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# A quantitative Erdős–Fuchs theorem and its generalization

by

Yong-Gao Chen (Nanjing) and Min Tang (Wuhu)

**1. Introduction.** Let $k \geq 2$ be a fixed integer and let $A = \{a_1 \leq a_2 \leq \cdots\}$ be an infinite sequence of nonnegative integers. We write $F(z) = \sum_{a\in A} z^a$, $A(n) = \sum_{a\in A,\, a\leq n} 1$ (counting repetitions). For $n = 0, 1, 2, \ldots$ let $r_k(A, n)$ denote the number of solutions of

$$a_{i_1} + \cdots + a_{i_k} \leq n.$$

In 1956, Erdős and Fuchs [1] proved the following result:

Theorem A. *If $A$ is an infinite sequence of nonnegative integers, then*

$$r_2(A, n) = cn + o(n^{1/4}(\log n)^{-1/2})$$

*cannot hold for any constant $c > 0$.*

Jurkat (unpublished), and later Montgomery and Vaughan [5] improved the Erdős–Fuchs theorem by eliminating the log power on the right-hand side:

Theorem B. *If $A$ is an infinite sequence of nonnegative integers, then*

$$r_2(A, n) = cn + o(n^{1/4})$$

*cannot hold for any constant $c > 0$.*

Up to now, the Erdős–Fuchs theorem has been extended in various directions. For other related problems, see [2], [3], [4] and [6]. Continuing this work, Tang [7] recently proved the following result.

Theorem C. *If $A$ is an infinite sequence of nonnegative integers and $k > 2$, then*

$$r_k(A, n) = cn + o(n^{1/4})$$

*cannot hold for any constant $c > 0$.*

In this paper, we obtain a stronger version of the above results:

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THEOREM 1.1. *If $A$ is an infinite sequence of nonnegative integers and $k \geq 2$, then for any constant $c > 0$ and any $\varepsilon > 0$,*

$$|r_k(A,n) - cn| \geq (h(k) - \varepsilon)(cn)^{1/4}$$

*holds for infinitely many positive integers $n$, where*

$$h(k) = \frac{4}{9}(25\pi)^{-1/4}([k/2]!)^{3/2} \quad \text{for } 2 \mid k$$

*and*

$$h(k) = 4(25\pi)^{-1/4} 3^{(1-4k)/(2k-2)} \frac{k-1}{k+2}\left(1 + \frac{1}{k+1}\right)^{3k/(2k-2)} ([k/2]!)^{3k/(2k-2)}$$

*for $2 \nmid k$. In particular, if $2 \nmid k$ and $k \geq 9$, then*

$$h(k) > \frac{4}{9}(25\pi)^{-1/4}([k/2]!)^{3/2}.$$

By a simple calculation we have

$$\frac{4}{9}(25\pi)^{-1/4} = 0.149\ldots,$$
$$h(3) > 0.0432, \quad h(5) > 0.276, \quad h(7) > 2.13.$$

Thus we have the following corollary.

COROLLARY 1.2. *If $A$ is an infinite sequence of nonnegative integers and $k \geq 2$, then for any constant $c > 0$,*

$$|r_k(A,n) - cn| \geq 0.04([k/2]!)^{3/2}(cn)^{1/4}$$

*holds for infinitely many positive integers $n$.*

Throughout this paper, let $z = re(\alpha)$, where $e(\alpha) = e^{2\pi i\alpha}$, $r = 1 - 1/N$, $N$ is a large positive integer and $\alpha$ is a real number.

## 2. Lemmas

LEMMA 2.1. *Let $m$ and $N$ be two positive integers. Then*

$$\int_0^1 |1-z|^{-2}\left|\frac{1-z^m}{1-z}\right|^2 d\alpha \leq \frac{1}{2}m^2 N(1 + o_N(1)).$$

*Proof.* We have

$$\int_0^1 |1-z|^{-2}\left|\frac{1-z^m}{1-z}\right|^2 d\alpha = \int_0^1 |1-z|^{-2}\Big|\sum_{j=0}^{m-1} z^j\Big|^2 d\alpha \le m^2\int_0^1 \left|\frac{1}{1-z}\right|^2 d\alpha$$
$$= m^2\int_0^1 \sum_{u=0}^{\infty} r^u e(u\alpha)\cdot\sum_{v=0}^{\infty} r^v e(-v\alpha)\,d\alpha$$
$$= m^2\sum_{n=0}^{\infty} r^{2n} = \frac{1}{2}m^2 N(1+o_N(1)). \ \blacksquare$$

Lemma 2.2. *Let $0<v<1$ and $\beta>0$. Then*

$$\Big|\sum_{n=0}^{\infty} n^\beta v^n - \Gamma(\beta+1)(-\log v)^{-\beta-1}\Big| \le e^{-\beta}\beta^\beta(-\log v)^{-\beta}.$$

*Proof.* Define $f(x)=x^\beta v^x$ $(x\ge 0)$. Then

$$f'(x)=\beta x^{\beta-1}v^x + x^\beta v^x\log v = 0 \ \Leftrightarrow\ x=-\beta(\log v)^{-1}.$$

It is clear that $f(x)$ is increasing for $0\le x\le -\beta(\log v)^{-1}$ and $f(x)$ is decreasing for $x\ge -\beta(\log v)^{-1}$. Let $k$ be the integer with $k\le -\beta(\log v)^{-1} < k+1$ and $b=-\beta(\log v)^{-1}$. Thus

$$f(n)\le \int_n^{n+1} f(x)\,dx, \qquad 0\le n<k,$$
$$f(k)\le \int_k^{b} f(x)\,dx + (k+1-b)f(b),$$
$$f(k+1)\le \int_b^{k+1} f(x)\,dx + (b-k)f(b),$$
$$f(n)\le \int_{n-1}^{n} f(x)\,dx, \qquad n>k+1,$$
$$f(n)\ge \int_{n-1}^{n} f(x)\,dx, \qquad 0<n\le k,$$
$$f(n)\ge \int_{n}^{n+1} f(x)\,dx, \qquad n\ge k+1.$$

Hence

$$\sum_{n=0}^{\infty} n^\beta v^n = \sum_{n=0}^{\infty} f(n) \le \int_0^{\infty} f(x)\,dx + f(b),$$

$$\sum_{n=0}^{\infty} n^\beta v^n = \sum_{n=0}^{\infty} f(n) \ge \int_0^\infty f(x)\,dx - \int_k^{k+1} f(x)\,dx \ge \int_0^\infty f(x)\,dx - f(b).$$

So

$$\Big|\sum_{n=0}^{\infty} n^\beta v^n - \int_0^\infty x^\beta v^x\,dx\Big| \le f(b).$$

Since

$$\int_0^\infty x^\beta v^x\,dx = \int_0^\infty x^\beta e^{x\log v}\,dx = \int_0^\infty (t(-\log v)^{-1})^\beta e^{-t}(-\log v)^{-1}\,dt$$
$$= (-\log v)^{-\beta-1}\int_0^\infty t^\beta e^{-t}\,dt = \Gamma(\beta+1)(-\log v)^{-\beta-1}$$

and

$$f(b) = b^\beta v^b = \beta^\beta(-\log v)^{-\beta}e^{-\beta} = e^{-\beta}\beta^\beta(-\log v)^{-\beta},$$

the proof is complete. ∎

Lemma 2.3. *Let $\beta > 0$ and $r = 1 - 1/N$, where $N$ is a large positive integer. Then*

$$\sum_{n=0}^{\infty} n^\beta r^{2n} = \Gamma(\beta+1)2^{-\beta-1}N^{\beta+1}(1+o_N(1)).$$

*Proof.* In Lemma 2.2, let $v = r^2$. Then

$$(-\log v)^{-\beta-1} = 2^{-\beta-1}N^{\beta+1}(1+o_N(1)), \qquad (-\log v)^{-\beta} = 2^{-\beta}N^\beta(1+o_N(1)),$$

and Lemma 2.2 yield the assertion. ∎

**3. Proof of Theorem 1.1.** Suppose that there exists an infinite sequence $A = \{a_1 \le a_2 \le \cdots\}$ of nonnegative integers, $k \ge 2$, $c > 0$, $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that $|r_k(A,n) - cn| < (h(k) - \varepsilon_0)(cn)^{1/4}$ for all $n \ge n_0$. By the assumption and

$$A^k(M) \ge \sum_{a_{i_1}+\cdots+a_{i_k}\le M} 1 = r_k(A,M),$$

we have

$$A(M) \ge \sqrt[k]{cM}(1+o_M(1)). \tag{3.1}$$

Let $\vartheta(n) = r_k(A,n) - cn$. Then, for $|z| < 1$, we have

$$\frac{1}{1-z}F^k(z) = \sum_{n=0}^{\infty} r_k(A,n)z^n = \frac{cz}{(1-z)^2} + \sum_{n=0}^{\infty}\vartheta(n)z^n.$$

That is,

$$F^k(z) = \frac{cz}{1-z} + (1-z)\sum_{n=0}^{\infty} \vartheta(n) z^n. \tag{3.2}$$

Using the idea of Jurkat, by differentiation of (3.2), we obtain

$$kF^{k-1}(z)F'(z) = \frac{c}{(1-z)^2} - \sum_{n=0}^{\infty} \vartheta(n) z^n + (1-z)\sum_{n=1}^{\infty} n\vartheta(n) z^{n-1}. \tag{3.3}$$

By (3.2), the assumption and Lemma 2.3 we have

$$\begin{aligned} F^k(r^2) &= \frac{cr^2}{1-r^2} + (1-r^2)\sum_{n=0}^{\infty} \vartheta(n) r^{2n} \\ &= \frac{c}{2}N(1+o_N(1)) + O\Big(\frac{1}{N}\sum_{n=0}^{\infty} n^{1/4} r^{2n}\Big) \\ &= \frac{c}{2}N(1+o_N(1)) + O\Big(\frac{1}{N}N^{5/4}\Big) = \frac{c}{2}N(1+o_N(1)). \end{aligned}$$

So

$$F(r^2) = \Big(\frac{c}{2}N\Big)^{1/k}(1+o_N(1)). \tag{3.4}$$

By (3.3), the assumption and Lemma 2.3 we have

$$\begin{aligned} &kF^{k-1}(r^2)F'(r^2) \\ &\quad = \frac{c}{(1-r^2)^2} - \sum_{n=0}^{\infty} \vartheta(n) r^{2n} + (1-r^2)\sum_{n=1}^{\infty} n\vartheta(n) r^{2n-2} \\ &\quad = \frac{c}{4}N^2(1+o_N(1)) + O\Big(\sum_{n=0}^{\infty} n^{1/4} r^{2n}\Big) + O\Big(\frac{1}{N}\sum_{n=0}^{\infty} n^{5/4} r^{2n}\Big) \\ &\quad = \frac{c}{4}N^2(1+o_N(1)) + O(N^{5/4}) + O\Big(\frac{1}{N}N^{9/4}\Big) \\ &\quad = \frac{c}{4}N^2(1+o_N(1)). \end{aligned} \tag{3.5}$$

By (3.4) and (3.5) we have

$$F'(r^2) = \frac{1}{k}2^{-1-1/k}c^{1/k}N^{1+1/k}(1+o_N(1)). \tag{3.6}$$

Let $\delta$ be a positive constant which will be determined later, $m=[\delta c^{-1/2}N^{1/2}]$ and let

$$J = \int_0^1 |kF^{k-1}(z)F'(z)| \cdot \left|\frac{1-z^m}{1-z}\right|^2 d\alpha,$$
$$J_1 = c\int_0^1 \frac{1}{|1-z|^2} \cdot \left|\frac{1-z^m}{1-z}\right|^2 d\alpha,$$
$$J_2 = \int_0^1 \left|\sum_{n=0}^{\infty} \vartheta(n) z^n\right| \cdot \left|\frac{1-z^m}{1-z}\right|^2 d\alpha,$$
$$J_3 = \int_0^1 \left|(1-z)\sum_{n=1}^{\infty} n\vartheta(n) z^{n-1}\right| \cdot \left|\frac{1-z^m}{1-z}\right|^2 d\alpha.$$

By (3.3), we have

$$J \le J_1 + J_2 + J_3. \tag{3.7}$$

To obtain a good lower bound of $J$, we need the following estimates. For $l \ge 1$, from (3.4), (3.6), $0 < F(r^4) < F(r^2)$ and $0 < F'(r^4) < F'(r^2)$, we have

$$\begin{aligned}
&\sum_{i_1,\dots,i_l \text{ pairwise distinct}} a_{i_1} r^{2a_{i_1}+\dots+2a_{i_l}} \\
&\quad \ge \sum_{i_1,\dots,i_l} a_{i_1} r^{2a_{i_1}+\dots+2a_{i_l}} - \sum_{1\le u<v\le l} \sum_{\substack{i_1,\dots,i_l \\ i_u=i_v}} a_{i_1} r^{2a_{i_1}+\dots+2a_{i_l}} \\
&\quad = r^2F'(r^2)(F(r^2))^{l-1} - (l-1)r^4F'(r^4)(F(r^2))^{l-2} \\
&\qquad - \frac{1}{2}(l-1)(l-2)r^2F'(r^2)F(r^4)(F(r^2))^{l-3} \\
&\quad = \frac{1}{k}\, 2^{-1-l/k} c^{l/k} N^{1+l/k}(1+o_N(1)) + O(N^{1+(l-1)/k}) \\
&\quad = \frac{1}{k}\, 2^{-1-l/k} c^{l/k} N^{1+l/k}(1+o_N(1)).
\end{aligned} \tag{3.8}$$

We also have

$$\sum_{t=0}^{m-1} r^{2t-1} \ge mr^{2m} = m(1+o_N(1)) \tag{3.9}$$

and by (3.1),

$$\begin{aligned}
&\sum_{\substack{-a+t-s=0,\, a\in A \\ 0\le s,t\le m-1}} r^{a+t+s-1} = \sum_{\substack{-a+t-s=0,\, a\in A \\ 0\le s,t\le m-1}} r^{2t-1} \\
&= \sum_{t=0}^{m-1} r^{2t-1} A(t) \ge \sum_{\sqrt{m}\le t<m} r^{2m}(ct)^{1/k}(1+o_N(1)) \\
&\ge r^{2m} c^{1/k}(1+o_N(1)) \int_{\sqrt{m}-1}^{m-1} t^{1/k}\, dt = \frac{k}{k+1} c^{1/k} m^{1+1/k}(1+o_N(1)).
\end{aligned} \tag{3.10}$$

Now we can give a lower bound of $J$.

If $2\mid k$, let $k=2l$; then by (3.8) and (3.9) we have

$$J=\frac{k}{r}\int_0^1\Big|zF'(z)(F(z))^{l-1}(\overline{F(z)})^l\Big(\sum_{t=0}^{m-1}z^t\Big)\Big(\sum_{s=0}^{m-1}\overline{z}^s\Big)\Big|\,d\alpha \tag{3.11}$$
$$\begin{aligned}
&\geq\frac{k}{r}\Big|\int_0^1 zF'(z)(F(z))^{l-1}(\overline{F(z)})^l\Big(\sum_{t=0}^{m-1}z^t\Big)\Big(\sum_{s=0}^{m-1}\overline{z}^s\Big)\,d\alpha\Big|\\
&=k\sum_{\substack{a_{i_1}+\dots+a_{i_l}-a_{i_{l+1}}-\dots-a_{i_{2l}}+t-s=0\\ 0\leq s,t\leq m-1}}a_{i_1}r^{a_{i_1}+\dots+a_{i_{2l}}+t+s-1}\\
&\geq k\cdot l!\sum_{i_1,\dots,i_l\text{ pairwise distinct}}a_{i_1}r^{2a_{i_1}+\dots+2a_{i_l}}\sum_{t=0}^{m-1}r^{2t-1}\\
&\geq k\cdot l!\,\frac{1}{k}\,2^{-1-l/k}c^{l/k}N^{1+l/k}m(1+o_N(1))\\
&=[k/2]!2^{-3/2}c^{1/2}mN^{3/2}(1+o_N(1))\\
&=[k/2]!2^{-3/2}\delta N^2(1+o_N(1)).
\end{aligned}$$

If $2\nmid k$, let $k=2l+1$; then by (3.8) and (3.10) we have

$$J=\frac{k}{r}\int_0^1\Big|zF'(z)(F(z))^{l-1}(\overline{F(z)})^{l+1}\Big(\sum_{t=0}^{m-1}z^t\Big)\Big(\sum_{s=0}^{m-1}\overline{z}^s\Big)\Big|\,d\alpha \tag{3.12}$$
$$\begin{aligned}
&\geq\frac{k}{r}\Big|\int_0^1 zF'(z)(F(z))^{l-1}(\overline{F(z)})^{l+1}\Big(\sum_{t=0}^{m-1}z^t\Big)\Big(\sum_{s=0}^{m-1}\overline{z}^s\Big)\,d\alpha\Big|\\
&=k\sum_{\substack{a_{i_1}+\dots+a_{i_l}-a_{i_{l+1}}-\dots-a_{i_{2l+1}}+t-s=0\\ 0\leq s,t\leq m-1}}a_{i_1}r^{a_{i_1}+\dots+a_{i_{2l+1}}+t+s-1}\\
&\geq k\cdot l!\sum_{i_1,\dots,i_l\text{ pairwise distinct}}a_{i_1}r^{2a_{i_1}+\dots+2a_{i_l}}\sum_{\substack{-a+t-s=0\\ a\in A\\ 0\leq s,t\leq m-1}}r^{a+t+s-1}\\
&\geq k\cdot l!\,\frac{1}{k}\,2^{-1-l/k}c^{l/k}N^{1+l/k}\frac{k}{k+1}c^{1/k}m^{1+1/k}(1+o_N(1))\\
&=[k/2]!2^{-3/2+1/(2k)}\frac{k}{k+1}\delta^{1+1/k}N^2(1+o_N(1)).
\end{aligned}$$

Now we give upper bounds of $J_1, J_2, J_3$.

By Lemma 2.1,

$$J_1<\frac{1}{2}cm^2N(1+o_N(1))=\frac{1}{2}\delta^2N^2(1+o_N(1)). \tag{3.13}$$

By Cauchy's inequality, Parseval's formula, the assumption and Lemma 2.3

we have

$$J_2 \le m^2 \int_0^1 \Big|\sum_{n=0}^{\infty} \vartheta(n) z^n\Big|\, d\alpha \le m^2 \Big(\int_0^1 \Big|\sum_{n=0}^{\infty} \vartheta(n) z^n\Big|^2 d\alpha\Big)^{1/2} \tag{3.14}$$
$$= m^2 \Big(\sum_{n=0}^{\infty} |\vartheta(n)|^2 r^{2n}\Big)^{1/2} = O\Big(m^2 \Big(\sum_{n=0}^{\infty} n^{1/2} r^{2n}\Big)^{1/2}\Big)$$
$$= O(m^2 N^{3/4}) = O(N^{7/4}).$$

Similarly,

$$J_3 = \int_0^1 \Big|\sum_{n=1}^{\infty} n\vartheta(n) z^{n-1}\Big| \cdot \Big|\frac{1-z^m}{1-z}(1-z^m)\Big|\, d\alpha$$
$$\le \Big(\int_0^1 \Big|\sum_{n=1}^{\infty} n\vartheta(n) z^{n-1}\Big|^2 d\alpha\Big)^{1/2} \cdot \Big(\int_0^1 \Big|\frac{1-z^m}{1-z}(1-z^m)\Big|^2 d\alpha\Big)^{1/2}$$
$$= \Big(\sum_{n=1}^{\infty} n^2 \vartheta^2(n) r^{2n-2}\Big)^{1/2} \cdot \Big((1+r^{2m}) \sum_{j=0}^{m-1} r^{2j}\Big)^{1/2}$$
$$\le (2m)^{1/2} \Big(\sum_{n=1}^{\infty} n^2 \vartheta^2(n) r^{2n-2}\Big)^{1/2}.$$

Furthermore, by the assumption and Lemma 2.3, we have

$$\sum_{n=1}^{\infty} n^2 \vartheta^2(n) r^{2n-2} = \sum_{n=1}^{n_0-1} n^2 \vartheta^2(n) r^{2n-2} + \sum_{n=n_0}^{\infty} n^2 \vartheta^2(n) r^{2n-2}$$
$$\le \sum_{n=1}^{n_0-1} n^2 \vartheta^2(n) r^{2n-2} + (h(k)-\varepsilon_0)^2 c^{1/2} \sum_{n=n_0}^{\infty} n^{5/2} r^{2n-2}$$
$$\le \sum_{n=1}^{n_0-1} n^2 \vartheta^2(n) r^{2n-2} + (h(k)-\varepsilon_0)^2 c^{1/2} r^{-2} \sum_{n=0}^{\infty} n^{5/2} r^{2n}$$
$$\le \Gamma(7/2) 2^{-7/2} (h(k)-\varepsilon_0)^2 c^{1/2} N^{7/2} (1+o_N(1))$$
$$\le \frac{15\sqrt{\pi}}{64\cdot\sqrt{2}} (h(k)-\varepsilon_0)^2 c^{1/2} N^{7/2} (1+o_N(1)).$$

Thus

$$J_3 \le \frac{1}{8}\sqrt{15}(2\pi)^{1/4} (h(k)-\varepsilon_0) c^{1/4} m^{1/2} N^{7/4} (1+o_N(1)) \tag{3.15}$$
$$= \frac{1}{8}\sqrt{15}(2\pi)^{1/4} (h(k)-\varepsilon_0) \delta^{1/2} N^2 (1+o_N(1)).$$

Case 1: $2 \mid k$. By (3.7), (3.11) and (3.13)–(3.15) we have

$$[k/2]!2^{-3/2}\delta N^2 \le \frac{1}{2}\delta^2 N^2 + O(N^{7/4}) + \frac{1}{8}\sqrt{15}(2\pi)^{1/4}(h(k)-\varepsilon_0)\delta^{1/2}N^2 + o(N^2).$$

Dividing by $N^2$ and letting $N \to \infty$, we have

$$[k/2]!2^{-3/2}\delta \le \frac{1}{2}\delta^2 + \frac{1}{8}\sqrt{15}(2\pi)^{1/4}(h(k)-\varepsilon_0)\delta^{1/2}.$$

So

$$h(k)-\varepsilon_0 \ge 8(15)^{-1/2}(2\pi)^{-1/4}\left([k/2]!2^{-3/2}\delta^{1/2} - \frac{1}{2}\delta^{3/2}\right).$$

Taking

$$\delta = \frac{1}{3\sqrt{2}}[k/2]!,$$

we have

$$h(k)-\varepsilon_0 \ge \frac{4}{9}(25\pi)^{-1/4}([k/2]!)^{3/2} = h(k),$$

a contradiction.

Case 2: $2 \nmid k$. By (3.7), (3.12) and (3.13)–(3.15) we have

$$[k/2]!2^{-3/2+1/(2k)}\frac{k}{k+1}\delta^{1+1/k}N^2 \le \frac{1}{2}\delta^2 N^2 + O(N^{7/4}) + \frac{1}{8}\sqrt{15}(2\pi)^{1/4}(h(k)-\varepsilon_0)\delta^{1/2}N^2 + o(N^2).$$

Dividing by $N^2$ and letting $N \to \infty$, we have

$$[k/2]!2^{-3/2+1/(2k)}\frac{k}{k+1}\delta^{1+1/k} \le \frac{1}{2}\delta^2 + \frac{1}{8}\sqrt{15}(2\pi)^{1/4}(h(k)-\varepsilon_0)\delta^{1/2}.$$

So

$$h(k)-\varepsilon_0 \ge 8(15)^{-1/2}(2\pi)^{-1/4}\left([k/2]!2^{-3/2+1/(2k)}\frac{k}{k+1}\delta^{1/2+1/k} - \frac{1}{2}\delta^{3/2}\right). \tag{3.16}$$

Taking

$$\delta = 3^{-k/(k-1)}\frac{1}{\sqrt{2}}\left(1+\frac{1}{k+1}\right)^{k/(k-1)}([k/2]!)^{k/(k-1)},$$

we have

$$\begin{aligned} h(k)-\varepsilon_0 &\ge 8(15)^{-1/2}(2\pi)^{-1/4}\left([k/2]!2^{-3/2+1/(2k)}\frac{k}{k+1}\delta^{1/2+1/k} - \frac{1}{2}\delta^{3/2}\right) \\ &= 4(25\pi)^{-1/4}3^{(1-4k)/(2k-2)}\frac{k-1}{k+2}\left(1+\frac{1}{k+1}\right)^{3k/(2k-2)}([k/2]!)^{3k/(2k-2)} \\ &= h(k), \end{aligned}$$

a contradiction. As a function of $\delta$, the right side of (3.16) has the largest value $h(k)$. Set

$$\delta_1 = \frac{1}{3\sqrt{2}}[k/2]!.$$

If $k \geq 9$, then $\sqrt{2}\,\delta_1 \geq 8$ and

$$\begin{aligned} h(k) &\geq 8(15)^{-1/2}(2\pi)^{-1/4}\left([k/2]!2^{-3/2+1/(2k)}\frac{k}{k+1}\delta_1^{1/2+1/k} - \frac{1}{2}\delta_1^{3/2}\right) \\ &> 8(15)^{-1/2}(2\pi)^{-1/4}\left([k/2]!2^{-3/2}\delta_1^{1/2} - \frac{1}{2}\delta_1^{3/2}\right) \\ &= \frac{4}{9}(25\pi)^{-1/4}([k/2]!)^{3/2}. \end{aligned}$$

This completes the proof of Theorem 1.1.

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Yong-Gao Chen
School of Mathematical Sciences
and Institute of Mathematics
Nanjing Normal University
Nanjing 210046, P.R. China
E-mail: ygchen@njnu.edu.cn

Min Tang
Department of Mathematics
Anhui Normal University
Wuhu 241000, P.R. China
E-mail: tmzzz2000@163.com

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