

## On $q$ -Euler numbers, $q$ -Salié numbers and $q$ -Carlitz numbers

by

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**1. Introduction.** The Euler numbers  $E_0, E_1, E_2, \dots$  are defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \frac{2e^x}{e^{2x} + 1} = \left( \frac{e^x + e^{-x}}{2} \right)^{-1} = \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right)^{-1};$$

they are all integers because of the recursion

$$\sum_{\substack{k=0 \\ 2|k}}^n \binom{n}{k} E_{n-k} = \delta_{n,0} \quad (n \in \mathbb{N} = \{0, 1, 2, \dots\}),$$

where  $\delta_{n,m}$  is the Kronecker symbol. It is easy to see that  $E_{2k+1} = 0$  for every  $k = 0, 1, 2, \dots$ . In 1871 Stern [St] obtained an interesting arithmetic property of the Euler numbers:

$$(1.1) \quad E_{2n+2^s} \equiv E_{2n} + 2^s \pmod{2^{s+1}} \quad \text{for any } n, s \in \mathbb{N};$$

equivalently we have

$$(1.1') \quad E_{2m} \equiv E_{2n} \pmod{2^{s+1}} \Leftrightarrow m \equiv n \pmod{2^s} \quad \text{for any } m, n, s \in \mathbb{N}.$$

Later Frobenius amplified Stern's proof in 1910, and several different proofs of (1.1) or (1.1') were given by Ernvall [E], Wagstaff [W] and Sun [Su]. Our first goal is to provide a complete  $q$ -analogue of the Stern congruence.

As usual we let  $(a; q)_n = \prod_{0 \leq k < n} (1 - aq^k)$  for every  $n \in \mathbb{N}$ , where an empty product is regarded to have value 1 and hence  $(a; q)_0 = 1$ . For  $n \in \mathbb{N}$  we set

$$[n]_q = \frac{1 - q^n}{1 - q} = \sum_{0 \leq k < n} q^k;$$

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this is the usual  $q$ -analogue of  $n$ . For any  $n, k \in \mathbb{N}$ , if  $k \leq n$  then we call

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{0 < r \leq n} [r]_q}{(\prod_{0 < s \leq k} [s]_q)(\prod_{0 < t \leq n-k} [t]_q)} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

a  $q$ -binomial coefficient; if  $k > n$  then we let  $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ . Obviously we have  $\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$ . It is easy to see that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad \text{for all } k, n = 1, 2, 3, \dots$$

By this recursion, each  $q$ -binomial coefficient is a polynomial in  $q$  with integer coefficients.

We define the  $q$ -Euler numbers  $E_n(q)$  ( $n \in \mathbb{N}$ ) by

$$(1.2) \quad \sum_{n=0}^{\infty} E_n(q) \frac{x^n}{(q; q)_n} = \left( \sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}} x^{2n}}{(q; q)_{2n}} \right)^{-1}.$$

Multiplying both sides by  $\sum_{n=0}^{\infty} q^{\binom{2n}{2}} x^{2n} / (q; q)_{2n}$ , we obtain the recursion

$$\sum_{\substack{k=0 \\ 2|k}}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} E_{n-k}(q) = \delta_{n,0} \quad (n \in \mathbb{N}),$$

which implies that  $E_n(q) \in \mathbb{Z}[q]$ . Observe that

$$\begin{aligned} \sum_{n=0}^{\infty} E_n(q) \frac{x^n}{\prod_{0 < k \leq n} [k]_q} &= \sum_{n=0}^{\infty} E_n(q) \frac{((1-q)x)^n}{(q; q)_n} \\ &= \left( \sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}} ((1-q)x)^{2n}}{(q; q)_{2n}} \right)^{-1} = \left( \sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}} x^{2n}}{\prod_{0 < k \leq 2n} [k]_q} \right)^{-1} \end{aligned}$$

and hence  $\lim_{q \rightarrow 1} E_n(q) = E_n$ .

The usual way to define a  $q$ -analogue of Euler numbers is as follows:

$$\sum_{n=0}^{\infty} \tilde{E}_n(q) \frac{x^n}{(q; q)_n} = \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{(q; q)_{2n}} \right)^{-1}.$$

(See, e.g., [GZ].) We assert that  $\tilde{E}_n(q) = q^{\binom{n}{2}} E_n(1/q)$ . In fact,

$$\begin{aligned} \sum_{n=0}^{\infty} q^{\binom{n}{2}} E_n(q^{-1}) \frac{x^n}{\prod_{0 < k \leq n} (1-q^k)} &= \sum_{n=0}^{\infty} E_n(q^{-1}) \frac{(-q^{-1}x)^n}{\prod_{0 < k \leq n} (1-q^{-k})} \\ &= \left( \sum_{n=0}^{\infty} \frac{q^{-\binom{2n}{2}} (-q^{-1}x)^{2n}}{\prod_{0 < k \leq 2n} (1-q^{-k})} \right)^{-1} = \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{\prod_{0 < k \leq 2n} (1-q^k)} \right)^{-1}. \end{aligned}$$

Recently, with the help of cyclotomic polynomials, Guo and Zeng [GZ] proved that if  $m, n, s, t \in \mathbb{N}$ ,  $m - n = 2^s t$  and  $2 \nmid t$  then

$$\tilde{E}_{2m}(q) \equiv q^{m-n} \tilde{E}_{2n}(q) \pmod{\prod_{r=0}^s (1 + q^{2^r t})}.$$

This is a partial  $q$ -analogue of Stern's result.

Using our  $q$ -analogue of Euler numbers, we are able to give below a complete  $q$ -analogue of the classical result of Stern.

**THEOREM 1.1.** *Let  $n, s, t \in \mathbb{N}$  and  $2 \nmid t$ . Then*

$$(1.3) \quad E_{2n}(q) - E_{2n+2^s t}(q) \equiv [2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}}.$$

The *Salié numbers*  $S_n$  ( $n \in \mathbb{N}$ ) are given by

$$\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = \frac{\cosh x}{\cos x} = \frac{(e^x + e^{-x})/2}{(e^{ix} + e^{-ix})/2} = \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right) / \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Multiplying both sides by  $\sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$  we get the recursion

$$\sum_{\substack{k=0 \\ 2|k}}^n \binom{n}{k} (-1)^{k/2} S_{n-k} = \frac{1 + (-1)^n}{2} \quad (n \in \mathbb{N}),$$

which implies that all Salié numbers are integers and  $S_{2k+1} = 0$  for all  $k \in \mathbb{N}$ .

By a sophisticated use of some deep properties of Bernoulli numbers, in 1965 Carlitz [C2] proved that  $2^n \mid S_{2n}$  for any  $n \in \mathbb{N}$  (which was first conjectured by Gandhi [G]). Recently Guo and Zeng [GZ] defined a  $q$ -analogue of Salié numbers in the following way:

$$\sum_{n=0}^{\infty} \tilde{S}_n(q) \frac{x^n}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2} x^{2n}}{(q; q)_{2n}} / \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(q; q)_{2n}}$$

and hence

$$\sum_{k=0}^n \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q (-1)^k \tilde{S}_{2n-2k}(q) = q^{n^2} \quad \text{for any } n \in \mathbb{N}.$$

They conjectured that  $(-q; q)_n = \prod_{0 < k \leq n} (1 + q^k)$  divides  $\tilde{S}_{2n}(q)$  (in  $\mathbb{Z}[q]$ ).

We define the  $q$ -Salié numbers by

$$(1.4) \quad \sum_{n=0}^{\infty} S_n(q) \frac{x^n}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} x^{2n}}{(q; q)_{2n}} / \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}} x^{2n}}{(q; q)_{2n}}.$$

Multiplying both sides by  $\sum_{n=0}^{\infty} (-1)^n q^{\binom{2n}{2}} x^{2n}/(q; q)_{2n}$  one finds

$$(1.5) \quad \sum_{k=0}^n \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q (-1)^k q^{\binom{2k}{2}} S_{2n-2k}(q) = q^{n(n-1)} \quad (n \in \mathbb{N}).$$

In this paper we are able to prove the following  $q$ -analogue of Carlitz's result concerning Salié numbers.

**THEOREM 1.2.** *Let  $n \in \mathbb{N}$ . Then  $(-q; q)_n = \prod_{0 < k \leq n} (1 + q^k)$  divides  $S_{2n}(q)$  in the ring  $\mathbb{Z}[q]$ .*

**COROLLARY 1.1.** *For any  $n \in \mathbb{N}$  we have  $(-q; q)_n \mid \tilde{S}_{2n}(q)$  in the ring  $\mathbb{Z}[q]$  as conjectured by Guo and Zeng.*

*Proof.* By Theorem 1.2,  $S_{2n}(q) = (-q; q)_n P_n(q)$  for some  $P_n(q) \in \mathbb{Z}[q]$ . Let  $m$  be a natural number not smaller than  $\deg P$ . Then  $q^m P(q^{-1}) \in \mathbb{Z}[q]$ . Since

$$q^{\binom{n+1}{2}} \prod_{0 < k \leq n} (1 + q^{-k}) = \prod_{0 < k \leq n} (1 + q^k),$$

$q^{m+\binom{n+1}{2}} S_{2n}(q^{-1})$  is in  $\mathbb{Z}[q]$  and divisible by  $(-q; q)_n$ . If the equality

$$\tilde{S}_{2n}(q) = q^{\binom{2n}{2}} S_{2n}(q^{-1})$$

holds, then  $q^m \tilde{S}_{2n}(q)$  is divisible by  $(-q; q)_n$  and hence so is  $\tilde{S}_{2n}(q)$  since  $q^m$  is relatively prime to  $(-q; q)_n$ .

Now let us explain why  $\tilde{S}_n(q) = q^{\binom{n}{2}} S_n(q^{-1})$  for any  $n \in \mathbb{N}$ . In fact,

$$\begin{aligned} \sum_{n=0}^{\infty} q^{\binom{n}{2}} S_n(q^{-1}) \frac{x^n}{\prod_{0 < k \leq n} (1 - q^k)} &= \sum_{n=0}^{\infty} S_n(q^{-1}) \frac{(-q^{-1}x)^n}{\prod_{0 < k \leq n} (1 - q^{-k})} \\ &= \sum_{n=0}^{\infty} \frac{q^{-n(n-1)} (-q^{-1}x)^{2n}}{\prod_{0 < k \leq 2n} (1 - q^{-k})} \bigg/ \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{2n}{2}} (-q^{-1}x)^{2n}}{\prod_{0 < k \leq 2n} (1 - q^{-k})} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2} x^{2n}}{\prod_{0 < k \leq 2n} (1 - q^k)} \bigg/ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{\prod_{0 < k \leq 2n} (1 - q^k)} = \sum_{n=0}^{\infty} \tilde{S}_n(q) \frac{x^n}{(q; q)_n}. \end{aligned}$$

This concludes our proof. ■

In 1956 Carlitz [C1] investigated the coefficients of

$$\frac{\sinh x}{\sin x} = \sum_{n=0}^{\infty} C_n \frac{x^n}{n!},$$

where

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

We call those numbers  $C_n$  ( $n \in \mathbb{N}$ ) *Carlitz numbers*. In 1965 Carlitz [C2] proved a conjecture of Gandhi [G] which states that  $2^n$  divides the numerator of  $C_{2n}$ .

Now we define *q-Carlitz numbers*  $C_n(q)$  ( $n \in \mathbb{N}$ ) by

$$(1.6) \quad \sum_{n=0}^{\infty} C_n(q) \frac{x^n}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} x^{2n+1}}{(q; q)_{2n+1}} \bigg/ \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)} x^{2n+1}}{(q; q)_{2n+1}}.$$

Multiplying both sides by  $\sum_{n=0}^{\infty} (-1)^n q^{n(2n+1)} x^{2n+1} / (q; q)_{2n+1}$  we get the recursion

$$(1.7) \quad \sum_{k=0}^n \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q (-1)^k q^{k(2k+1)} C_{2n-2k}(q) = q^{n(n-1)} \quad (n \in \mathbb{N}).$$

By (1.7) and induction,

$$[1]_q [3]_q \cdots [2n+1]_q C_{2n}(q) \in \mathbb{Z}[q];$$

in particular,  $(2n+1)!! C_{2n} \in \mathbb{Z}$ . If  $j, k \in \mathbb{N}$  and  $q^j = -1$ , then  $q^{j(2k+1)} = -1$  and hence  $q^{2k+1} \neq 1$ . Thus  $q^j + 1$  is relatively prime to  $1 - q^{2k+1}$  for any  $j, k \in \mathbb{N}$ , and hence  $(-q; q)_n = \prod_{0 < j \leq n} (1 + q^j)$  is relatively prime to the denominator of  $C_{2n}(q)$ . This basic property will be used later.

Here is our *q*-analogue of Carlitz's divisibility result concerning Carlitz numbers.

**THEOREM 1.3.** *For any  $n \in \mathbb{N}$ ,  $(-q; q)_n$  divides the numerator of  $C_{2n}(q)$ .*

Note that  $E_{2k+1}(q) = S_{2k+1}(q) = C_{2k+1}(q) = 0$  for all  $k \in \mathbb{N}$  because

$$\sum_{n=0}^{\infty} E_n(q) \frac{x^n}{(q; q)_n}, \quad \sum_{n=0}^{\infty} S_n(q) \frac{x^n}{(q; q)_n}, \quad \sum_{n=0}^{\infty} C_n(q) \frac{x^n}{(q; q)_n}$$

are even functions in  $x$ .

Our approach to *q*-Euler numbers, *q*-Salié numbers and *q*-Carlitz numbers is quite different from that of Guo and Zeng [GZ]. The proofs of Theorems 1.1–1.3 depend on new recursions for *q*-Euler numbers, *q*-Salié numbers and *q*-Carlitz numbers. In the next section we will prove Theorem 1.1. In Section 3 we establish an auxiliary theorem which implies that if  $l \in \mathbb{Z}$  and  $n \in \mathbb{N}$  then

$$(1.8) \quad \sum_{\substack{k \in \mathbb{Z} \\ 2k+l \geq 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2k+l \end{bmatrix}_q \equiv 0 \pmod{(-q; q)_n}.$$

(We can also substitute  $2n+1$  for  $2n$  in (1.8).) Section 4 is devoted to the proofs of Theorems 1.2 and 1.3 on the basis of Section 3.

## 2. Proof of Theorem 1.1

**LEMMA 2.1.** *For any  $n \in \mathbb{N}$  we have*

$$(2.1) \quad E_{2n}(q) = 1 - \sum_{0 < k \leq n} (-q; q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2(n-k)}(q).$$

*Proof.* Let us recall the following three known identities (cf. Theorem 10.2.1 and Corollary 10.2.2 of [AAR]):

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-x)^n}{(q; q)_n} = (x; q)_{\infty}$$

where  $(x; q)_{\infty} = \prod_{n=0}^{\infty} (1 - xq^n)$ ,

$$\sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_{\infty}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1; q)_n x^n}{(q; q)_n} = \frac{(-x; q)_{\infty}}{(x; q)_{\infty}}.$$

Observe that

$$\begin{aligned} \frac{1}{2} \sum_{n=0}^{\infty} E_n(q) \frac{x^n}{(q; q)_n} &= \left( \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} + \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (-x)^n}{(q; q)_n} \right)^{-1} \\ &= \frac{1}{(x; q)_{\infty} + (-x; q)_{\infty}} \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{2} \left( \sum_{n=0}^{\infty} E_n(q) \frac{x^n}{(q; q)_n} \right) \left( 1 + \sum_{n=0}^{\infty} \frac{(-1; q)_n x^n}{(q; q)_n} \right) \\ = \frac{1}{(x; q)_{\infty} + (-x; q)_{\infty}} \left( 1 + \frac{(-x; q)_{\infty}}{(x; q)_{\infty}} \right) = \frac{1}{(x; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n}. \end{aligned}$$

Comparing the coefficients of  $x^n$  we obtain

$$\frac{1}{2} E_n(q) + \frac{1}{2} \sum_{k=0}^n (-1; q)_k \begin{bmatrix} n \\ k \end{bmatrix}_q E_{n-k}(q) = 1,$$

i.e.,

$$E_n(q) = 1 - \sum_{0 < k \leq n} (-q; q)_{k-1} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{n-k}(q).$$

Substituting  $2n$  for  $n$  in the last equality and recalling that  $E_{2j+1}(q) = 0$  for  $j \in \mathbb{N}$ , we immediately obtain the desired equality (2.1). ■

**COROLLARY 2.1.** *For any  $n \in \mathbb{N}$  we have*

$$(2.2) \quad E_{2n}(q) \equiv 1 \pmod{1+q}.$$

*Proof.* This follows from (2.1) because  $1+q$  divides  $(-q; q)_m$  for all  $m = 1, 2, 3, \dots$  ■

The following trick is simple but useful.

$$(2.3) \quad \prod_{k=0}^n (1 + q^{2^k}) = [2^{n+1}]_q \quad \text{for any } n \in \mathbb{N}.$$

In fact,

$$\begin{aligned} (1-q) \prod_{k=0}^n (1+q^{2^k}) &= (1-q^2) \prod_{0 < k \leq n} (1+q^{2^k}) \\ &= \dots = (1-q^{2^n})(1+q^{2^n}) = 1-q^{2^{n+1}}. \end{aligned}$$

LEMMA 2.2. *Let  $m, n, s, t$  be positive integers with  $2m \geq n$  and  $2 \nmid t$ . Then  $(-q; q)_m \left[ \begin{smallmatrix} 2^{st} \\ n \end{smallmatrix} \right]_q$  is divisible by  $(1+q)^{\lfloor (m-1)/2 \rfloor} [2^s]_{qt}$ , where we use  $\lfloor \alpha \rfloor$  to denote the greatest integer not exceeding a real number  $\alpha$ .*

*Proof.* Write  $n = 2^k l$  with  $k, l \in \mathbb{N}$  and  $2 \nmid l$ . Then

$$[n]_q = \frac{1-q^n}{1-q} = \frac{1-q^{2^k l}}{1-q^l} \cdot \frac{1-q^l}{1-q} = [2^k]_{q^l} [l]_q.$$

Obviously  $[2^k]_{q^l} = \prod_{0 \leq j < k} (1+q^{2^j l})$  divides  $(-q; q)_m = \prod_{j=1}^m (1+q^j)$  since  $m \geq n/2 = 2^{k-1} l$ . Thus  $[2^s]_{qt} = [2^s t]_q / [t]_q$  divides

$$[l]_q (-q; q)_m \left[ \begin{smallmatrix} 2^{st} \\ n \end{smallmatrix} \right]_q = \frac{(-q; q)_m}{[2^k]_{q^l}} [2^s t]_q \left[ \begin{smallmatrix} 2^{st} - 1 \\ n - 1 \end{smallmatrix} \right]_q.$$

Note that  $[2^s]_{qt} = \prod_{r=0}^{s-1} (1+q^{2^r t})$  is relatively prime to  $[l]_q = (1-q^l)/(1-q)$  since  $l \equiv 1 \pmod{2}$ . Therefore  $[2^s]_{qt}$  divides  $(-q; q)_m \left[ \begin{smallmatrix} 2^{st} \\ n \end{smallmatrix} \right]_q$ .

Clearly  $(1+q)^{\lfloor (m+1)/2 \rfloor}$  divides

$$\prod_{j=1}^{\lfloor (m+1)/2 \rfloor} (1+q^{2^{j-1}}) \cdot \prod_{j=1}^{\lfloor m/2 \rfloor} (1+q^{2^j}) = (-q; q)_m.$$

Since

$$[2^s]_{qt} = \frac{1-q^{2t}}{1-q^t} \cdot \frac{1-q^{2^{st}}}{1-q^{2t}} = (1+q) \sum_{j=0}^{t-1} (-q)^j \sum_{r=0}^{2^{s-1}-1} q^{2^{rt}}$$

and the sum  $\sum_{0 \leq j < t} (-q)^j \sum_{0 \leq r < 2^{s-1}} q^{2^{rt}}$  takes value  $2^{s-1} t \neq 0$  at  $q = -1$ , the polynomial  $[2^s]_{qt}$  is divisible by  $1+q$  but not by  $(1+q)^2$ . Therefore  $(1+q)^{\lfloor (m-1)/2 \rfloor} [2^s]_{qt}$  divides  $(-q; q)_m \left[ \begin{smallmatrix} 2^{st} \\ n \end{smallmatrix} \right]_q$  by the above. ■

*Proof of Theorem 1.1.* The case  $s = 0$  is easy. In fact,

$$E_{2n}(q) - E_{2n+2^0 t}(q) = E_{2n}(q) \equiv 1 = [2^0]_{qt} \pmod{(1+q)[2^0]_{qt}}$$

by Corollary 2.1.

To handle the case  $s > 0$ , we use induction on  $n$ . Assume that

$$(*) \quad E_{2m}(q) - E_{2m+2^s t}(q) \equiv [2^s]_{qt} \pmod{(1+q)[2^s]_{qt}} \quad \text{for } 0 \leq m < n.$$

(This holds trivially in the case  $n = 0$ .) In view of Lemma 2.1, we have

$$\begin{aligned}
& E_{2n}(q) - E_{2n+2st}(q) \\
&= \sum_{k=1}^{n+2^{s-1}t} (-q; q)_{2k-1} \left( \begin{bmatrix} 2n+2st \\ 2k \end{bmatrix}_q E_{2n+2st-2k}(q) - \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2n-2k}(q) \right),
\end{aligned}$$

where we set  $E_l(q) = 0$  for  $l < 0$ .

Let  $0 < k \leq n + 2^{s-1}t$ . Applying a  $q$ -analogue of the Chu–Vandermonde identity (cf. [AAR, Exercise 10.4(b)]), we find that

$$\begin{aligned}
& \begin{bmatrix} 2n+2st \\ 2k \end{bmatrix}_q E_{2n+2st-2k}(q) - \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2n-2k}(q) \\
&= E_{2n+2st-2k}(q) \sum_{j=0}^{2k} q^{(2n-j)(2k-j)} \begin{bmatrix} 2n \\ j \end{bmatrix}_q \begin{bmatrix} 2st \\ 2k-j \end{bmatrix}_q - \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2n-2k}(q) \\
&= E_{2n+2st-2k}(q) \sum_{j=0}^{2k-1} q^{(2n-j)(2k-j)} \begin{bmatrix} 2n \\ j \end{bmatrix}_q \begin{bmatrix} 2st \\ 2k-j \end{bmatrix}_q \\
&\quad + \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q (E_{2n+2st-2k}(q) - E_{2n-2k}(q)).
\end{aligned}$$

In view of the hypothesis (\*),

$$(-q; q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q (E_{2n+2st-2k}(q) - E_{2(n-k)}(q)) \equiv 0 \pmod{(1+q)[2^s]_{q^t}}.$$

In view of Lemma 2.2, if  $0 \leq j < 2k$  then  $(-q; q)_{2k-1} \begin{bmatrix} 2st \\ 2k-j \end{bmatrix}_q$  is divisible by  $(1+q)^{k-1}[2^s]_{q^t}$ . Therefore, if  $k > 1$  then  $(1+q)[2^s]_{q^t}$  divides

$$(-q; q)_{2k-1} \left( \begin{bmatrix} 2n+2st \\ 2k \end{bmatrix}_q E_{2n+2st-2k}(q) - \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q E_{2n-2k}(q) \right)$$

by the above. In the case  $k = 1$ ,

$$(-q; q)_{2k-1} \begin{bmatrix} 2st \\ 2k-1 \end{bmatrix}_q = (1+q)[2^s t]_q = (1+q)[2^s]_{q^t} [t]_q$$

and hence

$$\begin{aligned}
& (-q; q)_1 \left( \begin{bmatrix} 2n+2st \\ 2 \end{bmatrix}_q E_{2n+2st-2}(q) - \begin{bmatrix} 2n \\ 2 \end{bmatrix}_q E_{2n-2}(q) \right) \\
&\equiv (1+q) E_{2n+2st-2}(q) q^{(2n-0)(2-0)} \begin{bmatrix} 2n \\ 0 \end{bmatrix}_q \begin{bmatrix} 2st \\ 2 \end{bmatrix}_q \pmod{(1+q)[2^s]_{q^t}} \\
&\equiv E_{2n+2st-2}(q) q^{4n} \frac{1+q}{[2]_q} [2^s t]_q [2^s t - 1]_q \pmod{(1+q)[2^s]_{q^t}} \\
&\equiv E_{2n+2st-2}(q) q^{4n} [2^s]_{q^t} [t]_q (1+q[2^s t - 2]_q) \equiv [2^s]_{q^t} \pmod{(1+q)[2^s]_{q^t}};
\end{aligned}$$



in the last step we have noted that  $q^{4n} - 1, [t]_q - 1, [2^{st} - 2]_q$  are divisible by  $1 + q$ , and  $E_{2n+2^{st}-2}(q) \equiv 1 \pmod{1 + q}$  by Corollary 2.1.

Combining the above we obtain

$$E_{2n}(q) - E_{2n+2^{st}}(q) \equiv \sum_{k=1}^{n+2^{s-1}t} \delta_{k,1} [2^s]_{q^t} = [2^s]_{q^t} \pmod{(1 + q)[2^s]_{q^t}}.$$

This concludes the induction.

The proof of Theorem 1.1 is now complete. ■

REMARK 2.1. With a bit more effort we can prove the following more general result. For  $k = 1, 2, 3, \dots$  let

$$\sum_{n=0}^{\infty} E_n^{(k)}(q) \frac{x^n}{(q; q)_n} = \left( \sum_{n=0}^{\infty} q^{\binom{kn}{2}} \frac{x^{kn}}{(q; q)_{kn}} \right)^{-1}.$$

Given positive integers  $k, s, t$  with  $2 \nmid t$ , we have

$$E_{2k'n}^{(2k')}(q) - E_{2k'(n+2^{s-1}t)}^{(2k')}(q) \equiv (2k' - 1)[2^s]_{q^{k't}} \pmod{(1 + q^{k'})[2^s]_{q^{k't}}}$$

for all  $n \in \mathbb{N}$ , where  $k' = 2^{k-1}$ . This is a  $q$ -analogue of Conjecture 5.5 in [GZ].

### 3. An auxiliary theorem

THEOREM 3.1. For all  $m, n \in \mathbb{N}$ , both

$$(3.1) \quad S_n^m := \sum_{k=0}^n (-1)^k q^{k(k-1)+2m(n-k)} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q$$

and

$$(3.2) \quad T_n^m := \sum_{0 \leq k < n} (-1)^k q^{k(k-1)+2m(n-1-k)} \begin{bmatrix} 2n \\ 2k + 1 \end{bmatrix}_q$$

are divisible by  $(-q; q)_n = \prod_{0 < k \leq n} (1 + q^k)$  in the ring  $\mathbb{Z}[q]$ . Also, for any  $m, n \in \mathbb{N}$  and  $\delta \in \{0, 1\}$  we have the congruence

$$(3.3) \quad \sum_{k=0}^n (-1)^k q^{k(k+2m-1)} \begin{bmatrix} 2n \\ 2k + \delta \end{bmatrix}_q \equiv 0 \pmod{(-q; q)_n}.$$

*Proof.* (i) We use induction on  $n$  to prove the first part.

For any  $m \in \mathbb{N}$ , clearly both  $S_0^m = 1$  and  $T_0^m = 0$  are divisible by  $(-q; q)_0 = 1$ , also both  $S_1^m = q^{2m} - 1$  and  $T_1^m = [2]_q = 1 + q$  are multiples of  $(-q; q)_1 = 1 + q$ .

Now let  $n > 1$  be an integer and assume that  $(-q; q)_{n-1}$  divides both  $S_{n-1}^m$  and  $T_{n-1}^m$  for all  $m \in \mathbb{N}$ .

For each  $m \in \mathbb{Z}$  we have

$$\begin{aligned}
S_n^m &= \sum_{l=0}^n (-1)^{n-l} q^{(n-l)(n-l-1)+2ml} \begin{bmatrix} 2n \\ 2(n-l) \end{bmatrix}_q \\
&= (-1)^n q^{n(n-1)} \sum_{l=0}^n (-1)^l q^{l(l+1)-2ln+2lm} \begin{bmatrix} 2n \\ 2l \end{bmatrix}_q \\
&= (-1)^n q^{n(n-1)-2n(n-1-m)} S_n^{n-1-m} = (-1)^n q^{n(2m-n+1)} S_n^{n-1-m}.
\end{aligned}$$

In particular,

$$S_n^m = (-1)^n q^{n(n+1)} S_n^{-1} \quad \text{and} \quad S_n^{n-1} = (-1)^n q^{n(n-1)} S_n^0.$$

Similarly, for every  $m \in \mathbb{Z}$  we have

$$\begin{aligned}
T_n^m &= \sum_{l=0}^{n-1} (-1)^{n-1-l} q^{(n-1-l)(n-l-2)+2ml} \begin{bmatrix} 2n \\ 2(n-1-l)+1 \end{bmatrix}_q \\
&= (-1)^{n-1} q^{(n-1)(n-2)} \sum_{l=0}^{n-1} (-1)^l q^{l(l+1)-2l(n-1)+2lm} \begin{bmatrix} 2n \\ 2l+1 \end{bmatrix}_q \\
&= (-1)^{n-1} q^{(n-1)(2m-n+2)} T_n^{n-2-m}.
\end{aligned}$$

In particular,

$$T_n^{n-1} = (-1)^{n-1} q^{n(n-1)} T_n^{-1} \quad \text{and} \quad T_n^{n-2} = (-1)^{n-1} q^{(n-1)(n-2)} T_n^0.$$

For any  $m \in \mathbb{N}$ , clearly

$$\begin{aligned}
S_n^{m+1} - S_n^m &= \sum_{k=0}^n (-1)^k q^{k(k-1)+2m(n-k)} (q^{2(n-k)} - 1) \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \\
&= \sum_{k=0}^n (-1)^k q^{k(k-1)+2m(n-k)} (q^{2n} - 1) \begin{bmatrix} 2n-1 \\ 2k \end{bmatrix}_q \\
&= (q^{2n} - 1) \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)+2m(n-k)} q^{2k} \begin{bmatrix} 2n-2 \\ 2k \end{bmatrix}_q \\
&\quad + (q^{2n} - 1) \sum_{k=1}^{n-1} (-1)^k q^{k(k-1)+2m(n-k)} \begin{bmatrix} 2n-2 \\ 2k-1 \end{bmatrix}_q \\
&= (q^{2n} - 1) q^{2(m+n-1)} S_{n-1}^{m-1} - (q^{2n} - 1) q^{2(m+n-2)} T_{n-1}^{m-1} \\
&= (q^{2n} - 1) q^{2(m+n-2)} (q^2 S_{n-1}^{m-1} - T_{n-1}^{m-1})
\end{aligned}$$

and

$$\begin{aligned}
 qT_n^{m+1} - T_n^m &= \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)+2m(n-1-k)} (q^{2(n-1-k)+1} - 1) \begin{bmatrix} 2n \\ 2k+1 \end{bmatrix}_q \\
 &= \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)+2m(n-1-k)} (q^{2n} - 1) \begin{bmatrix} 2n-1 \\ 2k+1 \end{bmatrix}_q \\
 &= (q^{2n} - 1) \sum_{k=0}^{n-2} (-1)^k q^{k(k-1)+2m(n-1-k)} q^{2k+1} \begin{bmatrix} 2n-2 \\ 2k+1 \end{bmatrix}_q \\
 &\quad + (q^{2n} - 1) \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)+2m(n-1-k)} \begin{bmatrix} 2n-2 \\ 2k \end{bmatrix}_q \\
 &= (q^{2n} - 1) q^{2m+2n-3} T_{n-1}^{m-1} + (q^{2n} - 1) S_{n-1}^m,
 \end{aligned}$$

therefore by the induction hypothesis we have

$$S_n^{m+1} \equiv S_n^m \pmod{(-q; q)_n} \quad \text{and} \quad qT_n^{m+1} \equiv T_n^m \pmod{(-q; q)_n}.$$

(Note that  $q^{n(n-1)} S_{n-1}^{-1} = (-1)^{n-1} S_{n-1}^{n-1}$  and  $q^{(n-1)(n-2)} T_{n-1}^{-1} = (-1)^n T_{n-1}^{n-2}$  are both divisible by  $(-q; q)_{n-1}$  by the induction hypothesis.) Thus, if  $(-q; q)_n$  divides both  $S_n^0$  and  $T_n^0$  then it divides both  $S_n^m$  and  $T_n^m$  for every  $m = 0, 1, 2, \dots$

Observe that

$$\begin{aligned}
 S_n^0 &= \sum_{k=0}^n (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2n-2k \end{bmatrix}_q \\
 &= \sum_{k=1}^n (-1)^k q^{k(k-1)+2n-2k} \begin{bmatrix} 2n-1 \\ 2n-2k \end{bmatrix}_q + \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n-1 \\ 2n-2k-1 \end{bmatrix}_q \\
 &= \sum_{k=1}^n (-1)^k q^{k(k-1)} q^{2(2n-2k)} \begin{bmatrix} 2n-2 \\ 2n-2k \end{bmatrix}_q \\
 &\quad + \sum_{k=1}^{n-1} (-1)^k q^{k(k-1)} (q^{2n-2k} + q^{2n-2k-1}) \begin{bmatrix} 2n-2 \\ 2n-2k-1 \end{bmatrix}_q \\
 &\quad + \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n-2 \\ 2n-2k-2 \end{bmatrix}_q \\
 &= -q^{2n-2} S_{n-1}^1 - q^{2n-3} (1+q) T_{n-1}^0 + S_{n-1}^0
 \end{aligned}$$

and hence  $(-q; q)_{n-1}$  divides  $S_n^0$  by the induction hypothesis. Similarly,  $(-q; q)_{n-1}$  divides  $T_n^0 = -q^{2n-2} T_{n-1}^1 + (1+q) S_{n-1}^1 + T_{n-1}^0$ .

Since

$$(-1)^n q^{n(n-1)} S_n^0 = S_n^{n-1} \equiv S_n^0 \pmod{(-q; q)_n}$$

and

$$1 - (-1)^n q^{n(n-1)} \equiv 1 - (-1)^n (-1)^{n-1} = 2 \pmod{1 + q^n},$$

we must have  $S_n^0/(-q; q)_{n-1} \equiv 0 \pmod{1 + q^n}$  and hence  $(-q; q)_n \mid S_n^0$ . Similarly, as

$$q^{n-2} (-1)^{n-1} q^{(n-1)(n-2)} T_n^0 = q^{n-2} T_n^{n-2} \equiv T_n^0 \pmod{(-q; q)_n}$$

and  $1 - (-1)^{n-1} q^{n(n-2)} \equiv 2 \pmod{1 + q^n}$ , we have  $T_n^0/(-q; q)_{n-1} \equiv 0 \pmod{1 + q^n}$  and hence  $(-q; q)_n \mid T_n^0$ . This concludes our induction step and proves the first part.

(ii) Now fix  $m, n \in \mathbb{N}$  and  $\delta \in \{0, 1\}$ . We can verify (3.3) directly if  $n < 2$ . Below we assume  $n \geq 2$ . By a previous argument,

$$(-1)^n S_n^{m+n-1} = q^{n(2m+n-1)} S_n^{-m} = q^{n(n-1)} \sum_{k=0}^n (-1)^k q^{k(k+2m-1)} \left[ \begin{matrix} 2n \\ 2k \end{matrix} \right]_q$$

and

$$\begin{aligned} (-1)^{n-1} T_n^{m+n-2} &= q^{(n-1)(2m+n-2)} T_n^{-m} \\ &= q^{(n-1)(n-2)} \sum_{k=0}^{n-1} (-1)^k q^{k(k+2m-1)} \left[ \begin{matrix} 2n \\ 2k+1 \end{matrix} \right]_q. \end{aligned}$$

Thus, applying the first part we immediately get (3.3).

The proof of Theorem 3.1 is now complete. ■

REMARK 3.1. Theorem 3.1 is somewhat difficult and sophisticated, however it is easy to evaluate the sums

$$\sum_{k=0}^n (-1)^k \binom{2n}{2k} = \sum_{k=0}^{2n} \binom{2n}{k} \frac{i^k + (-i)^k}{2}$$

and

$$\sum_{0 \leq k < n} (-1)^k \binom{2n}{2k+1} = \sum_{k=0}^{2n} \binom{2n}{k} \frac{i^k - (-i)^k}{2i}.$$

Now let us explain why (1.8) holds for any  $l \in \mathbb{Z}$  and  $n \in \mathbb{N}$ . Write  $l = 2m + \delta$  with  $m \in \mathbb{Z}$  and  $\delta \in \{0, 1\}$ . Then

$$\sum_{\substack{k \in \mathbb{Z} \\ 2k+l \geq 0}} (-1)^k q^{k(k-1)} \left[ \begin{matrix} 2n \\ 2k+l \end{matrix} \right]_q = \sum_{k+m \in \mathbb{N}} (-1)^k q^{k(k-1)} \left[ \begin{matrix} 2n \\ 2(k+m) + \delta \end{matrix} \right]_q$$

$$\begin{aligned}
 &= \sum_{k \in \mathbb{N}} (-1)^{k-m} q^{(k-m)(k-m-1)} \begin{bmatrix} 2n \\ 2k + \delta \end{bmatrix}_q \\
 &= (-1)^m \sum_{k=0}^{n-\delta} (-1)^k q^{k(k-1)-2km+m(m+1)} \begin{bmatrix} 2n \\ 2k + \delta \end{bmatrix}_q.
 \end{aligned}$$

So (1.8) follows from Theorem 3.1. Note also that

$$\begin{aligned}
 \sum_{\substack{k \in \mathbb{Z} \\ 2k+l \geq 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n+1 \\ 2k+l \end{bmatrix}_q - \sum_{\substack{k \in \mathbb{Z} \\ 2k+l-1 \geq 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2k+l-1 \end{bmatrix}_q \\
 &= \sum_{\substack{k \in \mathbb{Z} \\ 2k+l \geq 0}} (-1)^k q^{k(k-1)+2k+l} \begin{bmatrix} 2n \\ 2k+l \end{bmatrix}_q \\
 &= q^l \sum_{\substack{k \in \mathbb{Z} \\ 2k+l-2 \geq 0}} (-1)^{k-1} q^{k(k-1)} \begin{bmatrix} 2n \\ 2k+l-2 \end{bmatrix}_q
 \end{aligned}$$

and thus

$$(3.4) \quad \sum_{\substack{k \in \mathbb{Z} \\ 2k+l \geq 0}} (-1)^k q^{k(k-1)} \begin{bmatrix} 2n+1 \\ 2k+l \end{bmatrix}_q \equiv 0 \pmod{(-q; q)_n}.$$

#### 4. Proofs of Theorems 1.2 and 1.3

LEMMA 4.1. *We have*

$$(4.1) \quad 1 + \sum_{n=1}^{\infty} (-q; q)_{2n-1} \frac{(-1)^n x^{2n}}{(q; q)_{2n}} = \sum_{k=0}^{\infty} q^{\binom{2k}{2}} \frac{(-1)^k x^{2k}}{(q; q)_{2k}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q; q)_{2l}}$$

and

$$(4.2) \quad \sum_{n=0}^{\infty} (-q; q)_{2n} \frac{(-1)^n x^{2n+1}}{(q; q)_{2n+1}} = \sum_{k=0}^{\infty} q^{\binom{2k+1}{2}} \frac{(-1)^k x^{2k+1}}{(q; q)_{2k+1}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q; q)_{2l}}.$$

*Proof.* Let  $\delta \in \{0, 1\}$ . Then

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{2k+\delta}{2}} x^{2k+\delta}}{(q; q)_{2k+\delta}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q; q)_{2l}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+\delta}}{(q; q)_{2n+\delta}} \sum_{k=0}^n q^{\binom{2k+\delta}{2}} \begin{bmatrix} 2n+\delta \\ 2k+\delta \end{bmatrix}_q.$$

By the  $q$ -binomial theorem (cf. [AAR, Corollary 10.2.2(c)]),

$$(x; q)_m = \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k \quad \text{for any } m \in \mathbb{N}.$$

Thus

$$\begin{aligned} 2 \sum_{k=0}^n q^{\binom{2k+\delta}{2}} \begin{bmatrix} 2n+\delta \\ 2k+\delta \end{bmatrix}_q &= \sum_{l=0}^{2n+\delta} q^{\binom{l}{2}} \begin{bmatrix} 2n+\delta \\ l \end{bmatrix}_q + \sum_{l=0}^{2n+\delta} (-1)^{\delta+l} q^{\binom{l}{2}} \begin{bmatrix} 2n+\delta \\ l \end{bmatrix}_q \\ &= (-1; q)_{2n+\delta} + (-1)^\delta (1; q)_{2n+\delta} \end{aligned}$$

and hence

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{2k+\delta}{2}} x^{2k+\delta}}{(q; q)_{2k+\delta}} &\sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q; q)_{2l}} \\ &= \sum_{n=0}^{\infty} \left( \frac{(-1)^n x^{2n+\delta}}{(q; q)_{2n+\delta}} \cdot \frac{(-1; q)_{2n+\delta} + (-1)^\delta (1; q)_{2n+\delta}}{2} \right) \\ &= \begin{cases} 1 + \sum_{n=1}^{\infty} (-q; q)_{2n-1} \frac{(-1)^n x^{2n}}{(q; q)_{2n}} & \text{if } \delta = 0, \\ \sum_{n=0}^{\infty} (-q; q)_{2n} \frac{(-1)^n x^{2n+1}}{(q; q)_{2n+1}} & \text{if } \delta = 1. \end{cases} \end{aligned}$$

We are done. ■

REMARK 4.1. (4.1) and (4.2) are  $q$ -analogues of the trigonometric identities

$$\frac{1 + \cos(2x)}{2} = \cos^2 x \quad \text{and} \quad \frac{\sin(2x)}{2} = \sin x \cos x$$

respectively.

LEMMA 4.2. *Let  $n \geq k \geq 1$  be integers. Then both  $(-q; q)_k \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q$  and  $(-q; q)_k \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q$  are divisible by*

$$(-q^{n-k+1}; q)_k = \prod_{j=1}^k (1 + q^{n-j+1}).$$

*Proof.* Observe that

$$\begin{aligned} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q &= \prod_{j=1}^{2k} \frac{1 - q^{2n-j+1}}{1 - q^j} = \prod_{j=1}^k \frac{(1 - q^{2n-2j+1})(1 - q^{2n-(2j-1)+1})}{(1 - q^{2j})(1 - q^{2j-1})} \\ &= \prod_{j=1}^k \frac{(1 - q^{n-j+1})(1 + q^{n-j+1})(1 - q^{2n-2j+1})}{(1 - q^j)(1 + q^j)(1 - q^{2j-1})} \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{j=1}^k (1 + q^{n-j+1})}{(-q; q)_k} \prod_{j=1}^k \frac{1 - q^{2n-2j+1}}{1 - q^{2j-1}} \end{aligned}$$

and hence

$$(-q^{n-k+1}; q)_k \mid (-q; q)_k \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \prod_{j=1}^k (1 - q^{2j-1}).$$

Recall that the polynomial  $(-q^{n-k+1}; q)_k = \prod_{n-k < l \leq n} (1 + q^l)$  is relatively prime to  $\prod_{j=1}^k (1 - q^{2j-1})$ . Therefore  $(-q^{n-k+1}; q)_k \mid (-q; q)_k \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q$ .

Since  $[2k + 1]_q$  is also relatively prime to  $(-q^{n-k+1}; q)_k$ , we have

$$(-q; q)_k \begin{bmatrix} 2n + 1 \\ 2k + 1 \end{bmatrix}_q = (-q; q)_k \frac{[2n + 1]_q}{[2k + 1]_q} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \equiv 0 \pmod{(-q^{n-k+1}; q)_k}.$$

This concludes the proof. ■

REMARK 4.2. Lemma 4.2 yields a trivial result as  $q \rightarrow 1$ .

*Proof of Theorem 1.2.* Clearly

$$\begin{aligned} f(x) &:= \left( \sum_{n=0}^{\infty} S_{2n}(q) \frac{x^{2n}}{(q; q)_{2n}} \right) \left( 1 + \sum_{n=1}^{\infty} (-q; q)_{2n-1} \frac{(-1)^n x^{2n}}{(q; q)_{2n}} \right) \\ &= \sum_{n=0}^{\infty} S_{2n}(q) \frac{x^{2n}}{(q; q)_{2n}} + \sum_{n=1}^{\infty} \frac{x^{2n}}{(q; q)_{2n}} \sum_{k=1}^n (-1)^k (-q; q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2n-2k}(q). \end{aligned}$$

On the other hand, by (4.1) we have

$$f(x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)} x^{2k}}{(q; q)_{2k}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q; q)_{2l}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q; q)_{2n}} \sum_{k=0}^n (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q.$$

Therefore

$$\begin{aligned} S_{2n}(q) + \sum_{0 < k \leq n} (-1)^k (-q; q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2n-2k}(q) \\ = (-1)^n \sum_{k=0}^n (-1)^k q^{k(k-1)} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q \equiv 0 \pmod{(-q; q)_n} \end{aligned}$$

with the help of (1.8) or Theorem 3.1. If  $(-q; q)_l \mid S_{2l}(q)$  for all  $0 \leq l < n$ , then

$$S_{2n}(q) \equiv - \sum_{0 < k \leq n} (-1)^k (-q; q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q S_{2n-2k}(q) \equiv 0 \pmod{(-q; q)_n}$$

since  $\prod_{0 < j \leq n-k} (1 + q^j)$  divides  $S_{2n-2k}(q)$  and  $\prod_{n-k < j \leq n} (1 + q^j)$  divides  $(-q; q)_{2k-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q$  by Lemma 4.2. Thus we have the desired result by induction. ■

REMARK 4.3. As  $q \rightarrow 1$  our new recursion for  $q$ -Salié numbers yields a useful recursion for Salié numbers:

$$S_{2n} + \sum_{0 < k \leq n} (-1)^k 2^{2k-1} \binom{2n}{2k} S_{2n-2k} = (-1)^n \sum_{k=0}^n (-1)^k \binom{2n}{2k},$$

from which Carlitz's result  $2^n \mid S_{2n}$  follows by induction.

*Proof of Theorem 1.3.* It is apparent that

$$\begin{aligned} g(x) &:= \left( \sum_{n=0}^{\infty} C_{2n}(q) \frac{x^{2n}}{(q; q)_{2n}} \right) \left( \sum_{n=0}^{\infty} (-q; q)_{2n} \frac{(-1)^n x^{2n+1}}{(q; q)_{2n+1}} \right) \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(q; q)_{2n+1}} \sum_{k=0}^n (-1)^k (-q; q)_{2k} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q C_{2n-2k}(q). \end{aligned}$$

On the other hand, (4.2) implies that

$$\begin{aligned} g(x) &= \sum_{k=0}^{\infty} \frac{q^{k(k-1)} x^{2k+1}}{(q; q)_{2k+1}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(q; q)_{2l}} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(q; q)_{2n+1}} \sum_{k=0}^n (-1)^{n-k} q^{k(k-1)} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q. \end{aligned}$$

Therefore we have the recurrence relation

$$\sum_{k=0}^n (-1)^k (-q; q)_{2k} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q C_{2n-2k}(q) = \sum_{k=0}^n (-1)^{n-k} q^{k(k-1)} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q.$$

The right-hand side of the last equality is a multiple of  $(-q; q)_n$  by (3.4). So we have

$$\sum_{k=0}^n (-1)^k (-q; q)_{2k} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q C_{2n-2k}(q) \equiv 0 \pmod{(-q; q)_n}.$$

Assume that  $(-q; q)_l$  divides the numerator of  $C_{2l}(q)$  for each  $0 \leq l < n$ . Then  $(-q; q)_n$  divides the numerator of  $(-q; q)_{2k} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q C_{2n-2k}(q)$  for each  $0 < k \leq n$ , because  $\prod_{0 < j \leq n-k} (1 + q^j)$  divides the numerator of  $C_{2n-2k}(q)$  and  $\prod_{n-k < j \leq n} (1 + q^j)$  divides  $(-q; q)_{2k} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_q$  by Lemma 4.2. Thus  $(-q; q)_n$  must also divide the numerator of  $\begin{bmatrix} 2n+1 \\ 1 \end{bmatrix}_q C_{2n}(q) = [2n+1]_q C_{2n}(q)$ . Recall that  $[2n+1]_q$  is relatively prime to  $(-q; q)_n$ . So the numerator of  $C_{2n}(q)$  is divisible by  $(-q; q)_n$ .

In view of the above, the desired result follows by induction on  $n$ . ■



REMARK 4.4. As  $q \rightarrow 1$  our new recursion for  $q$ -Carlitz numbers yields the following recurrence relation for Carlitz numbers:

$$\sum_{k=0}^n (-1)^k 2^{2k} \binom{2n+1}{2k+1} C_{2n-2k} = (-1)^n \sum_{k=0}^n (-1)^k \binom{2n+1}{2k+1}.$$

From this one can easily deduce the Carlitz congruence  $C_{2n} \equiv 0 \pmod{2^n}$ .

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### References

- [AAR] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.
- [C1] L. Carlitz, *The coefficients of  $\sinh x/\sin x$* , Math. Mag. 29 (1956), 193–197.
- [C2] —, *The coefficients of  $\cosh x/\cos x$* , Monatsh. Math. 69 (1965), 129–135.
- [E] R. Ernvall, *Generalized Bernoulli numbers, generalized irregular primes, and class number*, Ann. Univ. Turku. Ser. A I 178 (1979), 72 pp.
- [G] J. M. Gandhi, *The coefficients of  $\cosh x/\cos x$  and a note on Carlitz's coefficients of  $\sinh x/\sin x$* , Math. Mag. 31 (1958), 185–191.
- [GZ] V. J. W. Guo and J. Zeng, *Some arithmetic properties of the  $q$ -Euler numbers and  $q$ -Salié numbers*, European J. Combin. 27 (2006), 884–895.
- [St] M. A. Stern, *Zur Theorie der Eulerschen Zahlen*, J. Reine Angew. Math. 79 (1875), 67–98.
- [Su] Z. W. Sun, *On Euler numbers modulo powers of two*, J. Number Theory 115 (2005), 371–380.
- [W] S. S. Wagstaff, Jr., *Prime divisors of the Bernoulli and Euler numbers*, in: Number Theory for the Millennium, III (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, 357–374.

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