. This implies that all the $\{a_i, X_n(a_i)\}$ $(i \in S)$ are different and of order $n$. Hence every element of order $n$ in $K_2(E)$ is of the form $\{a_i, X_n(a_i)\}$ for some $i \in S$. ■

THEOREM 2.2. *Let $F$ be an unramified extension of $\mathbb{Q}_p$ and $E = F(\zeta_{p^m})$, where $\zeta_{p^m}$ is a primitive $p^m$th root of unity. Then every element of order $p^k$ $(1 \le k \le m)$ in $K_2(E)$ is of the form $\{a, X_{p^k}(a)\}$ for some $a \in E^*$ satisfying $X_{p^k}(a) \neq 0$.*

*Proof.* If $p = 2$ and $m = 1$, then $\mathbb{Q}_2(\zeta_2) = \mathbb{Q}_2$ and every element of order 2 in $K_2(\mathbb{Q}_2)$ is of the form

$$\{-1, a\} = \{a-1, a\} = \{a-1, 1+(a-1)\} = \{a-1, X_2(a-1)\}.$$

Hence in the following, we assume $m > 1$ if $p = 2$.

It is sufficient to prove the theorem for $k = m$. As the group $(K_2(F))_{\mathrm{tors}}$ is cyclic of order $(q-1)p^m$, it follows that every element of it of order $p^k$ is the $p^{m-k}$th power of an element of order $p^m$. If we prove that every element in $K_2(F)$ of order $p^m$ has the form $\{a, X_{p^m}(a)\}$, then raising such elements to the power $p^{m-k}$ we get all elements of order $p^k$. We have the identity

$$X_{p^m}(x) = X_{p^k}(x^{p^{m-k}}).$$

Hence

$$\{a, X_{p^m}(a)\}^{p^{m-k}} = \{a^{p^{m-k}}, X_{p^k}(a^{p^{m-k}})\} = \{a_1, X_{p^k}(a_1)\},$$

with $a_1 = a^{p^{m-k}}$, and the claim follows.

Let $\zeta$ be a primitive $p^m$th root of unity, $\zeta_p = \zeta^{p^{m-1}}$ a primitive $p$th root of unity, $\pi = 1 - \zeta$ a uniformizer of $E$. Let $(\ ,\ )$ be the $p^m$th norm residue symbol for $E$. Let $\mathrm{Tr}_m$ be the trace from $E$ to $\mathbb{Q}_p$. Then by the law of

Artin–Hasse ([2, p. 94]) and Sharifi's formulae in Section 2 of [6], we have $(\pi, 1-\pi^{p^m}) = \zeta^C$, where

$$C = -(1+\delta)\bigg(\sum_{p^m\nmid i} \mathrm{Tr}_m\Big(\frac{\pi^i}{i}\,\frac{1}{1-\zeta^i}\Big) + \frac{p^m+1}{2}\sum_{p^m\mid i}\mathrm{Tr}_m\Big(\frac{\pi^i}{i}\Big)\bigg) \pmod{p^m\mathbb{Z}_p},$$

with $\delta = 0$ if $p$ is odd and $\delta = 2^{m-1}$ if $p = 2$. Let

$$A = \sum_{p^m\nmid i} \mathrm{Tr}_m\Big(\frac{\pi^i}{i}\,\frac{1}{1-\zeta^i}\Big), \qquad B = \frac{p^m+1}{2}\sum_{p^m\mid i}\mathrm{Tr}_m\Big(\frac{\pi^i}{i}\Big).$$

We will prove the following two assertions:

(1) $B \equiv 0 \pmod p$ if $p$ is odd, $B \equiv 1 \pmod p$ if $p = 2$;
(2) $A \equiv -1 \pmod p$ if $p$ is odd, $A \equiv 0 \pmod p$ if $p = 2$.

Let $b_i = \mathrm{Tr}_m(\pi^i/i)$. If $i = p^m$, then

$$\begin{aligned}
b_{p^m} &= \frac{1}{p^m}\,\mathrm{Tr}_m\bigg(\sum_{i=0}^{p^m}\binom{p^m}{i}(-\zeta)^i\bigg)\\
&= \frac{1}{p^m}\,\mathrm{Tr}_m\bigg(\sum_{j=0}^{p}\binom{p^m}{jp^{m-1}}(-\zeta)^{jp^{m-1}} + \sum_{p^{m-1}\nmid i}\binom{p^m}{i}(-\zeta)^i\bigg)\\
&= \frac{1}{p^m}\,\mathrm{Tr}_m\bigg(\sum_{j=0}^{p}\binom{p^m}{jp^{m-1}}(\zeta_p)^j(-1)^{jp^{m-1}}\bigg)\\
&= \Big(1-\frac{1}{p}\Big)(1+(-1)^p) + \frac{1}{p}\bigg(\sum_{j=1}^{p-1}\binom{p^m}{jp^{m-1}}(-1)^{jp^{m-1}+1}\bigg).
\end{aligned}$$

If $p$ is odd, then it is very easy to prove that $b_{p^m} = 0$. If $p = 2$, then $b_{2^m} = 1 - \frac{1}{2}\binom{2^m}{2^{m-1}}$. We will prove that $b_{2^m} \equiv 2 \pmod 4$. Since $\sum_{i=0}^{2^m}\binom{2^m}{i} = 2^{2^m}$ is divisible by 8 (note that we have assumed $m > 1$ in this case) and $4 \mid \binom{2^m}{i}$ if $i \neq 0,\ 2^{m-1}$, or $2^m$, we have

$$8 \,\Big|\, \sum_{i=0}^{2^m}\binom{2^m}{i} - 2\sum_{i=1}^{2^{m-1}-1}\binom{2^m}{i} = 2 + \binom{2^m}{2^{m-1}}.$$

Hence $\binom{2^m}{2^{m-1}} \equiv 6 \pmod 8$, which implies $b_{2^m} = 1 - \frac{1}{2}\binom{2^m}{2^{m-1}} \equiv 2 \pmod 4$. So $b_{p^m} = 0$ for odd $p$; and $b_{p^m} \equiv 2 \pmod 4$ for $p = 2$.

By the proof of Lemma 2 in [6], the different of $E/\mathbb{Q}_p$ is $\mathfrak{D} = (p^m/(1-\zeta_p)) = (\pi^{(mp^m-(m+1)p^{m-1})})$. Hence for any $\alpha \in \mathfrak{D}^{-1}$, $\mathrm{Tr}_m(\alpha) \in \mathbb{Z}_p$. Let $v$ be the standard discrete valuation of $E$, i.e., $v(\pi) = 1$. Then $v(\pi^i/i)$ is an increasing function of $i$. So if $p^m \mid i$, $i > p^m$ and $p$ is odd, then

$$v\left(\frac{1}{p^2}\,\frac{\pi^i}{i}\right) \geq 2p^m - (p^m - p^{m-1}) - 2(p^m - p^{m-1}) = 3p^{m-1} - p^m$$
$$\geq (m+1)p^{m-1} - mp^m.$$

If $p^m \mid i$, $i > p^m$ and $p = 2$, then

$$v\left(\frac{1}{p^2}\,\frac{\pi^i}{i}\right) \geq 2p^m - (m+1)(p^m - p^{m-1}) - 2(p^m - p^{m-1})$$
$$= (m+3)p^{m-1} - (m+1)p^m = (m+1)p^{m-1} - mp^m.$$

In both cases, we have

$$v\left(\frac{1}{p^2}\,\frac{\pi^i}{i}\right) \geq (m+1)p^{m-1} - mp^m.$$

So

$$\frac{1}{p^2}\,\frac{\pi^i}{i} \in \mathfrak{D}^{-1},$$

which implies $p^2 \mid b_i$. Hence (1) is proved.

Let

$$a_i = \frac{\pi^i}{i}\,\frac{1}{1-\zeta^i}.$$

Then

$$\sum_{p^m \nmid i} \mathrm{Tr}_m(a_i) = \sum_{t=0}^{m-1} \sum_{\substack{j=1 \\ p \nmid j}}^{\infty} \mathrm{Tr}_m(a_{p^t j}).$$

We will study this sum by the considering the following three cases.

(a) If $t < m-1$ and $p \nmid j$, then the ideal

$$\left(\frac{1}{p}\,a_{p^t j}\right) = \left(\pi^{p^t(j-1)+(t+1)(p^{m-1}-p^m)}\right)$$

is contained in the ideal $\mathfrak{D}^{-1} = (\pi^{-mp^m+(m+1)p^{m-1}})$. Hence

$$\mathrm{Tr}_m\left(\frac{1}{p}\,a_{p^t j}\right) \in \mathbb{Z}_p,$$

which implies $\mathrm{Tr}_m(a_{p^t j}) \in p\mathbb{Z}_p$ for every $t < m-1$.

(b) If $t = m-1$ and $j > 1$, $p \nmid j$, then the ideal

$$\left(\frac{1}{p}\,a_{p^{m-1} j}\right) = \left(\pi^{p^{m-1}(j-1)+m(p^{m-1}-p^m)}\right)$$

is contained in $\mathfrak{D}^{-1}$ for $p^{m-1}(j-1)+m(p^{m-1}-p^m) \geq (-mp^m+(m+1)p^{m-1})$. Hence $\mathrm{Tr}_m(a_{p^t j}) \in p\mathbb{Z}_p$ if $0 \leq t \leq m-1$ or $j > 1$, where $j$ satisfies $p \nmid j$.

(c) If $t = m-1$ and $j = 1$, then

$$\begin{aligned}
\mathrm{Tr}_m(a_{p^{m-1}}) &= \frac{1}{p^{m-1}} \mathrm{Tr}_{F(\zeta_p)/F}\left(\frac{1}{1-\zeta_p}\,(\mathrm{Tr}_{E/F(\zeta_p)}(1-\zeta)^{p^{m-1}})\right) \\
&= \frac{1}{p^{m-1}} \mathrm{Tr}_{F(\zeta_p)/F}\left(\frac{1}{1-\zeta_p}\left(\mathrm{Tr}_{E/F(\zeta_p)}\left(\sum_{i=0}^{p^{m-1}} \binom{p^{m-1}}{i}(-\zeta)^i\right)\right)\right) \\
&= \mathrm{Tr}_{F(\zeta_p)/F}\left(\frac{1+\zeta_p(-1)^{p^{m-1}}}{1-\zeta_p}\right),
\end{aligned}$$

for $\mathrm{Tr}_{E/F(\zeta_p)}((-\zeta)^i) = 0$ if $i$ is not 0 or $p^{m-1}$. Hence $\mathrm{Tr}_m(a_{p^{m-1}}) = 0$ if $p = 2$, and $\mathrm{Tr}_m(a_{p^{m-1}}) = p-1$ if $p$ is odd. Hence (2) is proved.

So we always have $C \equiv 1 \pmod p$, which implies $(\pi, 1-\pi^{p^m})$ is of order $p^m$. Since $(\pi, 1-\pi^{p^m}) = (\pi, 1-\pi^{p^{m-1}})(\pi, X_{p^m}(\pi))$ and $(\pi, 1-\pi^{p^{m-1}})$ is of order $p^{m-1}$, we know that $(\pi, X_{p^m}(\pi))$ is of order $p^m$. So $\{\pi, X_{p^m}(\pi)\}$ is of order $p^m$ in $K_2(E)$.

Note that the Galois group $\mathrm{Gal}(E/F)$ acts transitively on the set of all primitive $p^m$th roots of unity. And the $p^m$th norm residue symbol $(\ ,\ )$ is commutative with the Galois action. Hence $\{\{\sigma(\pi), X_{p^m}(\sigma(\pi))\} \mid \sigma \in \mathrm{Gal}(E/F)\} = \{a \in K_2(E) \mid a \text{ is of order } p^m\}$. ∎

**Acknowledgements.** The author is grateful to the referee for helpful comments, and to Hourong Qin and Kejian Xu for helpful discussion. Thanks also go to the Morningside Center of Mathematics in Beijing for hospitality.

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Department of Mathematics
Nanjing University
Nanjing, China
E-mail: guoxj@nju.edu.cn

*Received on 3.7.2006*
*and in revised form on 24.8.2006* (5231)