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On sums of seven cubes of almost primes

by

KOICHI KAWADA (Morioka)

1. Introduction. Shortly after the celebrated work of Vinogradov on the sum of three primes in 1937, Hua showed that every sufficiently large odd integer can be written as a sum of nine cubes of primes, and established an asymptotic formula for the number of such representations, taking advantage of Vinogradov's method on estimating exponential sums over primes (see Hua [12] for a proof). The corresponding result for the sum of eight or fewer cubes of primes seems to be still now beyond the grasp of the existing technology. In the series of papers [3] and [4], Brüdern came up with an approach to research in this direction, which is based on a combination of the circle method and the linear sieve. In particular, he succeeded in proving that every sufficiently large integer can be written as a sum of seven cubes of almost primes. To state this theorem precisely, we recall the familiar terminology from sieve theory. When a natural number n has at most r prime divisors counted according to multiplicity, n is called a P_r, or a P_r-*number*. Then Brüdern [4] established that every sufficiently large integer n can be written as

$$n = p^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + y^3, \tag{1.1}$$

where p is a prime, $x_1, \ldots, x_5$ are P_5-numbers, and y is a P_{69}. This result may be also regarded as an interesting refinement of Linnik's seven cube theorem.

As Brüdern mentioned in [2], various combinations of almost primes may be substituted in this theorem. Namely, for various choices of natural numbers $r_1, \ldots, r_7$, one may prove that every sufficiently large integer n can be written in the form

$$n = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3, \tag{1.2}$$

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where x_j is a P_{r_j} for $1 \le j \le 7$. As regards this kind of conclusions, the author takes a great interest in the following three questions:

(I) Make the maximum of $\{r_1, \dots, r_7\}$ as small as possible.
(II) Make $r_1 + \dots + r_7$ as small as possible.
(III) Make the number of primes in the representation (that is, the number of variables x_j with $r_j = 1$) as many as possible.

The purpose of this article is to discuss these problems.

Prior to the statement of our result, we make a relevant observation on solutions of the associated congruence

$$n \equiv x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 \pmod q,$$

namely, if, and only if, n is odd and $9 \nmid n$, this congruence has a solution with $(q, x_1x_2\cdots x_7) = 1$ for every natural number q (see Brüdern [4, p. 217]). This condition concerns our problem naturally, because if n violates it, then the numbers x_j in the representation (1.2) receive some restriction on their prime factors. In relation to this issue, we introduce the numbers $a_{n,2}$ and $a_{n,3}$ defined by

$$a_{n,2} = \begin{cases} 1 & \text{when } n \text{ is odd}, \\ 2 & \text{when } n \text{ is even}, \end{cases} \qquad a_{n,3} = \begin{cases} 1 & \text{when } 9 \nmid n, \\ 3 & \text{when } 9 \mid n. \end{cases} \tag{1.3}$$

These numbers $a_{n,2}$ and $a_{n,3}$ are defined so that for all natural numbers n and q, the congruence

$$n \equiv x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + (a_{n,2}x_6)^3 + (a_{n,3}x_7)^3 \pmod q$$

may have a solution with $(q, x_1x_2\cdots x_7) = 1$ (see Section 4 below). Under this notation, Brüdern [4] showed the aforementioned result on (1.1) with $y = a_{n,2}a_{n,3}y'$ for a P_{67}-number y', to be precise.

Now, by using the numbers $a_{n,2}$ and $a_{n,3}$, we may state the main theorem of this paper, which provides the best answers at present to questions (I) and (II) above.

THEOREM 1. *Every sufficiently large integer n can be written as*

$$n = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + (a_{n,2}x_6)^3 + (a_{n,3}x_7)^3, \tag{1.4}$$

with a P_4-number x_1, P_3-numbers $x_2, \dots, x_6$, and a P_2-number x_7. In particular, every sufficiently large integer can be written as a sum of seven cubes of P_4-numbers.

In this conclusion, we may actually restrict the variables x_2, x_3, x_4 and x_6 to the products of exactly three distinct primes, and x_7 to the product of two distinct primes. This shall be transparent from our proof below.

To prove Theorem 1, we basically follow the strategy of Brüdern [4], and we remark here on what we add to his proof. There are two points on the circle method, and one more on sieve methods. The first thing is our

Lemma 2 below. This lemma is within reach of the methods of Vaughan [18], but seems to be new. The second one is concerned with Vaughan's iterative method restricted to minor arcs. Brüdern [4] appealed to its simplified form in Vaughan [19], and we may gain by the method in the form of Vaughan [17]. These technical efforts enlarge the level of distribution in our application of the linear sieve. Finally, the third point is that we use the idea of switching principle due to Iwaniec [13] and Chen [10], instead of the weighted sieve employed by Brüdern [4].

One may regard Lemma 5 below as the essence of this work. This lemma may be easily applied to other combinations of almost primes. If one confines one variable in (1.4) to a prime as in the Theorem of Brüdern [4], then one may for example derive the following conclusion from the lemma.

THEOREM 2. *Every sufficiently large integer n can be written in the form* (1.4) *with a prime number x_1, a P_6-number x_2, and P_3-numbers $x_3, \ldots, x_7$.*

We then turn to question (III). After all, it seems hard for now to confine four variables to primes in the representation (1.2), but three variables can be restricted to primes. Actually, we point out that the following result had been implicitly obtained in literature.

THEOREM 3. *Every sufficiently large integer can be written as a sum of three cubes of primes and four cubes of natural numbers.*

This is in fact immediate from the work of Vaughan [16] and Brüdern [1]. For a large integer n, let N be the number of natural numbers of the form $n - p_1^3 - p_2^3 - p_3^3$ with primes p_j. Then, for any fixed $\varepsilon > 0$, the lower bound $N \gg n^{8/9-\varepsilon}$ follows by the method of Vaughan [16], while Brüdern [1] proved that, except for $O(n^{37/42+\varepsilon})$ possible exceptions, all natural numbers less than n can be written as a sum of four cubes of natural numbers. Since $8/9 > 37/42$, Theorem 3 follows.

Moreover, it is possible to confine the remaining four cubes of natural numbers in Theorem 3 to those of almost primes. In view of the argument of Brüdern [1], the best result in this direction would be obtained, presumably, via the ideas of vector sieve of Brüdern and Fouvry [5]. But it is possible to obtain a conclusion of this kind via the sieve procedure of this paper. To this end, however, one needs to make minor modifications to our Lemma 5 so that especially the lemma becomes compatible with application of the Theorem of Vaughan [16], and also to construct appropriate sets of almost primes according to the methods of Wooley [21] and [22]. These tasks require some additional work in practice, and thus, for now, we just announce the next theorem without proof. In the theorem, we include an example of a conclusion in which two variables are confined to primes, following the referee's suggestion.

Theorem 4. *Every sufficiently large integer* n *can be written in the form* (1.4) *with each of the following restrictions on the variables*:

(i) x_1, x_2 *and* x_3 *are prime numbers*, x_4 *is a* P_{23}*-number*, *and* x_5, x_6 *and* x_7 *are* P_9*-numbers.*

(ii) x_1 *and* x_2 *are prime numbers*, x_3, x_4 *and* x_5 *are* P_6*-numbers*, *and* x_6 *and* x_7 *are* P_5*-numbers.*

Here we pause to respond to an enquiry of the referee about sums of cubes of what is called smooth numbers. Since smooth numbers are natural numbers that have small prime factors only, the latter question may be regarded as the opposite one to our main topic, in a sense. Such a problem was recently discussed by Harcos [11] and Brüdern and Wooley [9], and in particular Brüdern and Wooley [9] proved that every large natural number n can be written as a sum of eight cubes of natural numbers, each of which has no prime factor exceeding $\exp(c(\log n \log\log n)^{1/2})$ with some positive constant c.

As for the sum of seven cubes, it follows from Theorem 1.2 of Wooley [21], without substantial difficulty, that every large integer n can be written in the form (1.2) with confining $x_1, \ldots, x_6$ to smooth numbers in the sense just mentioned. But if we focus on the largest prime factor of $x_1 x_2 \cdots x_7$ in the representation (1.2), then it seems that no non-trivial result has appeared in the literature hitherto. On the latter problem, we may provide the following conclusion, as a by-product of our Lemma 5 below.

Theorem 5. *Let* $\mathscr{A}(P,R)$ *denote the set of natural numbers up to* P *having no prime factor exceeding* R, *and put*

$$\xi = \frac{\sqrt{2833}-43}{41}, \qquad \phi = \frac{14\xi - 1}{8\xi + 21}, \qquad \theta = \frac{4(5 - 16\xi - \xi^2)}{8\xi + 21}.$$

Then for each sufficiently small positive number ε, *there exists an* η *with* $0 < \eta < \varepsilon$ *such that every sufficiently large integer* n *can be written in the form*

$$n = x^3 + (\varpi_1 x_1)^3 + (\varpi_2 x_2)^3 + (\varpi_3 x_3)^3 + y_1^3 + y_2^3 + y_3^3,$$

where $n^{\phi/3} < \varpi_j \le 2n^{\phi/3}$, $x_j \in \mathscr{A}(n^{(1-\phi)/3}, n^\eta)$, $y_j \in \mathscr{A}(n^{1/3}, n^\eta)$ *for* $j = 1$, 2 *and* 3, *and* x *is a natural number* $\le n^{1/3}$ *that has a divisor* d *with* $n^{\theta/3-\varepsilon} < d \le 2n^{\theta/3-\varepsilon}$.

In particular, *therefore*, *every large* n *can be written as the sum of seven cubes of natural numbers having no prime factor exceeding* $n^{(1-\theta)/3+\varepsilon}$.

We remark that $(1-\theta)/3$ is slightly smaller than 0.278413, and that the above value of ξ comes from the theorem of Wooley [22]. In fact, there is no difficulty to alter the proof of Lemma 5 below so that one can derive

Theorem 5 using the latter theorem of Wooley [22], and the proof of this theorem is therefore omitted.

We use the standard notation in number theory, but we mention the following. The letter p, with or without subscript, always denotes a prime number. We adopt the familiar convention on the letter ε, that is, each statement involving ε holds for any fixed, small positive value of ε. We write $e(\alpha) = \exp(2\pi i\alpha)$, and $\|\alpha\|$ denotes the distance from α to the nearest integer.

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2. Auxiliary integrals. Throughout what follows, P and M denote positive, large real numbers. We denote by $\mathscr{P}(M)$ the set of prime numbers ϖ satisfying

$$M < \varpi \le 2M, \qquad \varpi \equiv 2 \pmod 3.$$

In Sections 2–4, we suppose that $\mathscr{A}$ is a set of natural numbers up to $2P/M$, and that every number in $\mathscr{A}$ has no prime divisor in $\mathscr{P}(M)$. In the proof of Theorem 1, we shall take $\mathscr{A}$ as a certain set of P_2-numbers satisfying the latter restriction. We then define

$$\mathscr{A}_M = \{\varpi x : \varpi \in \mathscr{P}(M),\ x \in \mathscr{A},\ P < \varpi x \le 2P\}. \tag{2.1}$$

When $\mathscr{B}$ is a finite set of natural numbers, we write

$$g(\alpha; \mathscr{B}) = \sum_{x \in \mathscr{B}} e(x^3\alpha).$$

We also write

$$h(\alpha, \varpi) = \sum_{\substack{x \in \mathscr{A} \\ P/\varpi < x \le 2P/\varpi}} e((\varpi x)^3\alpha),$$

for $\varpi \in \mathscr{P}(M)$, so that one has

$$g(\alpha; \mathscr{A}_M) = \sum_{\varpi \in \mathscr{P}(M)} h(\alpha, \varpi). \tag{2.2}$$

Further we set

$$I = \int_0^1 |g(\alpha; \mathscr{A})|^6 \, d\alpha.$$

We begin with providing a couple of auxiliary mean value estimates. These are minor modifications of Lemmata 2 and 3 of Vaughan [19], suitable for our later argument.

Lemma 1. *If* $M \le P^{1/7}$, *then*

$$\int_0^1 |g(\alpha;\mathscr{A}_M)|^6\,d\alpha \ll P^{3+\varepsilon}M^2 + P^{7/6+\varepsilon}M^{3/2}I^{2/3}.$$

Proof. On putting

$$I_0(\varpi_1,\ldots,\varpi_6) = \int_0^1 \prod_{j=1}^6 h((-1)^j\alpha,\varpi_j)\,d\alpha,$$

by (2.2) we have

$$\int_0^1 |g(\alpha;\mathscr{A}_M)|^6\,d\alpha = \sum_{\varpi_1,\ldots,\varpi_6\in\mathscr{P}(M)} I_0(\varpi_1,\ldots,\varpi_6). \tag{2.3}$$

By using Hölder's inequality, and then by considering the underlying diophantine equations, one finds that

$$\begin{aligned}\int_0^1 \prod_{j=1}^6 |h(\alpha,\varpi_j)|\,d\alpha &\le \prod_{j=1}^6 \Big(\int_0^1 |h(\alpha,\varpi_j)|^6\,d\alpha\Big)^{1/6} \\ &\le \prod_{j=1}^6 \Big(\int_0^1 |g(\varpi_j^3\alpha;\mathscr{A})|^6\,d\alpha\Big)^{1/6} = I,\end{aligned} \tag{2.4}$$

whence $|I_0(\varpi_1,\ldots,\varpi_6)| \le I$. Also, in view of the underlying diophantine equations and the assumption that $\mathscr{A} \subset [1,2P/M]$, it follows by Hua's inequality ([20, Lemma 2.5]) that

$$I \le \Big(\int_0^1 |g(\alpha;\mathscr{A})|^4\,d\alpha\Big)^{1/2}\Big(\int_0^1 |g(\alpha;\mathscr{A})|^8\,d\alpha\Big)^{1/2} \ll \Big(\frac{P}{M}\Big)^{7/2+\varepsilon}. \tag{2.5}$$

Therefore it is easily seen that

$$\sum_{\substack{\varpi_1,\ldots,\varpi_6\in\mathscr{P}(M)\\ \mathrm{card}\{\varpi_1,\ldots,\varpi_6\}\le 2}} I_0(\varpi_1,\ldots,\varpi_6) \ll M^2 I \ll P^{7/6+\varepsilon}M^{5/6}I^{2/3}. \tag{2.6}$$

Next, when i and j are integers with $1 \le i < j \le 6$, write $\mathscr{W}_{i,j}$ for the set of sextuplets $(\varpi_1,\ldots,\varpi_6)\in\mathscr{P}(M)^6$ such that for every k satisfying $1\le k\le 6$, $k\ne i$ and $k\ne j$, one has $\varpi_k\ne\varpi_i$ and $\varpi_k\ne\varpi_j$. A simple combinatorial argument reveals that whenever $\mathrm{card}\{\varpi_1,\ldots,\varpi_6\}\ge 3$, one can choose two from the latter six primes that are different from the remaining four, in other words, $(\varpi_1,\ldots,\varpi_6)$ belongs to some $\mathscr{W}_{i,j}$. Since $I_0(\varpi_1,\ldots,\varpi_6)$

is always non-negative as the number of solutions of a certain diophantine equation, one thus finds that

$$\sum_{\substack{\varpi_1,\dots,\varpi_6\in\mathscr{P}(M)\\ \mathrm{card}\{\varpi_1,\dots,\varpi_6\}\ge 3}} I_0(\varpi_1,\dots,\varpi_6) \le \sum_{1\le i<j\le 6}\ \sum_{(\varpi_1,\dots,\varpi_6)\in\mathscr{W}_{i,j}} I_0(\varpi_1,\dots,\varpi_6).$$

For $(\varpi_1,\dots,\varpi_6)\in\mathscr{W}_{i,j}$, we denote the four primes ϖ_k in the sextuplet with $k\ne i,j$ by $\varpi_{k_1},\dots,\varpi_{k_4}$, and observe that the inner sum above does not exceed

$$\sum_{\varpi_{k_1},\dots,\varpi_{k_4}\in\mathscr{P}(M)} \int_0^1 |h(\alpha,\varpi_{k_1})\cdots h(\alpha,\varpi_{k_4})| \times \Big|\sum_{\substack{\varpi_i\in\mathscr{P}(M)\\ \varpi_i\ne\varpi_{k_1},\dots,\varpi_{k_4}}} h(\alpha,\varpi_i)\Big|\Big|\sum_{\substack{\varpi_j\in\mathscr{P}(M)\\ \varpi_j\ne\varpi_{k_1},\dots,\varpi_{k_4}}} h(\alpha,\varpi_j)\Big|\,d\alpha.$$

Also one has $|h(\alpha,\varpi_{k_1})\cdots h(\alpha,\varpi_{k_4})| \le |h(\alpha,\varpi_{k_1})|^4+\dots+|h(\alpha,\varpi_{k_4})|^4$. Therefore, by symmetry, and by considering the underlying diophantine equation together with the assumption on the set $\mathscr{A}$, we deduce that

$$\sum_{\substack{\varpi_1,\dots,\varpi_6\in\mathscr{P}(M)\\ \mathrm{card}\{\varpi_1,\dots,\varpi_6\}\ge 3}} I_0(\varpi_1,\dots,\varpi_6) \ll \sum_{\varpi_1,\dots,\varpi_4\in\mathscr{P}(M)} \int_0^1 |h(\alpha,\varpi_1)|^4 \Big|\sum_{\substack{\varpi\in\mathscr{P}(M)\\ \varpi\ne\varpi_1,\dots,\varpi_4}} h(\alpha,\varpi)\Big|^2 d\alpha \ll M^3V, \tag{2.7}$$

where V denotes the number of solutions of

$$x_1^3-x_2^3=\varpi_1^3(y_1^3+y_2^3-y_3^3-y_4^3),$$

subject to

$$1\le x_1,x_2\le 2P,\quad \varpi_1\in\mathscr{P}(M),\quad \varpi_1\nmid x_1x_2,\quad y_1,\dots,y_4\in\mathscr{A}.$$

Note here that for every solution counted by V, one necessarily has $x_1\equiv x_2 \pmod{\varpi_1^3}$, because $\varpi_1\equiv 2 \pmod 3$. Thus an upper bound for V is essentially given by Lemma 3.7 of Vaughan [18]. To confirm this, we first modify the definition of $T_s(P,R,\theta)$ of Vaughan [18] for $k=s=3$, substituting our set $\mathscr{A}$ for the set $\mathscr{A}(P^{1-\theta},R)$ at (2.6) of [18]. One may easily observe that even after this alteration, Lemma 3.7 of [18] is still valid on simply replacing $S_3(Q,R)$ by our I, without substantial change in the proof. We then apply the latter lemma to $T_3(2^7P,2,\theta)$ (on setting $k=3$ and $M=P^\theta$) which is a trivial upper bound for our V. Note that the requirement for this application of Vaughan's lemma is $2M\le(2^7P)^{1/7}$, which is ensured by our assumption. In this way, the proof of Lemma 3.7 of Vaughan [18] may be straightforwardly modified to establish the inequality

$$V \ll P^{3+\varepsilon}M^{-1} + P^{7/6+\varepsilon}M^{-3/2}I^{2/3}. \tag{2.8}$$

The desired conclusion follows from (2.3) and (2.6)–(2.8). ∎

Lemma 2. *If $M \le P^{1/7}$, then*

$$\int_0^1 \Big(\sum_{\varpi\in\mathscr{P}(M)} |h(\alpha,\varpi)|\Big)^6 \, d\alpha \ll P^{3+\varepsilon}M^3 + P^{7/6+\varepsilon}M^2I^{2/3}.$$

Proof. We may basically follow the pattern of the proof of the previous lemma. In this case, we put

$$I_1(\varpi_1,\ldots,\varpi_6) = \int_0^1 \prod_{j=1}^6 |h(\alpha,\varpi_j)| \, d\alpha,$$

and observe that

$$\int_0^1 \Big(\sum_{\varpi\in\mathscr{P}(M)} |h(\alpha,\varpi)|\Big)^6 \, d\alpha = \sum_{\varpi_1,\ldots,\varpi_6\in\mathscr{P}(M)} I_1(\varpi_1,\ldots,\varpi_6). \tag{2.9}$$

But it is immediate from (2.4) and (2.5) that

$$\sum_{\substack{\varpi_1,\ldots,\varpi_6\in\mathscr{P}(M)\\ \mathrm{card}\{\varpi_1,\ldots,\varpi_6\}\le 2}} I_1(\varpi_1,\ldots,\varpi_6) \ll M^2 I \ll P^{7/6+\varepsilon}M^{5/6}I^{2/3}. \tag{2.10}$$

To estimate the remaining part, we first imitate the argument leading to (2.7), and then use Cauchy's inequality and consider the underlying diophantine equation. In this way, we may observe that

$$\sum_{\substack{\varpi_1,\ldots,\varpi_6\in\mathscr{P}(M)\\ \mathrm{card}\{\varpi_1,\ldots,\varpi_6\}\ge 3}} I_1(\varpi_1,\ldots,\varpi_6) \tag{2.11}$$
$$\ll M \sum_{\varpi_1,\ldots,\varpi_4\in\mathscr{P}(M)} \sum_{\substack{\varpi\in\mathscr{P}(M)\\ \varpi\ne\varpi_1,\ldots,\varpi_4}} \int_0^1 |h(\alpha,\varpi_1)|^4 |h(\alpha,\varpi)|^2 \, d\alpha \ll M^4 V_1,$$

where V_1 denotes the number of solutions of

$$\varpi^3(x_1^3 - x_2^3) = \varpi_1^3(x_3^3 + x_4^3 - x_5^3 - x_6^3) \tag{2.12}$$

subject to

$$x_1,\ldots,x_6 \in \mathscr{A}, \quad \varpi,\varpi_1 \in \mathscr{P}(M), \quad \varpi \ne \varpi_1. \tag{2.13}$$

Let V_2 and V_3 be the numbers of solutions counted by V_1 with $x_1 = x_2$ and $x_1 > x_2$, respectively, so that by symmetry one has

$$V_1 \ll V_2 + V_3. \tag{2.14}$$

When $x_1 = x_2$, the equation is equivalent to $x_3^3 + x_4^3 = x_5^3 + x_6^3$. Since $\mathscr{A}$ is a set of natural numbers up to $2P/M$, it is easy to see that

$$V_2 \ll (P/M)M^2(P/M)^{2+\varepsilon} \ll P^{3+\varepsilon}M^{-1}. \tag{2.15}$$

To estimate V_3, we work harder than in the proof of Lemma 3 of Vaughan [19], appealing to estimates in Vaughan [18]. We first note that the conditions (2.12) and (2.13) imply $x_1 \equiv x_2 \pmod{\varpi_1^3}$, since $\varpi_1 \equiv 2 \pmod 3$ and $\varpi_1 \nmid x_1x_2$ by our conventions on $\mathscr{P}(M)$ and $\mathscr{A}$. Next, put $h = (x_1 - x_2)/\varpi_1^3$ and $x = x_1 + x_2$ for solutions counted by V_3. Then $1 \le h \le 2PM^{-4}$ and $1 \le x \le 4P/M$, while the equation (2.12) gives the relation

$$\varpi^3 h(3x^2 + h^2\varpi_1^6) = 4(x_3^3 + x_4^3 - x_5^3 - x_6^3).$$

Therefore, on introducing the exponential sum

$$F(\alpha) = \sum_{\varpi\in\mathscr{P}(M)} \sum_{1\le h\le 2PM^{-4}} \sum_{1\le x\le 4P/M} e(3h\varpi^3x^2\alpha) \sum_{M<m\le 2M} e(h^3\varpi^3m^6\alpha),$$

we may find that

$$V_3 \le \int_0^1 F(\alpha)|g(4\alpha;\mathscr{A})|^4\,d\alpha \le \Big(\int_0^1 |F(\alpha)|^3\,d\alpha\Big)^{1/3} I^{2/3}. \tag{2.16}$$

To estimate $F(\alpha)$, we consider the sums

$$F_1(\alpha) = \sum_{\varpi\in\mathscr{P}(M)} \sum_{1\le h\le 2PM^{-4}} \Big|\sum_{1\le x\le 4P/M} e(3h\varpi^3x^2\alpha)\Big|^2, \tag{2.17}$$

$$F_2(\alpha) = \sum_{\varpi\in\mathscr{P}(M)} \sum_{1\le h\le 2PM^{-4}} \Big|\sum_{M<m\le 2M} e(h^3\varpi^3m^6\alpha)\Big|^2. \tag{2.18}$$

Our treatment of $F_1(\alpha)$ is based on Lemma 3.1 of Vaughan [18], and to make the comparison easier, we temporarily write

$$P_1 = 2P/M, \quad H_1 = 2PM^{-4} = P_1/M^3, \quad Q_1 = 2PM^{-2} = P_1/M,$$

which, respectively, correspond to P, H and Q in the latter lemma of Vaughan [18]. Now according to Dirichlet's theorem ([20, Lemma 2.1]), for $\varpi \in \mathscr{P}(M)$, we take integers r and b such that

$$1 \le r \le 2P_1H_1, \quad (r, b) = 1, \quad |r\varpi^3\alpha - b| \le (2P_1H_1)^{-1}.$$

Then Lemma 3.1 of Vaughan [18] yields the estimate

$$\sum_{1\le h\le H_1} \Big|\sum_{1\le x\le 2P_1} e(3h\varpi^3x^2\alpha)\Big|^2 \ll \frac{P_1^{2+\varepsilon}H_1}{r + Q_1^3|r\varpi^3\alpha - b|} + P_1^{1+\varepsilon}H_1. \tag{2.19}$$

Our next trouble will be to play a simultaneous game with the rational approximations of α and of $\varpi^3\alpha$, with ϖ varying. To this end, we define q to be the least natural number such that $\|q\alpha\| \le (P_1H_1)^{-1}$ for each α. It is convenient to regard q as a function of α for a while. Note that one has always $q \le P_1 H_1$ by Dirichlet's theorem, and the integer a determined by $\|q\alpha\| = |q\alpha - a|$ is necessarily coprime to q. So if the above integer r does not exceed $P_1/20$, then

$$\begin{aligned}|ar\varpi^3 - qb| &\le r\varpi^3|q\alpha - a| + q|r\varpi^3\alpha - b| \\ &\le (P_1/20)(2M)^3(P_1H_1)^{-1} + P_1H_1(2P_1H_1)^{-1} \\ &= 2M^7P^{-1}/5 + 1/2 < 1,\end{aligned}$$

since $M \le P^{1/7}$, whence $\varpi^3 a/q = b/r$. In this case, therefore, we have

$$r = q/(q,\varpi^3), \qquad |r\varpi^3\alpha - b| = \|q\alpha\|\varpi^3/(q,\varpi^3).$$

Unless $r \le P_1/20$, on the other hand, the right hand side of (2.19) is $O(P_1^{1+\varepsilon}H_1)$. Hence, on putting

$$\xi(q) = \sum_{\varpi\in\mathscr{P}(M)} (q,\varpi^3), \qquad \Psi(\alpha) = (q + (P/M)^3\|q\alpha\|)^{-1},$$

we deduce from (2.17) and (2.19) that

$$F_1(\alpha) \ll P_1^{2+\varepsilon}H_1\xi(q)\Psi(\alpha) + P_1^{1+\varepsilon}H_1M. \tag{2.20}$$

As regards $\xi(q)$, we have

$$\xi(q) = \sum_{\substack{\varpi\in\mathscr{P}(M)\\ \varpi\nmid q}} 1 + \sum_{\substack{\varpi\in\mathscr{P}(M)\\ \varpi\mid q}} (q,\varpi^3) \ll M^3\log(2q). \tag{2.21}$$

We also see that $\xi(q) \ge \operatorname{card}\mathscr{P}(M) \gg M(\log M)^{-1}$, by the prime number theorem for arithmetic progressions. Therefore the right hand side of (2.20) is $O(P_1^{1+\varepsilon}H_1M)$ when $q > P_1M^2$, and is $O(P_1^{2+\varepsilon}H_1\xi(q)\Psi(\alpha))$ when $\Psi(\alpha) > P_1^{-1}$. Accordingly we define

$$F_3(\alpha) = \begin{cases} P_1^2H_1\xi(q)\Psi(\alpha) & \text{when } q \le P_1M^2, \\ 0 & \text{when } q > P_1M^2, \end{cases}$$

$$F_4(\alpha) = \begin{cases} P_1H_1M & \text{when } \Psi(\alpha) \le P_1^{-1}, \\ 0 & \text{when } \Psi(\alpha) > P_1^{-1}, \end{cases}$$

and note that the conditions $q > P_1M^2$ and $\Psi(\alpha) > P_1^{-1}$ are incompatible. Then we may express the inequality (2.20) as

$$F_1(\alpha) \ll P^\varepsilon(F_3(\alpha) + F_4(\alpha)). \tag{2.22}$$

Turning to $F_2(\alpha)$, we put $l = h\varpi$ in the definition (2.18). Since every natural number l has at most $O(\log l)$ prime divisors, we see that

$$F_2(\alpha) \ll (\log P)\sum_{1\le l\le 4PM^{-3}}\Big|\sum_{M<m\le 2M} e(l^3m^6\alpha)\Big|^2.$$

The sum on the right hand side can be estimated by Lemma 3.4 of Vaughan [18], on setting $H = 4PM^{-3}$, $R = 2$ and $X = P_1H_1$ in that lemma. Making use of $\Psi(\alpha)$ defined above, one may eventually find that the lemma of Vaughan yields the estimate

$$F_2(\alpha) \ll P^{1+\varepsilon}M^{-1}\Psi(\alpha)^{1/3} + P^{1+\varepsilon}M^{-2}. \tag{2.23}$$

Now we write

$$V_4 = \int_0^1 (F_3(\alpha)F_2(\alpha))^{3/2}\,d\alpha, \qquad V_5 = \int_0^1 (F_4(\alpha)F_2(\alpha))^{1/2}|F(\alpha)|^2\,d\alpha.$$

Since $|F(\alpha)| \le F_1(\alpha)^{1/2}F_2(\alpha)^{1/2}$, we deduce from (2.22) that

$$\int_0^1 |F(\alpha)|^3\,d\alpha \ll P^\varepsilon \int_0^1 (F_3(\alpha)F_2(\alpha))^{1/2}|F(\alpha)|^2\,d\alpha + P^\varepsilon V_5.$$

But the first term on the right hand side does not exceed

$$P^\varepsilon V_4^{1/3}\Big(\int_0^1 |F(\alpha)|^3\,d\alpha\Big)^{2/3}$$

by Hölder's inequality, so ultimately we have

$$\int_0^1 |F(\alpha)|^3\,d\alpha \ll P^\varepsilon(V_4 + V_5). \tag{2.24}$$

We first handle V_5. On the one hand, we deduce from (2.23) and the definition of $F_4(\alpha)$ that

$$F_4(\alpha)F_2(\alpha) \ll P_1H_1M \cdot P^{1+\varepsilon}M^{-2} \ll P^{3+\varepsilon}M^{-6}$$

for all $\alpha \in [0,1]$, because unless $F_4(\alpha) = 0$, one has $\Psi(\alpha) \le P_1^{-1} \le M^{-3}$ by our assumption. On the other hand, by considering the number of solutions of the underlying diophantine equation, we may see that

$$\int_0^1 |F(\alpha)|^2\,d\alpha \ll P^\varepsilon M(PM^{-4})(P/M)M = P^{2+\varepsilon}M^{-3}.$$

These two estimates lead to the bound

$$V_5 \ll (P^{3+\varepsilon}M^{-6})^{1/2}P^{2+\varepsilon/2}M^{-3} = P^{7/2+\varepsilon}M^{-6}. \tag{2.25}$$

To evaluate V_4, we use (2.23), recall the definitions, and assort the integral by the value of q. Then we see straightforwardly that (discarding the definition of q as the function of α at this point)

$$V_4 \ll P^{6+\varepsilon}M^{-12} \sum_{1\le q\le PM} \xi(q)^{3/2}(M^{3/2}I_2(q) + I_{3/2}(q)), \tag{2.26}$$

where we put, for $\sigma = 2$ and $3/2$,

$$I_\sigma(q) = \int_0^1 (q + (P/M)^3\|q\alpha\|)^{-\sigma}\, d\alpha.$$

For these values of σ, we have simply

$$I_\sigma(q) \le 2q \int_0^{(2q)^{-1}} (q + (P/M)^3 q\alpha)^{-\sigma}\, d\alpha \ll q^{1-\sigma}(P/M)^{-3}. \tag{2.27}$$

Next, applying Hölder's inequality to the expression for $\xi(q)$ that appeared in (2.21), we have

$$\xi(q)^{3/2} \ll M^{3/2} + (\log P)^{1/2} \sum_{\substack{\varpi\in\mathscr{P}(M)\\ \varpi\mid q}} (q,\varpi^3)^{3/2}.$$

So, in view of the trivial estimate $(q,\varpi^3)^{3/2} \le (q,\varpi^3)(\varpi^3)^{1/2}$, it follows that

$$\xi(q)^{3/2} \ll (\log P)M^{3/2} \sum_{\varpi\in\mathscr{P}(M)} (q,\varpi^3).$$

Thus simple calculation reveals, for $\sigma = 2$ and $3/2$, that

$$\begin{aligned}\sum_{1\le q\le PM} \xi(q)^{3/2} q^{1-\sigma} &\ll M^{3/2}(\log P) \sum_{\varpi\in\mathscr{P}(M)} \sum_{1\le q\le PM} (q,\varpi^3) q^{1-\sigma} \\ &\ll M^{5/2}(PM)^{2-\sigma+\varepsilon}.\end{aligned} \tag{2.28}$$

By (2.26)–(2.28), we conclude that

$$V_4 \ll P^{3+\varepsilon} M^{-9}(M^4 + P^{1/2}M^3) \ll P^{7/2+\varepsilon} M^{-6}. \tag{2.29}$$

The proof of the lemma is now completed immediately by combining (2.9)–(2.11), (2.14)–(2.16), (2.24), (2.25) and (2.29). ∎

3. Minor arc estimates. The basis of this section is Vaughan's iterative method restricted to minor arcs, devised by Vaughan in [17, §§5–8]. This idea was summarised by Brüdern and Wooley in the shape of Lemma 4 of [8]. Our later argument shall require to generalise this abstracted lemma of Brüdern and Wooley [8] just slightly. In fact, Lemma 4 of Brüdern and Wooley [8] corresponds to the case $m = 1$ in the next lemma, but it should be noted that our lower limit for M is relaxed in comparison with the corresponding one in Lemma 4 of [8] by virtue of a trick of Vaughan (see the last inequality on p. 214 of Vaughan [19]).

Although one needs essentially nothing beyond the work of Vaughan [17] and [19] to prove the next lemma, we nonetheless present an account for the convenience of the reader.

LEMMA 3. *Let X and M be real numbers with $1 \le M \le X^{1/7}$, let $S : \mathbb{R} \to [0, \infty)$ be a Riemann integrable function of period* 1, *and for $\varpi \in \mathscr{P}(M)$, put*

$$f_\varpi(\alpha) = \sum_{\substack{X<x\le 2X \\ \varpi \nmid x}} e(x^3\alpha).$$

Moreover let m be a natural number, and denote by $\mathfrak{m}$ the set of real numbers $\alpha \in [0, 1)$ such that whenever q is a natural number with $\|qm\alpha\| \le M^3X^{-2}$, one has $q > XM^3$. Then

$$\sum_{\substack{\varpi\in\mathscr{P}(M) \\ \varpi\nmid m}} \int_{\mathfrak{m}} |f_\varpi(m\alpha)|^2 S(\varpi^3\alpha)\, d\alpha \ll X^{3/2+\varepsilon} M^{-5/2} \int_0^1 S(\alpha)\, d\alpha.$$

Proof. Let $\mathfrak{n}$ denote the set of real numbers $\alpha \in [0, 1)$ such that whenever q is a natural number with $\|qm\alpha\| \le M^3X^{-2}$, then $q > X/8$. If $\varpi \in \mathscr{P}(M)$ and $\|rm\alpha\varpi^3\| \le M^3X^{-2}$ for some integer r with $1 \le r \le X/8$, then α cannot belong to $\mathfrak{m}$ because $r\varpi^3 \le XM^3$. We therefore notice that whenever $\alpha \in \mathfrak{m}$ and $\varpi \in \mathscr{P}(M)$, one may write $\alpha\varpi^3 = \beta + k$ with $\beta \in \mathfrak{n}$ and an integer k with $0 \le k < \varpi^3$. This observation leads to the inequality

$$\int_{\mathfrak{m}} |f_\varpi(m\alpha)|^2 S(\varpi^3\alpha)\, d\alpha \le \varpi^{-3} \sum_{k=0}^{\varpi^3-1} \int_{\mathfrak{n}} \left| f_\varpi\left(\frac{m(\beta+k)}{\varpi^3}\right)\right|^2 S(\beta)\, d\beta,$$

for $\varpi \in \mathscr{P}(M)$. By the definition of the function f_ϖ, we see that

$$\varpi^{-3} \sum_{k=0}^{\varpi^3-1} \left| f_\varpi\left(\frac{m(\beta+k)}{\varpi^3}\right)\right|^2 = \sum\sum_{\substack{X<x,y\le 2X \\ \varpi\nmid xy \\ m(x^3-y^3)\equiv 0 \,(\mathrm{mod}\, \varpi^3)}} e(m(x^3-y^3)\beta/\varpi^3).$$

When $\varpi \equiv 2 \pmod 3$ and $\varpi \nmid m$, the congruence appearing in the summation condition is tantamount to $x \equiv y \pmod{\varpi^3}$, so that the double sum in question is

$$\sum_{\substack{r=1 \\ \varpi\nmid r}}^{\varpi^3} \Big| \sum_{\substack{X<x\le 2X \\ x\equiv r \,(\mathrm{mod}\, \varpi^3)}} e(mx^3\beta/\varpi^3)\Big|^2 \le \sum_{r=1}^{\varpi^3} \Big| \sum_{\substack{X<x\le 2X \\ x\equiv r \,(\mathrm{mod}\, \varpi^3)}} e(mx^3\beta/\varpi^3)\Big|^2 = \Psi_\varpi(\beta),$$

say. We thus find that

$$\sum_{\substack{\varpi\in\mathscr{P}(M) \\ \varpi\nmid m}} \int_{\mathfrak{m}} |f_\varpi(m\alpha)|^2 S(\varpi^3\alpha)\, d\alpha \le \int_{\mathfrak{n}} \sum_{\varpi\in\mathscr{P}(M)} \Psi_\varpi(\alpha) S(\alpha)\, d\alpha. \tag{3.1}$$

Also, denoting the real part of a complex number ζ by $\Re\zeta$, we have

$$\Psi_\varpi(\alpha) = 2\Re \sum\sum_{\substack{X<y<x\le 2X \\ x\equiv y \pmod{\varpi^3}}} e(m(x^3-y^3)\alpha/\varpi^3) + O(X)$$

for each $\varpi \in \mathscr{P}(M)$, and by the substitution $h = (x-y)/\varpi^3$ and $z = x+y$, the double sum on the right hand side becomes

$$\sum_{1\le h\le XM^{-3}} e\left(\frac{1}{4}\, m\alpha h^3\varpi^6\right) \sum_{\substack{2X+h\varpi^3<z\le 4X-h\varpi^3 \\ z\equiv h \pmod 2}} e\left(\frac{3}{4}\, m\alpha h z^2\right). \tag{3.2}$$

Concerning the last inner sum over z, we write

$$F(\beta,\gamma;h) = \sum_{\substack{2X<z\le 4X \\ z\equiv h \pmod 2}} e\left(\frac{3}{4}\,\beta z^2 - \gamma z\right), \qquad K(\gamma, r) = \sum_{2X+r<z\le 4X-r} e(\gamma z),$$

and $\eta = X^{-2}$. Then the inner sum at (3.2) may be expressed as

$$\int_\eta^{1-\eta} F(hm\alpha,\gamma;h)K(\gamma,h\varpi^3)\,d\gamma + O(1).$$

Accordingly it follows that

$$\begin{aligned} \sum_{\varpi\in\mathscr{P}(M)} \Psi_\varpi(\alpha) = 2\Re \int_\eta^{1-\eta} \sum_{1\le h\le XM^{-3}} F(hm\alpha,\gamma;h) \\ \times \sum_{\varpi\in\mathscr{P}(M)} K(\gamma,h\varpi^3) e\left(\frac{1}{4}\, m\alpha h^3\varpi^6\right) d\gamma + O(XM). \end{aligned} \tag{3.3}$$

Here we have

$$K(\gamma,h\varpi^3) = \frac{e(\gamma)}{e(\gamma)-1}\,(e(\gamma([4X]-h\varpi^3)) - e(\gamma([2X]+h\varpi^3))),$$

when $2h\varpi^3 \le [4X]-[2X]$, and $K(\gamma,h\varpi^3) = 0$ otherwise. Therefore, on defining

$$G_h(\varrho,\sigma) = \sum_{\substack{\varpi\in\mathscr{P}(M) \\ 2h\varpi^3\le[4X]-[2X]}} e\left(\frac{1}{4}\,\varrho\varpi^6 + \sigma\varpi^3\right),$$

one easily obtains the inequality

$$\sum_{\varpi\in\mathscr{P}(M)} K(\gamma,h\varpi^3) e\left(\frac{1}{4}\, m\alpha h^3\varpi^6\right) \ll \|\gamma\|^{-1} \sum_{\theta\in\{1,-1\}} |G_h(\alpha m h^3, \theta\gamma h)|,$$

whence by (3.3),

$$(3.4)\qquad \sum_{\varpi\in\mathscr{P}(M)} \Psi_\varpi(\alpha) \ll XM + (\log X)\sup_{\gamma,\theta\in\mathbb{R}} \sum_{1\le h\le XM^{-3}} |F(\alpha mh,\gamma;h)G_h(\alpha mh^3,\theta\gamma h)|.$$

We then derive from Lemma 7 of Vaughan [17] (regarding our αm as α in that lemma) that

$$\sup_{\substack{\alpha\in\mathfrak{n}\\ \gamma\in\mathbb{R}}} \sum_{1\le h\le XM^{-3}} |F(\alpha mh,\gamma;h)|^2 \ll (XM^{-3})X^{1+\varepsilon} = X^{2+\varepsilon}M^{-3}.$$

Also, as Brüdern and Wooley pointed out on p. 27 of [8], the proof of Lemma 8 of Vaughan [17] yields the bound

$$\sup_{\substack{\alpha\in\mathfrak{n}\\ \gamma\in\mathbb{R}}} \sum_{1\le h\le XM^{-3}} |G_h(\alpha mh^3,\gamma h)|^2 \ll X^\varepsilon(XM^{-3})M = X^{1+\varepsilon}M^{-2},$$

whenever $(XM^{-3})^{3/4}M^2 \le (XM^{-3})M$. Hence, by (3.4) and Cauchy's inequality, we have

$$\sup_{\alpha\in\mathfrak{n}} \sum_{\varpi\in\mathscr{P}(M)} \Psi_\varpi(\alpha) \ll X^{3/2+\varepsilon}M^{-5/2},$$

and the conclusion of the lemma is immediate from (3.1). ∎

The next lemma is our variant of Lemma 10 of Vaughan [17].

Lemma 4. *Let P and M be large real numbers, and d be a natural number satisfying $P^{1/10} < M \le (P/d)^{1/7}$. Let $\mathscr{A}_M$ and I be as in the preamble to Lemma* 1, *and define the function*

$$f(\alpha;d) = \sum_{P/d<x\le 2P/d} e(d^3x^3\alpha).$$

Moreover, denote by $\mathfrak{m}_d$ the set of real numbers $\alpha\in[0,1)$ such that whenever q is a natural number with $\|qd^3\alpha\| \le M^3(d/P)^2$, one has $q > PM^3/d$. Then

$$\int_{\mathfrak{m}_d} |f(\alpha;d)^2 g(\alpha;\mathscr{A}_M)^6|\,d\alpha \ll (PM/d)^{3/2+\varepsilon}(P^3+MI).$$

Proof. Recalling the expression (2.2) for $g(\alpha;\mathscr{A}_M)$, we first eliminate from it the primes ϖ dividing d. Note that the number of such primes is $O(\log P)$, and that $|f(\alpha;d)| \ll (P/d)^{3/4+\varepsilon}$ for $\alpha\in\mathfrak{m}_d$ by Weyl's inequality

([20, Lemma 2.4]). Thus it follows by Hölder's inequality that

$$(3.5)\quad \int\limits_{\mathfrak{m}_d} |f(\alpha;d)|^2 \Big|\sum_{\substack{\varpi\in\mathscr{P}(M)\\ \varpi|d}} h(\alpha,\varpi)\Big|^6 d\alpha \ll \left(\frac{P}{d}\right)^{(3+\varepsilon)/2} \sum_{\substack{\varpi\in\mathscr{P}(M)\\ \varpi|d}} \int_0^1 |h(\alpha,\varpi)|^6\, d\alpha$$
$$\ll (P/d)^{3/2+\varepsilon} I.$$

We then denote by $\mathscr{P}_d(M)$ the set of $\varpi\in\mathscr{P}(M)$ with $\varpi\nmid d$. For each prime $\varpi\in\mathscr{P}_d(M)$, moreover, we put $f_0(\alpha;d,\varpi)=f(\alpha;d\varpi)$ for convenience, and write

$$f_1(\alpha;d,\varpi)=\sum_{\substack{P/d<x\le 2P/d\\ \varpi\nmid x}} e(d^3x^3\alpha),$$

so that $f(\alpha;d)=f_0(\alpha;d,\varpi)+f_1(\alpha;d,\varpi)$. Accordingly we set

$$U_j=\int\limits_{\mathfrak{m}_d}\Big(\sum_{\varpi\in\mathscr{P}_d(M)} |f_j(\alpha;d,\varpi)^{1/3}h(\alpha,\varpi)|\Big)^6 d\alpha$$

for $j=0$ and 1, and see that

$$(3.6)\qquad \int\limits_{\mathfrak{m}_d} |f(\alpha;d)|^2\Big|\sum_{\varpi\in\mathscr{P}_d(M)} h(\alpha,\varpi)\Big|^6 d\alpha \ll U_0+U_1.$$

Next, by Dirichlet's theorem ([20, Lemma 2.1]), there is a natural number $r\le (P/d)^2M^{-3}$ satisfying $\|rd^3\varpi^3\alpha\|\le M^3(d/P)^2$. But whenever $\alpha\in\mathfrak{m}_d$ and $\varpi\in\mathscr{P}_d(M)$, one necessarily has $r>P/(8d)$ in view of the assumption on $\mathfrak{m}_d$, and consequently it follows by Weyl's inequality ([20, Lemma 2.4]) that

$$|f(\alpha;d\varpi)|\ll\left(\frac{P}{d\varpi}\right)^{1+\varepsilon}\left(r^{-1}+\frac{d\varpi}{P}+r\left(\frac{d\varpi}{P}\right)^3\right)^{1/4}\ll\left(\frac{P}{dM}\right)^{3/4+\varepsilon}.$$

The latter bound and Lemma 2 yield the estimate

$$(3.7)\qquad U_0\ll (P/(dM))^{3/2+\varepsilon}(P^3M^3+P^{7/6}M^2I^{2/3}).$$

As regards U_1, we have by Hölder's inequality

$$(3.8)\qquad U_1\ll M^5\sum_{\varpi\in\mathscr{P}_d(M)}\int\limits_{\mathfrak{m}_d} |f_1(\alpha;d,\varpi)^2h(\alpha,\varpi)^6|\,d\alpha.$$

At this point, some comments may be in order. We could substitute the function $\sum_{\varpi\in\mathscr{P}(M)} g(\varpi^3\alpha;\mathscr{A})$ for $g(\alpha;\mathscr{A}_M)$, without any substantial change in our proof, as in Vaughan [17] and Brüdern [4]. If we did that, we would have $g(\varpi^3\alpha;\mathscr{A})$ instead of $h(\alpha,\varpi)$ at (3.8), and then the argument about U_1 would be slightly easier, because the integral on the right hand side of (3.8) may be estimated directly by Lemma 3. We have nonetheless adopted $g(\alpha;\mathscr{A}_M)$ in our situation, with a view to facilitating the description of a routine argument on the major arc contribution below. In any case, this is not essential at all. We may transform $h(\alpha,\varpi)$ into $g(\varpi^3\alpha;\mathscr{A})$ essentially, removing dependence upon ϖ from the summation condition in the definition of $h(\alpha,\varpi)$. This is done by the technique of Vaughan which is contained also in the proof of the previous lemma.

We define the functions

$$K_\varpi(\gamma)=\sum_{P/\varpi<x\le 2P/\varpi} e(x\gamma),\qquad g(\alpha,\gamma;\mathscr{A})=\sum_{x\in\mathscr{A}} e(x^3\alpha+x\gamma),$$
$$K^*(\gamma)=\min\{P/M,\ \|\gamma\|^{-1}\}.$$

By orthogonality, we see plainly

$$h(\alpha,\varpi)=\int_0^1 g(\varpi^3\alpha,\gamma;\mathscr{A})K_\varpi(-\gamma)\,d\gamma.$$

For $M<\varpi\le 2M$, we know the bound $|K_\varpi(-\gamma)|\ll K^*(\gamma)$, and via an application of Hölder's inequality, we have

$$|h(\alpha,\varpi)|^6\ll\Big(\int_0^1 K^*(\gamma)\,d\gamma\Big)^5\Big(\int_0^1 K^*(\gamma)|g(\varpi^3\alpha,\gamma;\mathscr{A})|^6\,d\gamma\Big).$$

But it follows immediately from the definition that

$$\int_0^1 K^*(\gamma)\,d\gamma\ll\int_0^{M/P}\frac{P}{M}\,d\gamma+\int_{M/P}^{1/2}\frac{1}{\gamma}\,d\gamma\ll\log P, \tag{3.9}$$

so we deduce from (3.8) that

$$U_1\ll M^{5+\varepsilon}\int_0^1 K^*(\gamma)\sum_{\varpi\in\mathscr{P}_d(M)}\int_{\mathfrak{m}_d}|f_1(\alpha;d,\varpi)^2 g(\varpi^3\alpha,\gamma;\mathscr{A})^6|\,d\alpha\,d\gamma.$$

Now we apply Lemma 3 with $X=P/d$, $m=d^3$, $\mathfrak{m}=\mathfrak{m}_d$ and $S(\alpha)=|g(\alpha,\gamma;\mathscr{A})|^6$. We may then observe that $f_1(\alpha;d,\varpi)$ coincides with $f_\varpi(m\alpha)$ in Lemma 3, and that

$$\int_0^1|g(\alpha,\gamma;\mathscr{A})|^6\,d\alpha\le\int_0^1|g(\alpha;\mathscr{A})|^6\,d\alpha=I,$$

in view of orthogonality. Therefore Lemma 3 yields the estimate

$$\sum_{\varpi\in\mathscr{P}_d(M)}\int_{\mathfrak{m}_d}|f_1(\alpha;d,\varpi)^2 g(\varpi^3\alpha,\gamma;\mathscr{A})^6|\,d\alpha\ll(P/d)^{3/2+\varepsilon}M^{-5/2}I,$$

which is enough to conclude that

$$U_1\ll(P/d)^{3/2+\varepsilon}M^{5/2}I, \tag{3.10}$$

by virtue of (3.9) again. Since $M \gg P^{1/10}$, we see that

$$P^{7/6}M^{-1}I^{2/3} \ll PM^{2/3}I^{2/3} = (P^3)^{1/3}(MI)^{2/3} \ll P^3 + MI,$$

and the lemma follows from (3.5)–(3.7) and (3.10). ∎

4. The circle method and the linear sieve. This section is devoted to establishing the base of our proof, by applying the circle method and the linear sieve. Previous to its statement, we introduce some notation, and record preliminary results on it.

Throughout, let n be a given, large natural number, and put

$$P = \frac{1}{2}n^{1/3}, \qquad L = \log P. \tag{4.1}$$

We define

$$u(\beta) = \int_P^{2P} e(\beta t^3)\,dt, \qquad v(\beta) = \int_P^{2P} \frac{e(\beta t^3)}{\log t}\,dt.$$

By partial integration one may derive the familiar bounds

$$u(\beta) \ll P(1 + P^3|\beta|)^{-1}, \qquad v(\beta) \ll P(\log P)^{-1}(1 + P^3|\beta|)^{-1}. \tag{4.2}$$

We then define

$$\mathfrak{I}(n) = \int_{-\infty}^{\infty} u(\beta)v(\beta)^6 e(-n\beta)\,d\beta.$$

The absolute convergence of the latter integral is ensured by (4.2). One may also show that

$$P^4(\log P)^{-6} \ll \mathfrak{I}(n) \ll P^4(\log P)^{-6}. \tag{4.3}$$

In fact, the upper bound follows immediately by (4.2), and the lower bound may be confirmed by using Fourier's inversion formula (for example, see p. 46 of [6], the proof of the lower bound for $I(n)$).

We next introduce the exponential sums

$$S(q, a) = \sum_{r=1}^{q} e(ar^3/q), \qquad S^*(q, a) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e(ar^3/q).$$

Then, recalling the numbers $a_{n,2}$ and $a_{n,3}$ defined by (1.3), we define, for an integer d,

$$A_d(q, n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S(q, ad^3)S^*(q, a)^4 S^*(q, a_{n,2}^3 a)S^*(q, a_{n,3}^3 a)}{q\varphi(q)^6}\, e\Big(-\frac{an}{q}\Big),$$

where $\varphi(q)$ denotes Euler's totient function. Lemma 8.5 of Hua [12] states that $S^*(q, a) \ll q^{1/2+\varepsilon}$ whenever $(q, a) = 1$, and obviously the same bound holds even if a is replaced by $8a$ or $27a$. Thus, using the trivial bound $|S(q, ad^3)| \le q$, we have

$$A_d(q,n) \ll \varphi(q)q(q^{1/2+\varepsilon/7})^6 q^{-1}\varphi(q)^{-6} \ll q^{-2+\varepsilon}. \tag{4.4}$$

Then we write

$$\mathfrak{S}_d(n) = \sum_{q=1}^{\infty} A_d(q,n), \qquad \mathfrak{S}(n) = \mathfrak{S}_1(n),$$

observing that the infinite series converges absolutely by (4.4). All the necessary information on these singular series $\mathfrak{S}_d(n)$ is essentially recorded in Brüdern [4], and we recall it here. First, the function $A_d(q,n)$ is multiplicative with respect to q (see Lemma 8.1 of Hua [12] and Lemmata 2.10 and 2.11 of Vaughan [20]). We next define $\gamma = \gamma(p)$, by $\gamma = 1$ for $p \ne 3$ and $\gamma = 2$ for $p = 3$. Then by Lemma 8.3 of Hua [12], $S^*(p^l, a) = 0$ whenever $l > \gamma$ and $p \nmid a$, whence $A_d(p^l, n) = 0$ whenever $l > \gamma$. When $0 \le l \le \gamma$, one finds that $A_d(p^l, n)$ is equal to either $A_0(p^l, n)$ or $A_1(p^l, n)$ according as $p \mid d$ or not. Thus, on putting

$$B_d(p,n) = \sum_{l=0}^{\gamma} A_d(p^l, n),$$

we have

$$\mathfrak{S}_d(n) = \prod_p B_d(p,n) = \prod_{p \mid d} B_0(p,n) \prod_{p \nmid d} B_1(p,n). \tag{4.5}$$

Imitating the proof of Lemma 2.12 of Vaughan [20], one may see that

$$\begin{aligned} B_d(p,n) &= \sum_{a=1}^{p^\gamma} \frac{S(p^\gamma, ad^3) S^*(p^\gamma, a)^4 S^*(p^\gamma, a_{n,2}^3 a) S^*(p^\gamma, a_{n,3}^3 a)}{p^\gamma \varphi(p^\gamma)^6}\, e\Big(-\frac{an}{p^\gamma}\Big) \\ &= \varphi(p^\gamma)^{-6} M_d(p,n), \end{aligned} \tag{4.6}$$

where $M_d(p,n)$ denotes the number of solutions of the congruence

$$(dx)^3 + y_1^3 + \cdots + y_4^3 + (a_{n,2}y_5)^3 + (a_{n,3}y_6)^3 \equiv n \pmod{p^\gamma} \tag{4.7}$$

with $1 \le x \le p^\gamma$, $1 \le y_j \le p^\gamma$ and $p \nmid y_j$ for $1 \le j \le 6$. Further we write $M_d^*(p,n)$ for the number of solutions counted by $M_d(p,n)$ with the additional restriction $p \nmid x$. As for the latter, all we shall need is to check that one always has $M_1^*(p,n) \ge 1$, in the case $d = 1$. In fact, this is trivial for $p = 2$, and may be confirmed directly by hand for $p = 3$, and here is the point of the definitions of $a_{n,2}$ and $a_{n,3}$. When $p \ge 5$, on the other hand, we notice by repeated application of the Cauchy–Davenport theorem (see Vaughan [20, Lemma 2.14]) that at least $\min\{p, 6(p-1)/(3, p-1) - 5\}$ residue classes modulo p may be represented by $x^3 + y_1^3 + \cdots + y_4^3 + (a_{n,2}y_5)^3$

with $p\nmid xy_1\cdots y_5$. But one easily has $6(p-1)/(3,p-1)-5\geq p$ for $p\geq 5$, so it follows that $M_1^*(p,n)\geq 1$ in this case as well.

Now, on denoting by $M'(p,n)$ the number of solutions of

$$y_1^3+\cdots+y_4^3+(a_{n,2}y_5)^3+(a_{n,3}y_6)^3\equiv n \pmod{p^\gamma}$$

with $1\leq y_j\leq p^\gamma$ and $p\nmid y_j$ for $1\leq j\leq 6$, we have, of course, $M_0(p,n)=p^\gamma M'(p,n)$. Also it is trivial that there are $p^{\gamma-1}M'(p,n)$ solutions counted by $M_1(p,n)$ with $p\,|\,x$. In addition, we know $M_1^*(p,n)\geq 1$, whence

$$M_1(p,n)\geq p^{\gamma-1}M'(p,n)+1=p^{-1}M_0(p,n)+1. \tag{4.8}$$

So we have, in particular, $B_1(p,n)\geq p^{-12}$ by (4.6), while we see that $B_d(p,n)=1+O(p^{-2+\varepsilon})$ by (4.4). Therefore, by (4.5), we have

$$1\ll\mathfrak{S}(n)\ll 1. \tag{4.9}$$

Now we may define

$$\omega(d)=\frac{\mathfrak{S}_d(n)}{\mathfrak{S}(n)}=\prod_{p|d}\frac{B_0(p,n)}{B_1(p,n)}=\prod_{p|d}\frac{M_0(p,n)}{M_1(p,n)}. \tag{4.10}$$

Apparently $\omega(d)$ is multiplicative, and by (4.4), (4.6) and (4.8), we have for all primes p,

$$0\leq\omega(p)<p \quad\text{and}\quad \omega(p)=1+O(p^{-2+\varepsilon}). \tag{4.11}$$

These results will allow us to apply the linear sieve to our problem.

We further require some notation concerning the linear sieve. For $z\geq 2$, we define

$$\Pi(z)=\prod_{p<z}p,\qquad V(z)=\prod_{p<z}(1-\omega(p)/p),$$

and note that, by (4.8) and (4.11), one has

$$(\log z)^{-1}\ll V(z)\ll(\log z)^{-1}. \tag{4.12}$$

We denote the Euler constant by γ_0, and recall the well known functions $\phi_0(s)$ and $\phi_1(s)$ associated with the linear sieve, which are defined by

$$\phi_0(s)=0 \quad\text{and}\quad \phi_1(s)=2e^{\gamma_0}/s \quad\text{for } 0<s\leq 2,$$

and by the differential-difference equations

$$(s\phi_0(s))'=\phi_1(s-1) \quad\text{and}\quad (s\phi_1(s))'=\phi_0(s-1) \quad\text{for } s\geq 2.$$

In particular, it is known that for $2\leq s\leq 3$, one has

$$\phi_0(s)=2e^{\gamma_0}s^{-1}\log(s-1),\qquad \phi_1(s)=2e^{\gamma_0}s^{-1}. \tag{4.13}$$

Finally we introduce a special notation which facilitates the statement of the next lemma. For positive real numbers X and δ with $2\leq X\leq P$, we define $\mathfrak{A}(X,\delta)$ to be the family of sets $\mathscr{B}$ having the following property:

$\mathscr{B}$ is a set of natural numbers, and there exists a function $w(\beta)$ of a real variable β such that

$$\left| w(\beta) - \delta \int_X^{2X} \frac{e(\beta t^3)}{\log t}\, dt \right| \le \delta X L^{-3/2}, \tag{4.14}$$

and whenever integers q and a satisfy $|\alpha - a/q| \le 27L^{500}P^{-3}$ and $1 \le q \le L^{500}$ (here q and a may not be coprime), then

$$|g(\alpha;\mathscr{B}) - \varphi(q)^{-1}S^*(q,a)w(\alpha - a/q)| \le \delta P L^{-2000}.$$

Now we can write down the next lemma, using the notation introduced here and in the preamble to Lemma 1 (in particular, recall $\mathscr{A}_M$ and I).

LEMMA 5. *Let M, D and z be real numbers satisfying*

$$\max\{P^{1/10}, P^{1/9}D^{-1/3}\} < M \le (P/D)^{1/7}, \qquad D \ge z \ge 2. \tag{4.15}$$

Let $\delta_0, \ldots, \delta_3$ be positive real numbers, and $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$ be sets of natural numbers up to $2P$. Suppose that $\mathscr{A}_M \in \mathfrak{A}(P,\delta_0)$, $\mathscr{B}_1 \in \mathfrak{A}(P,\delta_1)$ and $\mathscr{B}_j \in \mathfrak{A}(P/a_{n,j}^3, \delta_j)$ for $j = 2$ and 3. Write $\Delta = \delta_0^3\delta_1\delta_2\delta_3(a_{n,2}a_{n,3})^{-1}$ and

$$J = \int_0^1 |g(\alpha;\mathscr{B}_1)g(a_{n,2}^3\alpha;\mathscr{B}_2)g(a_{n,3}^3\alpha;\mathscr{B}_3)|^2\, d\alpha.$$

Finally, write $R(n,z)$ for the number of representations of n in the form

$$n = x^3 + x_1^3 + x_2^3 + x_3^3 + y_1^3 + (a_{n,2}y_2)^3 + (a_{n,3}y_3)^3, \tag{4.16}$$

where $P < x \le 2P$, $(x, \Pi(z)) = 1$, and

$$x_j \in \mathscr{A}_M, \quad y_j \in \mathscr{B}_j, \quad \text{for } 1 \le j \le 3. \tag{4.17}$$

Then, on putting $s = (\log D)/(\log z)$, we have

$$R(n,z) > (\phi_0(s) + O((\log L)^{-3/10}))\Delta V(z)\mathfrak{S}(n)\mathfrak{J}(n) - E, \tag{4.18}$$

$$R(n,z) < (\phi_1(s) + O((\log L)^{-3/10}))\Delta V(z)\mathfrak{S}(n)\mathfrak{J}(n) + E, \tag{4.19}$$

with

$$E \ll P^4L^{-50} + P^{3/4+\varepsilon}M^{3/4}D^{1/4}J^{1/2}(P^{3/2} + (MI)^{1/2} + (PI)^{1/3}M^{3/2}D^{1/4}). \tag{4.20}$$

Proof. We define $R_d(n)$ to be the number of representations of n in the form (4.16) with $P < x \le 2P$, $d \mid x$ and (4.17). For now, we denote by $r_n(x)$ the number of solutions of (4.16) subject to (4.17), regarding (4.16) as an equation in x_j, y_j $(1 \le j \le 3)$, for given n and x. Then we have the trivial formulae

$$R_d(n) = \sum_{\substack{P<x\le 2P\\ x\equiv 0 \,(\mathrm{mod}\, d)}} r_n(x), \qquad R(n,z) = \sum_{\substack{P<x\le 2P\\ (x,\Pi(z))=1}} r_n(x).$$

Therefore, as in Brüdern [3] and [4], estimates for $R(n,z)$ may be derived from appropriate information on $R_d(n)$ via sieve theory.

We investigate $R_d(n)$ via the circle method. We write

$$G(\alpha)=g(\alpha;\mathscr{A}_M)^3 g(\alpha;\mathscr{B}_1)g(a_{n,2}^3\alpha;\mathscr{B}_2)g(a_{n,3}^3\alpha;\mathscr{B}_3),$$

for short, and define, for $\mathfrak{B}\subset[0,1)$,

$$R_d(n;\mathfrak{B})=\int\limits_{\mathfrak{B}} f(\alpha;d)G(\alpha)e(-n\alpha)\,d\alpha,$$

recalling the function $f(\alpha;d)$ defined in the statement of Lemma 4. We write $\mathfrak{P}$ for the set of real numbers $\alpha\in[0,1)$ such that there exist coprime integers q and a satisfying

$$|\alpha-a/q|\le L^{500}P^{-3},\quad 1\le q\le L^{500},\quad 0\le a\le q, \tag{4.21}$$

and we put $\mathfrak{p}=[0,1)\setminus\mathfrak{P}$. Then we have

$$R_d(n)=R_d(n;[0,1))=R_d(n;\mathfrak{P})+R_d(n;\mathfrak{p}). \tag{4.22}$$

The evaluation of $R_d(n;\mathfrak{P})$ is routine. We temporarily put $\mathscr{B}_0=\mathscr{A}_M$, and $a_{n,0}=a_{n,1}=1$ for convenience. By the assumptions, there exist functions $w_0(\beta),\dots,w_3(\beta)$ such that whenever α, q and a satisfy (4.21), one has

$$g(a_{n,j}^3\alpha;\mathscr{B}_j)=\frac{S^*(q,a_{n,j}^3a)}{\varphi(q)}\,w_j(a_{n,j}^3(\alpha-a/q))+O(\delta_jPL^{-2000}) \tag{4.23}$$

for $j=0,\dots,3$. The functions $w_j(\beta)$ also satisfy the conditions corresponding to (4.14). Since

$$\int\limits_{P/a_{n,j}^3}^{2P/a_{n,j}^3}\frac{e(\beta a_{n,j}^3t^3)}{\log t}\,dt=a_{n,j}^{-1}\int\limits_{P}^{2P}\frac{e(\beta t^3)}{\log(t/a_{n,j})}\,dt=a_{n,j}^{-1}v(\beta)+O(PL^{-2}),$$

we have

$$w_j(a_{n,j}^3\beta)=\delta_ja_{n,j}^{-1}v(\beta)+O(\delta_jPL^{-3/2}). \tag{4.24}$$

Here we pause to compute the integral

$$\widetilde{\mathfrak{I}}(n)=\int\limits_{-L^{500}/P^3}^{L^{500}/P^3}u(\beta)w_0(\beta)^3w_1(\beta)w_2(a_{n,2}^3\beta)w_3(a_{n,3}^3\beta)e(-n\beta)\,d\beta.$$

By (4.2) and (4.24), we have simply

$$\widetilde{\mathfrak{I}}(n)=\varDelta\int\limits_{-L^{500}/P^3}^{L^{500}/P^3}u(\beta)v(\beta)^6e(-n\beta)\,d\beta+O\biggl(\int\limits_0^{L^{500}/P^3}\frac{\varDelta P^7L^{-13/2}}{1+P^3\beta}\,d\beta\biggr),$$

and then, making use of (4.2) and (4.3), we see that

$$\widetilde{\mathfrak{I}}(n)=\varDelta\mathfrak{I}(n)+O(\varDelta P^4L^{-13/2}\log L)=\varDelta\mathfrak{I}(n)(1+O(L^{-1/3})). \tag{4.25}$$

Next, by Theorem 4.1 of Vaughan [20], whenever α, q and a satisfy (4.21), one has

$$f(\alpha;d) = q^{-1}S(q,ad^3)\int_{P/d}^{2P/d} e(\beta(dt)^3)\,dt + O(L^{500+\varepsilon}) \tag{4.26}$$
$$= (dq)^{-1}S(q,ad^3)u(\beta) + O(d^{-1}PL^{-2000})$$

for $1 \le d \le PL^{-2501}$. Now we note that $D \le P^{3/10}$ by (4.15), and the measure of $\mathfrak{P}$ is $O(P^{-3}L^{1500})$. Thus it follows straightforwardly from (4.23) and (4.26) that

$$R_d(n;\mathfrak{P}) = d^{-1}\widetilde{\mathfrak{J}}(n)\sum_{1\le q\le L^{500}} A_d(q,n) + O(\Delta d^{-1}P^4L^{-10})$$

for $1 \le d \le D$, and the formula is still valid when we replace the last sum over q by the series $\mathfrak{S}_d(n)$, in view of (4.4), (4.3) and (4.25). Inserting the latter result for $R_d(n;\mathfrak{P})$ into (4.22), and recalling (4.10), we conclude thus far that for $1 \le d \le D$, one has

$$R_d(n) = d^{-1}\omega(d)\mathfrak{S}(n)\widetilde{\mathfrak{J}}(n) + O(\Delta d^{-1}P^4L^{-10}) + R_d(n;\mathfrak{p}).$$

By this formula and the conditions on the multiplicative function $\omega(d)$ recorded in (4.11), one may now apply the linear sieve in order to estimate $R(n,z)$. Here we use Rosser's linear sieve in the form of Theorem 9 of Motohashi [15], and obtain the lower bound

$$R(n,z) > (\phi_0(s) + O((\log L)^{-3/10}))V(z)\mathfrak{S}(n)\widetilde{\mathfrak{J}}(n)$$
$$- \sum_{1\le d\le D} (|R_d(n;\mathfrak{p})| + O(\Delta d^{-1}P^4L^{-10})),$$

together with the corresponding upper bound, in which ϕ_0 and the minus sign on the right hand side are replaced respectively by ϕ_1 and the plus sign, and the inequality sign is of course reversed. Recalling (4.25), and observing by (4.3), (4.9) and (4.12) that

$$\sum_{1\le d\le D} O(\Delta d^{-1}P^4L^{-10}) \ll \Delta V(z)\mathfrak{S}(n)\mathfrak{J}(n)L^{-2},$$

we notice that the desired inequalities (4.18) and (4.19) are valid with

$$E = \sum_{1\le d\le D} |R_d(n;\mathfrak{p})|. \tag{4.27}$$

Thus it only remains to establish the estimate (4.20) for this sum E.

For each natural number d, we denote by $\mathfrak{M}_d$ the set of real numbers $\alpha \in [0,1)$ such that there exists a natural number $q \le PM^3/d$ satisfying $\|qd^3\alpha\| \le M^3(d/P)^2$, and put $\mathfrak{m}_d = [0,1)\setminus\mathfrak{M}_d$, noting that this $\mathfrak{m}_d$ satisfies the requirement in Lemma 4. It is easy to check that $\mathfrak{P} \subset \mathfrak{M}_d$ whenever

$1 \le d \le D$, and we have

$$R_d(n;\mathfrak{p}) = R_d(n;\mathfrak{p}\cap\mathfrak{M}_d) + R_d(n;\mathfrak{m}_d). \tag{4.28}$$

As for $R_d(n;\mathfrak{m}_d)$, Schwarz's inequality yields the estimate

$$|R_d(n;\mathfrak{m}_d)| \le J^{1/2}\Big(\int_{\mathfrak{m}_d} |f(\alpha;d)^2 g(\alpha;\mathscr{A}_M)^6|\,d\alpha\Big)^{1/2},$$

and we may apply Lemma 4 to the latter integral in parentheses. Accordingly we have

$$\begin{aligned}\sum_{1\le d\le D}|R_d(n;\mathfrak{m}_d)| &\ll \sum_{1\le d\le D} J^{1/2}((PM/d)^{3/2+\varepsilon}(P^3+MI))^{1/2} \\ &\ll P^{3/4+\varepsilon}M^{3/4}D^{1/4}J^{1/2}(P^{3/2}+M^{1/2}I^{1/2}).\end{aligned} \tag{4.29}$$

We turn to $R_d(n;\mathfrak{p}\cap\mathfrak{M}_d)$. By the definition of $\mathfrak{M}_d$, if $\alpha\in\mathfrak{M}_d$, then there exist integers r and b satisfying

$$1\le r\le PM^3/d, \quad |rd^3\alpha - b| \le M^3(d/P)^2, \quad (r,b)=1. \tag{4.30}$$

One may simply confirm that this pair of r and b is unique for given d and α. Therefore, for $\alpha\in\mathfrak{M}_d$, we may define

$$f^*(\alpha;d) = r^{-1}S(r,b)\int_{P/d}^{2P/d} e(t^3(d^3\alpha - b/r))\,dt,$$

and then, by Theorem 4.1 of Vaughan [20], we have

$$f(\alpha;d) = f^*(\alpha;d) + O((PM^3/d)^{1/2+\varepsilon}). \tag{4.31}$$

To estimate $f^*(\alpha;d)$, we introduce the multiplicative function $\kappa(q)$ by defining

$$\kappa(p^{3u+1}) = 2p^{-u-1/2}, \quad \kappa(p^{3u+2}) = \kappa(p^{3u+3}) = p^{-u-1},$$

for non-negative integers u. It is known that whenever $(r,b)=1$, one has $S(r,b)/r \ll \kappa(r)$, by Lemmata 4.3–4.5 of Vaughan [20]. Combining this bound with an estimate corresponding to the first inequality in (4.2), we have

$$f^*(\alpha;d) \ll Pd^{-1}\kappa(r)(1+(P/d)^3|d^3\alpha - b/r|)^{-1}. \tag{4.32}$$

Next, we denote by $\mathfrak{M}_d'$ the set of real numbers $\alpha\in\mathfrak{M}_d$ for which the integers r and b determined by (4.30) satisfy the stronger constraints

$$1\le r\le 2^{-1}(P/(dM^3))^{3/2}, \quad |rd^3\alpha - b| \le 2^{-1}(d/(PM^3))^{3/2}. \tag{4.33}$$

Noticing that $\kappa(r)\ll r^{-1/3}$ by the definition, we observe by (4.32) that when $\alpha\in\mathfrak{M}_d\setminus\mathfrak{M}_d'$, one has

$$f^*(\alpha;d) \ll (P/d)(r+(P/d)^3|rd^3\alpha - b|)^{-1/3} \ll (PM^3/d)^{1/2}.$$

Accordingly we define

$$f^\dagger(\alpha;d)=\begin{cases} f^*(\alpha;d) & \text{when } \alpha\in\mathfrak{M}'_d,\\ 0 & \text{when } \alpha\notin\mathfrak{M}'_d,\end{cases}$$

and remark that for $\alpha\in\mathfrak{M}_d$, the formula (4.31) is valid with $f^\dagger(\alpha;d)$ in place of $f^*(\alpha;d)$. Thus naturally we have

$$|R_d(n;\mathfrak{p}\cap\mathfrak{M}_d)|\ll\int_{\mathfrak{p}}|f^\dagger(\alpha;d)G(\alpha)|\,d\alpha+(PM^3/d)^{1/2+\varepsilon}\int_0^1|G(\alpha)|\,d\alpha,$$

whence

$$\sum_{1\le d\le D}|R_d(n;\mathfrak{p}\cap\mathfrak{M}_d)|\ll\int_{\mathfrak{p}}|F(\alpha)G(\alpha)|\,d\alpha+P^\varepsilon\widetilde{E},\tag{4.34}$$

where we write

$$F(\alpha)=\sum_{1\le d\le D}|f^\dagger(\alpha;d)|,\qquad \widetilde{E}=(PDM^3)^{1/2}\int_0^1|G(\alpha)|\,d\alpha.$$

Hereafter, for $X\ge1$, we denote by $\mathfrak{N}(X)$ the set of real numbers $\alpha\in[0,1)$ such that there exist integers q and a satisfying

$$1\le q\le X,\qquad |q\alpha-a|\le XP^{-3},\qquad (q,a)=1.\tag{4.35}$$

We shall estimate $F(\alpha)$ for a given α. This concerns only natural numbers $d\le D$ with $\alpha\in\mathfrak{M}'_d$, in view of the definitions of $f^\dagger$ and F, and for each such d (if any), we take coprime integers r and b satisfying (4.33). Since our assumption (4.15) implies $DM^{-3}\le PM^{-10}<1$, we have swiftly

$$rd^3<2^{-1}P^{3/2},\qquad |rd^3\alpha-b|<2^{-1}P^{-3/2},\tag{4.36}$$

which tells that there are q and a satisfying (4.35) with $X=2^{-1}P^{3/2}$. One may easily check that this pair of q and a is unique for the given α, so that one has $b/(rd^3)=a/q$ and $r=q/(q,d^3)$, and thus

$$f^\dagger(\alpha;d)\ll Pd^{-1}\kappa(q/(q,d^3))(1+P^3|\alpha-a/q|)^{-1},$$

by (4.32). Since we have

$$\sum_{1\le d\le D}d^{-1}\kappa(q/(q,d^3))\ll q^\varepsilon\kappa(q)L,$$

by Lemma 2.3 of Kawada and Wooley [14] (note that our $\kappa(q)$ is almost the same as the function $w_3(q)$ in the notation of [14], and indeed one has $\kappa(q)\le w_3(q)\ll q^\varepsilon\kappa(q)$), we conclude that whenever $\alpha\in\mathfrak{N}(2^{-1}P^{3/2})$, q and a satisfy (4.35) with $X=2^{-1}P^{3/2}$, then one has

$$F(\alpha)\ll q^\varepsilon\kappa(q)PL(1+P^3|\alpha-a/q|)^{-1}.\tag{4.37}$$

In addition, we have already observed implicitly that $F(\alpha)=0$ unless $\alpha\in\mathfrak{N}(2^{-1}P^{3/2})$. In fact, our argument around (4.36) reveals that if $\alpha\in\mathfrak{M}'_d$ for some d with $1 \le d \le D$, then $\alpha \in \mathfrak{N}(2^{-1}P^{3/2})$. Or, if $\alpha \notin \mathfrak{N}(2^{-1}P^{3/2})$, then $f^{\dagger}(\alpha; d) = 0$ for $1 \le d \le D$.

By (4.37) with the trivial bound $\kappa(q) \ll q^{-1/3}$, we find that

$$\sup_{\alpha \in [0,1) \setminus \mathfrak{N}(X)} |F(\alpha)| \ll PLX^{\varepsilon - 1/3}. \tag{4.38}$$

In particular, unless $\alpha \in \mathfrak{N}(P)$, we have $F(\alpha) \ll P^{2/3+\varepsilon}$, but we see that $P^{2/3} \ll (PDM^3)^{1/2}$ by (4.15). Consequently, it follows from (4.34) that

$$\sum_{1 \le d \le D} |R_d(n; \mathfrak{p} \cap \mathfrak{M}_d)| \ll \int_{\mathfrak{p} \cap \mathfrak{N}(P)} |F(\alpha)G(\alpha)| \, d\alpha + P^{\varepsilon} \widetilde{E}. \tag{4.39}$$

Moreover, by Schwarz's inequality and Lemma 1, we observe that

$$\begin{aligned} \widetilde{E} &\ll (PDM^3)^{1/2} J^{1/2} \Big(\int_0^1 |g(\alpha; \mathscr{A}_M)|^6 \, d\alpha \Big)^{1/2} \\ &\ll P^{1/2+\varepsilon} M^{5/2} D^{1/2} J^{1/2} (P^{3/2} + P^{7/12} M^{-1/4} I^{1/3}). \end{aligned} \tag{4.40}$$

We next put

$$T_1 = \int_0^1 |g(\alpha; \mathscr{A}_M)|^8 \, d\alpha, \qquad T_2 = \int_0^1 \prod_{j=0}^{3} |g(a_{n,j}^3 \alpha; \mathscr{B}_j)|^2 \, d\alpha,$$

recalling the convenient convention on $\mathscr{B}_0$, $a_{n,0}$ and $a_{n,1}$, and set

$$\Upsilon = \int_{\mathfrak{p} \cap \mathfrak{N}(P)} |F(\alpha)^3 g(\alpha; \mathscr{A}_M)^2| \, d\alpha.$$

The bounds $T_j \ll P^5$ for $j = 1$ and 2 follow immediately from Theorem 2 of Vaughan [17], by considering the underlying diophantine equations. Meanwhile, the technique of estimating integrals like Υ may be found in a number of articles in this area nowadays, and it indeed yields the estimate

$$\Upsilon \ll P^2 L^{-150}. \tag{4.41}$$

Assuming the latter bound for the moment, we deduce via application of Hölder's inequality that

$$\int_{\mathfrak{p} \cap \mathfrak{N}(P)} |F(\alpha)G(\alpha)| \, d\alpha \ll \Upsilon^{1/3} T_1^{1/6} T_2^{1/2} \ll P^4 L^{-50}. \tag{4.42}$$

The required bound (4.20) now follows straightforwardly from (4.27)–(4.29), (4.39), (4.40) and (4.42). (It may be rather helpful to note that one has $P^{1/2} M^{5/2} D^{1/2} \ll P^{3/4} M^{3/4} D^{1/4}$ by (4.15), when we compare the right hand side of (4.29) with that of (4.40).)

What remains now is to confirm (4.41). To this end, we put

$$\Upsilon_1(X) = \int_{\mathfrak{N}(2X)} |F(\alpha)^2 g(\alpha; \mathscr{A}_M)^2| \, d\alpha,$$

and estimate it for $1 \le X \le P$. By (4.37),

$$\Upsilon_1(X) \ll P^2L^2X^\varepsilon \sum_{1\le q\le 2X} \kappa(q)^2 \times \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{-1/(qP^2)}^{1/(qP^2)} (1+P^3|\beta|)^{-2}|g(a/q+\beta;\mathscr{A}_M)|^2\,d\beta.$$

We see that

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} |g(a/q+\beta;\mathscr{A}_M)|^2 = \sum_{x,y\in\mathscr{A}_M} e((x^3-y^3)\beta) \sum_{\substack{a=1\\(a,q)=1}}^{q} e((x^3-y^3)a/q),$$

and the last inner sum is at most (q, x^3-y^3) in modulus by the well known estimate for the Ramanujan sum. Thus we have

$$\Upsilon_1(X) \ll P^2L^2X^\varepsilon \sum_{1\le q\le 2X} \kappa(q)^2 \sum_{1\le x,y\le 2P} (q,x^3-y^3) \int_{-\infty}^{\infty} (1+P^3|\beta|)^{-2}\,d\beta.$$

But it was shown in the proof of Lemma 3.3 of Brüdern, Kawada and Wooley [7] that

$$\sum_{1\le q\le 2X} \kappa(q)^2 \sum_{1\le x,y\le 2P} (q,x^3-y^3) \ll P^2X^\varepsilon$$

for $1 \le X \le P$ (see (3.9), (3.10) and the displayed inequality following (3.10) of [7]). So we have

$$\Upsilon_1(X) \ll P^2L^2X^{\varepsilon/2}\cdot P^2X^{\varepsilon/2}\cdot P^{-3} \ll PL^2X^\varepsilon.$$

Now we write $\mathfrak{N}^*(X) = \mathfrak{N}(2X)\setminus\mathfrak{N}(X)$. Then, combining the last result with (4.38), we deduce for $1 \le X \le P$ that

$$\int_{\mathfrak{N}^*(X)} |F(\alpha)^3g(\alpha;\mathscr{A}_M)^2|\,d\alpha \ll \sup_{\alpha\in\mathfrak{N}^*(X)} |F(\alpha)|\,\Upsilon_1(X) \ll P^2L^3X^{\varepsilon-1/3}.$$

Since $\mathfrak{N}(L^{500}) \subset \mathfrak{P}$, we notice that $\mathfrak{p}\cap\mathfrak{N}(P)$ is covered by the union of $\mathfrak{N}^*(X)$ for $X = 2^kL^{500}$ with integers k such that $L^{500} \le X \le P$. Hence, by summing the last inequality over such values of X, we obtain (4.41) at once, and the proof of the lemma is complete at last. ∎

5. Switching principle. Now that Lemma 5 is established, our strategy may be transparent. We apply the latter lemma with the sets $\mathscr{A}$ and $\mathscr{B}_j$ taken to be appropriate sets of almost primes. We shall define these sets so that the integrals I and J are immediately estimated by the mean value theorems recorded in Brüdern [2]. We require that the bound (4.20) gets a shape like $E \ll P^4L^{-50}$, and this demand gives restrictions on D in terms of P and M. At this stage we may determine the optimal value of M, so that D, the parameter called the level of distribution, may be as large as possible. Then, with an appropriate choice of z, we apply Lemma 5 and the idea of switching principle. In this way one may get various conclusions similar to Theorem 1.

Here we fix our setting for the proof of Theorem 1. First we define parameters by

$$M = 2P^{1/8}, \quad D = P^{9/122}, \quad z = P^{1/40},$$

recalling (4.1). It is trivial that the requirements (4.15) are fulfilled. Next, let $\Omega(x)$ denote the number of prime divisors of x, counted according to multiplicity, and for a natural number r, introduce the set

$$\mathscr{C}_r = \{P < x \le 2P : (x, \Pi(z)) = 1,\ \Omega(x) = r\}.$$

Hereafter we use the symbol $x \sim X$ as a shorthand for $X < x \le 2X$, and then define various sets of almost primes as follows:

$$\mathscr{B}_1 = \bigcup_{r=1}^{3} \mathscr{C}_r, \quad \mathscr{B}_1' = \bigcup_{r=5}^{40} \mathscr{C}_r,$$

$$\mathscr{B}_2 = \{p_1p_2p_3 : p_1 \sim P^{15/113},\ p_2 \sim P^{14/113},\ p_3 \sim P/(a_{n,2}p_1p_2)\},$$

$$\mathscr{B}_3 = \{p_1p_2 : p_1 \sim P^{1/7},\ p_2 \sim P/(a_{n,3}p_1)\},$$

$$\mathscr{A} = \{p_1p_2 : p_1 \sim (P/M)^{1/7},\ 1 \le p_2 \le 2(P/(Mp_1))\}.$$

For this set $\mathscr{A}$, moreover, we define $\mathscr{A}_M$ by (2.1), noting that every number in $\mathscr{A}$ has indeed no prime divisor in $\mathscr{P}(M)$, as we supposed in Section 2.

By the Theorem of Brüdern [2], we have

$$I = \int_0^1 |g(\alpha; \mathscr{A})|^6 \, d\alpha \ll (P/M)^{23/7+\varepsilon}. \tag{5.1}$$

Also, the Proposition of Brüdern [2] means that for any set $\mathscr{B}$ of natural numbers up to $2P$, one has

$$\int_0^1 |g(\alpha; \mathscr{B})^2 g(a_{n,j}^3\alpha; \mathscr{B}_j)^4| \, d\alpha \ll P^{\lambda_j+\varepsilon}$$

for $j = 2$ and 3, with $\lambda_2 = 369/113$ and $\lambda_3 = 23/7$, and an obvious application of Schwarz's inequality leads to the bound

$$\int_0^1 |g(\alpha; \mathscr{B}) g(a_{n,2}^3\alpha; \mathscr{B}_2) g(a_{n,3}^3\alpha; \mathscr{B}_3)|^2 \, d\alpha \ll P^{2591/791+\varepsilon}. \tag{5.2}$$

This provides upper bounds for integrals corresponding to J, when we apply Lemma 5 with the current sets $\mathscr{B}_2$ and $\mathscr{B}_3$.

The above choice of M results from the optimisation mentioned in the opening paragraph of this section. Actually, making use of (5.1) and (5.2) to estimate I and J, we find, after simple calculation, that the inequality (4.20) turns into

$$E \ll P^4L^{-50} + P^{4-467/25312+\varepsilon}D^{1/4} + P^{4-2983/75936+\varepsilon}D^{1/2} \ll P^4L^{-50}.$$

In the definition of D, therefore, the exponent $9/122$ could be any fixed real number strictly less than (but appropriately near to) $467/6328$. Further, one may say that our choice of z is not optimal if one takes interest in better quality of the lower bound for the number of representations in Theorem 1 (that is, to get the largest possible value in place of the constant 0.86 in (5.8) below). In the latter sense, the best choice of z would be approximately $P^{0.02113}$.

We next check some conditions concerning the notation $\mathfrak{A}(X,\delta)$ introduced in the preamble to Lemma 5, although this task may be regarded as a basic exercise in this area. We take $\mathscr{A}_M$ as an example. By the Siegel–Walfisz Theorem and partial summation, one may show that whenever $\alpha = a/q+\beta$ with $1\le q\le L^{500}$ and $|\beta|\le 27L^{500}P^{-3}$ (even if $(q,a)>1$), then one has

$$\sum_{p_2\sim P/(\varpi p_1)} e((\varpi p_1p_2)^3\alpha) = \frac{S^*(q,(\varpi p_1)^3a)}{\varphi(q)}\int_{P/(\varpi p_1)}^{2P/(\varpi p_1)} \frac{e((\varpi p_1t)^3\beta)}{\log t}\,dt + O\biggl(\frac{P}{\varpi p_1}\,L^{-3000}\biggr),$$

for $\varpi\in\mathscr{P}(M)$ and $p_1\sim(P/M)^{1/7}$ (refer to Hua [12, Lemmata 7.14 and 7.15]). Since $(q,\varpi p_1)=1$, one has $S^*(q,(\varpi p_1)^3a)=S^*(q,a)$, so by a change of variable in the integral (rewrite ϖp_1t as t again), one obtains

$$g(\alpha;\mathscr{A}_M) = \frac{S^*(q,a)}{\varphi(q)}\int_P^{2P}\delta_0(t)\,\frac{e(\beta t^3)}{\log t}\,dt + O(PL^{-3000}),$$

where we put

$$\delta_0(t) = \sum_{\varpi\in\mathscr{P}(M)}\ \sum_{p_1\sim(P/M)^{1/7}} \frac{\log t}{\varpi p_1\log(t/(\varpi p_1))}.$$

But, for $P\le t\le 2P$, one has $\log t=\log P+O(1)$, whence $\delta_0(t)=\delta_0(P)(1+O(L^{-1}))$. On putting $\delta_0=\delta_0(P)$, one has $\delta_0\gg L^{-2}$, and one may thus conclude that $\mathscr{A}_M\in\mathfrak{A}(P,\delta_0)$. Similarly, on putting

$$\delta_2 = \sum_{\substack{p_1\sim P^{15/113}\\ p_2\sim P^{14/113}}} \frac{\log P}{p_1p_2\log(P/(p_1p_2))},\qquad \delta_3 = \sum_{p_1\sim P^{1/7}}\frac{\log P}{p_1\log(P/p_1)},$$

one may confirm that $\mathscr{B}_j \in \mathfrak{A}(P/a_{n,j}, \delta_j)$ for $j = 2$ and 3, together with the bounds $\delta_2 \gg L^{-2}$ and $\delta_3 \gg L^{-1}$.

To deal with the sets $\mathscr{B}_1$ and $\mathscr{B}_1'$, we first define $C_1(s) = 1$ or 0 according as $s \geq 1$ or $s < 1$, and then define $C_r(s)$ inductively for $r \geq 2$ by

$$C_r(s) = \int_r^{\max\{s,r\}} \frac{C_{r-1}(t-1)}{t-1}\, dt.$$

We may notice that $\mathscr{C}_r \in \mathfrak{A}(P, C_r(40))$ by Lemma 2.2 of Brüdern and Kawada [6] (with a trifling modification on the length of the interval in the latter lemma). Consequently, on writing

$$\delta_1 = \sum_{r=1}^{3} C_r(40), \qquad \delta_1' = \sum_{r=5}^{40} C_r(40),$$

we see that $\mathscr{B}_1 \in \mathfrak{A}(P, \delta_1)$ and $\mathscr{B}_1' \in \mathfrak{A}(P, \delta_1')$.

With respect to the constants δ_1 and δ_1', we shall require numerical estimates. We have $C_1(40) = 1$ and $C_2(40) = \log 39$ by the definition, while numerical integration gives the bounds

$$C_3(40) > 5.914, \qquad C_4(40) > 5.676.$$

On the other hand, Brüdern and Kawada [6] pointed out (see (6.35) of [6]) that as a consequence of simple application of the linear sieve, one may derive the estimate

$$\sum_{r=1}^{40} C_r(40) \leq 40e^{-\gamma_0}\phi_1(40) < 40e^{-\gamma_0}(1 + 10^{-9}),$$

recalling the notation introduced in the preamble to Lemma 5. Thus we may confirm the numerical bounds

$$\delta_1 > 10.577, \qquad \delta_1' < 6.205. \tag{5.3}$$

By the definitions and preliminary results above, we state that when we apply Lemma 5 with these sets $\mathscr{A}_M$, $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$, we have

$$\varDelta = \delta_0^3 \delta_1 \delta_2 \delta_3 (a_{n,2} a_{n,3})^{-1} \gg L^{-9},$$

and the estimate (4.20) implies that

$$E \ll P^4 L^{-50} \ll \varDelta V(z) \mathfrak{S}(n) \mathfrak{J}(n) L^{-34}, \tag{5.4}$$

by recalling (4.3), (4.9) and (4.12). So the inequalities (4.18) and (4.19) are valid with E deleted in this case, and in this respect the situation is the same when we replace $\mathscr{B}_1$ with $\mathscr{B}_1'$.

Proof of Theorem 1. Under the notation fixed in this section, now we write $R(n, z)$ for the number of representations of n in the form

$$(5.5)\qquad n = x^3 + x_1^3 + x_2^3 + x_3^3 + y_1^3 + (a_{n,2}y_2)^3 + (a_{n,3}y_3)^3,$$

subject to $P < x \le 2P$, $(x, \Pi(z)) = 1$, and

$$(5.6)\qquad x_1, x_2, x_3 \in \mathscr{A}_M, \quad y_1 \in \mathscr{B}_1, \quad y_2 \in \mathscr{B}_2, \quad y_3 \in \mathscr{B}_3.$$

Then, in view of (5.4), Lemma 5 gives the lower bound

$$(5.7)\quad R(n,z) > \left(\phi_0\left(\frac{180}{61}\right) + O((\log L)^{-3/10})\right)\frac{\delta_0^3\delta_1\delta_2\delta_3}{a_{n,2}a_{n,3}}\, V(z)\mathfrak{S}(n)\mathfrak{J}(n).$$

Next we write $R'(n,z)$ for the number of representations of n in the form (5.5) subject to $x \in \mathscr{B}_1'$ and (5.6). By our choice of z, when $P < x \le 2P$ and $(x, \Pi(z)) = 1$, one necessarily has $\Omega(x) \le 40$. Therefore, if we could show that $R(n,z) - R'(n,z) > 0$, then we conclude that n can be written in the form (5.5) with the variables satisfying (5.6) and $\Omega(x) \le 4$. This clearly means that the conclusion of Theorem 1 is true, because every number in $\mathscr{A}_M$, $\mathscr{B}_1$ or $\mathscr{B}_2$ is P_3, and every member of $\mathscr{B}_3$ is P_2.

Removing the implicit condition $\Omega(y_1) \le 3$ from the definition of $R'(n,z)$, we denote the number of such representations by $R''(n,z)$. Namely, $R''(n,z)$ is the number of representations of n in the form (5.5) subject to $P < y_1 \le 2P$, $(y_1, \Pi(z)) = 1$, and

$$x_1, x_2, x_3 \in \mathscr{A}_M, \qquad x \in \mathscr{B}_1', \qquad y_2 \in \mathscr{B}_2, \qquad y_3 \in \mathscr{B}_3.$$

Trivially we see that $R'(n,z) \le R''(n,z)$. We apply Lemma 5 to $R''(n,z)$, exchanging the roles of x and y_1 in the apparent manner. By the note following (5.4), we thus obtain the upper bound

$$R''(n,z) < \left(\phi_1\left(\frac{180}{61}\right) + O((\log L)^{-3/10})\right)\frac{\delta_0^3\delta_1'\delta_2\delta_3}{a_{n,2}a_{n,3}}\, V(z)\mathfrak{S}(n)\mathfrak{J}(n).$$

By (4.13), (5.7) and the last inequality, we deduce that

$$\begin{aligned} R(n,z) - R'(n,z) &\ge R(n,z) - R''(n,z) \\ &> \left(\delta_1 \log\frac{119}{61} - \delta_1' + O((\log L)^{-3/10})\right)\phi_1\left(\frac{180}{61}\right) \\ &\quad \times \delta_0^3\delta_2\delta_3(a_{n,2}a_{n,3})^{-1}V(z)\mathfrak{S}(n)\mathfrak{J}(n). \end{aligned}$$

But by (5.3), a modicum of computation shows that

$$(5.8)\qquad \delta_1 \log\frac{119}{61} - \delta_1' > 0.86,$$

whence $R(n,z) - R'(n,z) > 0$ as required, and Theorem 1 follows. ∎

Finally, we record the necessary changes in the above proof required to establish Theorem 2. To this end, we set $M = P^{2277/18419}$, $D = P^{1/11}$ and $z = P^{1/33}$, for instance. We define the sets $\mathscr{B}_2$, $\mathscr{A}$ and $\mathscr{A}_M$ as above, but redefine $\mathscr{B}_1$ to be the set of primes $p \sim P$, and $\mathscr{B}_1'$ to be the set of natural numbers $x \sim P$ satisfying $(x, \Pi(z)) = 1$ and $\Omega(x) \geq 7$. Moreover, $\mathscr{B}_3$ is redefined to be the set identical with $\mathscr{B}_2$ in this case, except for the trivial modification that $a_{n,2}$ is now replaced by $a_{n,3}$ in the definition of $\mathscr{B}_2$ above. Then Theorem 2 can be proved in the same way as Theorem 1 with these alterations, by appealing to the numerical bounds $C_3(33) > 5.214$, $C_4(33) > 4.596$, $C_5(33) > 2.689$, and $C_6(33) > 1.118$, so we omit the further details.

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Department of Mathematics
Faculty of Education
Iwate University
Morioka, 020-8550 Japan
E-mail: kawada@iwate-u.ac.jp

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