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A hybrid of theorems of Goldbach and Piatetski-Shapiro

by

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1. Introduction. It is well known that almost all sufficiently large even integers can be written as a sum of two primes. We state this in the form that for almost all sufficiently large even integer n,

$$\sum_{n=p_1+p_2}(\log p_1)(\log p_2)=(1+o(1))C(n)n, \tag{1.1}$$

where

$$C(n)=\frac{n}{\phi(n)}\prod_{p\nmid n}\left(1-\frac{1}{(p-1)^2}\right). \tag{1.2}$$

It is interesting to find more familiar thin sets of primes which serve this purpose. An example is the set of Piatetski-Shapiro primes of type γ which are of the form $[n^{1/\gamma}]$. We denote this set by P_γ.

For the counting function of P_γ, Piatetski-Shapiro [8] first proved that for $11/12<\gamma\le 1$ (the case $\gamma>1$ is trivial),

$$P_\gamma(x)=\sum_{\substack{p\le x\\ p=[n^{1/\gamma}]}}1=(1+o(1))\,\frac{x^\gamma}{\log x}. \tag{1.3}$$

Heath-Brown [3] extended the range to $662/755<\gamma\le 1$. The best result is due to Liu and Rivat [6].

In this paper we shall apply the sieve method combined with the circle method to prove the following theorems.

THEOREM 1. *If γ is fixed with $8/9<\gamma\le 1$, then for almost all sufficiently large even integers n,*

$$T_1(n)=\frac{1}{\gamma}\sum_{\substack{p_1+p_2=n\\ p_1\in P_\gamma}}p_1^{1-\gamma}(\log p_1)(\log p_2)=(1+o(1))C(n)n.$$

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Theorem 2. *If γ_1, γ_2 are fixed with $27/29 < \gamma_i \le 1$, then for almost all sufficiently large even integers n,*

$$T_2(n) = \frac{1}{\gamma_1\gamma_2} \sum_{\substack{p_1+p_2=n \\ p_i \in P_{\gamma_i}}} p_1^{1-\gamma_1} p_2^{1-\gamma_2} (\log p_1)(\log p_2) = (1+o(1))C(n)n.$$

Theorem 3. *If γ is fixed with $21/23 < \gamma \le 1$, then for almost all sufficiently large even integers n,*

$$T(n) = \sum_{\substack{p_1+p_2=n \\ p_i \in P_\gamma}} 1 \ge \frac{\varrho_0 C(n) n^{2\gamma-1}}{\log^2 n},$$

where ϱ_0 is a definite positive constant.

Throughout this paper, we always assume that n, N are sufficiently large even integers and ε is a sufficiently small positive constant. Assume that c, c_1, c_2 are positive constants which have different values at different places. $m \sim M$ means that there are positive constants c_1 and c_2 such that $c_1 M < m \le c_2 M$. We also assume that γ is fixed with $21/23 < \gamma \le 1$ and that

$$N(d) = [-d^\gamma] - [-(d+1)^\gamma]. \tag{1.4}$$

2. Some preliminary lemmas. In the following, we assume that

$$H = N^{1-\gamma+\Delta+8\varepsilon}. \tag{2.1}$$

By the discussion in [1], the asymptotic formula, valid for $0 \le \Delta \le 1-\gamma$,

$$\sum_{N/10 < p \le N} N(p) e(\alpha p) = \gamma \sum_{N/10 < p \le N} p^{\gamma-1} e(\alpha p) + O(N^{\gamma-\Delta-5\varepsilon}), \tag{2.2}$$

depends on the fact that for $J \le H$ and $0 \le u \le 1$,

$$\min\left(1, \frac{N^{1-\gamma}}{J}\right) \sum_{h\sim J} \Big| \sum_{n \sim N} \Lambda(n) e(\alpha n + h(n+u)^\gamma) \Big| \ll N^{1-\Delta-6\varepsilon}. \tag{2.3}$$

Lemma 1 ([1, Proposition 2]). *Assume that $N^{1-\gamma+2\Delta+30\varepsilon} \ll M \ll N^{5\gamma-4-6\Delta-120\varepsilon}$ and that $a(m), b(k) = O(1)$. Then for $J \le H$ and $0 \le u \le 1$, we have*

$$\min\left(1, \frac{N^{1-\gamma}}{J}\right) \sum_{h\sim J} \Big| \sum_{m\sim M} \sum_{km\sim N} a(m) b(k) e(\alpha km + h(km+u)^\gamma) \Big| \ll N^{1-\Delta-10\varepsilon}.$$

Lemma 2 ([1, Proposition 3]). *Assume that $M \ll N^{4\gamma-3-5\Delta-50\varepsilon}$, $a(m) = O(1)$ and*

$$6(1-\gamma) + \tfrac{19}{3}\Delta < 1. \tag{2.4}$$

Then for $J \le H$ *and* $0 \le u \le 1$, *we have*

$$\min\left(1, \frac{N^{1-\gamma}}{J}\right) \sum_{h\sim J} \Big| \sum_{m\sim M} a(m) \sum_{km\sim N} e(\alpha km + h(km+u)^{\gamma}) \Big| \ll N^{1-\Delta-10\varepsilon}.$$

LEMMA 3. *We have*

$$\sum_{N/10<p\le N} N(p)e(\alpha p) = \gamma \sum_{N/10<p\le N} p^{\gamma-1} e(\alpha p) + O(N^{\gamma-5\varepsilon}). \tag{2.5}$$

Proof. Taking $\Delta = 0$ and $V = N^{1-\gamma+30\varepsilon}$ in the proof of Lemma 4 of [5] yields the assertion.

We define $w(u)$ as the continuous solution of the equations

$$w(u) = 1/u, \qquad 1 \le u \le 2, \tag{2.6}$$

$$(uw(u))' = w(u-1), \quad u > 2. \tag{2.7}$$

$w(u)$ is called *Buchstab's function*; it plays an important role in finding asymptotic formulas in the sieve method. In particular,

$$w(u) = \begin{cases} \dfrac{1+\log(u-1)}{u}, & 2 \le u \le 3, \\[2ex] \dfrac{1+\log(u-1)}{u} + \dfrac{1}{u} \displaystyle\int_2^{u-1} \frac{\log(t-1)}{t}\, dt, & 3 \le u \le 4. \end{cases} \tag{2.8}$$

LEMMA 4 ([5, Lemma 8]). *We have the following bounds*:

(1) $w(u) \ge 0.5607$ *for* $u \ge 2.47$;
(2) $w(u) \le 0.5644$ *for* $u \ge 3$.

LEMMA 5 ([5, Lemma 9]). *Assume that* $\mathcal{E} = \{n : x < n \le 2x\}$ *and that* $z \le x$. *Let*

$$P(z) = \prod_{p<z} p.$$

Then for sufficiently large x *and* z, *we have*

$$S(\mathcal{E}, z) = \sum_{\substack{x<n\le 2x \\ (n,P(z))=1}} 1 = \left(w\left(\frac{\log x}{\log z}\right) + O(\varepsilon)\right)\frac{x}{\log z}.$$

3. The proofs of Theorems 1 and 2. The reduction of Theorems 1 and 2 to the estimate (1.1) is by means of the identity

$$f_1 f_2 - g_1 g_2 = (f_1 - g_1) f_2 + g_1 (f_2 - g_2). \tag{3.1}$$

We let $g_1 = g_2 = g = \sum_{p<n} e(\alpha p) \log p$. Then the sum in (1.1) is given by

$$R(n) = \int_0^1 g^2(\alpha) e(-\alpha n)\, d\alpha.$$

We let, for $1 \le i \le 2$,

$$f_i(\alpha) = \frac{1}{\gamma_i} \sum_{p<n} N(p) e(\alpha p) p^{1-\gamma_i} \log p,$$

and

$$T(n) = \int_0^1 f_1(\alpha) f_2(\alpha) e(-\alpha n)\, d\alpha.$$

Thus $T(n) = R(n) + E$, where $E = \int_0^1 (f_1 f_2 - g_1 g_2) e(-\alpha n)\, d\alpha$.

By the Parseval theorem, Cauchy's inequality and (3.1) we have

$$(3.2) \qquad \sum_{N<n\le 2N} |E|^2 \le \int_0^1 |f_1 f_2 - g_1 g_2|^2\, d\alpha$$
$$\ll \sup_\alpha |f_1 - g|^2 \int_0^1 |f_2|^2\, d\alpha + \sup_\alpha |f_2 - g|^2 \int_0^1 |g|^2\, d\alpha.$$

Since

$$\int_0^1 |f_2|^2\, d\alpha \ll n^{2-\gamma_2} \log n, \qquad \int_0^1 |g|^2\, d\alpha \ll n \log n,$$

we require, for $1 \le i \le 2$, an estimate

$$\sup_\alpha |f_i - g| \ll n^{1-\delta_i-\varepsilon}$$

for some $\varepsilon > 0$, where $\delta_2 = 0$, $\delta_1 = \frac{1}{2}(1-\gamma_2)$, and then we have

$$\sum_{N<n\le 2N} |E|^2 \ll N^{3-\varepsilon}.$$

Hence Theorems 1 and 2 follow from

THEOREM 4. *Let γ, δ satisfy $0 < \gamma \le 1$, $0 < \delta$ and $9(1-\gamma) + 11\delta < 1$. Then uniformly in α, we have*

$$\frac{1}{\gamma} \sum_{\substack{p<N \\ p=[n^{1/\gamma}]}} e(\alpha p) p^{1-\gamma} \log p = \sum_{p<N} e(\alpha p) \log p + O(N^{1-\delta}),$$

where the implied constant may depend on γ and δ only.

Proof. If $9(1-\gamma)+11\delta < 1$, then $1-4(1-\gamma)-5\delta-\varepsilon > 5(1-\gamma)+6\delta+\varepsilon$ provided ε is sufficiently small, hence by Propositions 2 and 3 of [1] we can take $a = 1-(1-\gamma)-2\delta-\varepsilon$ in Proposition 3 of [1]. The conditions (3.7) and (3.8) of [1] are satisfied by Section 6 of [1], the condition (3.9) $1-a < c/2$ of [1] follows for sufficiently small ε. Since $3(1-\gamma)+6\delta < 1$, by Propositions 1, 2 and 3 of [1], the conclusion follows.

4. Mean value formulas in the sieve method. From now on we assume $21/23 < \gamma \le 27/29$.

LEMMA 6. *Assume that* $M, K \ll N^{7/23}$ *and that* $a(m), b(k) = O(1)$. *Let*

$$I(n) = \sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}} \frac{\gamma^2(n_1n_2)^{\gamma-1}}{\log n_2}. \tag{4.1}$$

Then for $N < n \le 2N$, *except for* $O(N\log^{-2} N)$ *values*, *we have*

$$\sum_{\substack{m\sim M,\, k\sim K\\ (m,n)=(k,n)=1}} a(m)b(k)\Bigg(\sum_{\substack{n=mkl+p_2\\ n/10<mkl\le n\\ n/10<p_2\le n}} N(mkl)N(p_2) - \frac{1}{\phi(mk)}\, I(n)\Bigg) = O\bigg(\frac{N^{2\gamma-1}}{\log^{20} N}\bigg).$$

Proof. We have

$$\begin{aligned}\varSigma_1 &= \sum_{\substack{m\sim M,\, k\sim K\\ (m,n)=(k,n)=1}} a(m)b(k) \sum_{\substack{n=mkl+p_2\\ n/10<mkl\le n\\ n/10<p_2\le n}} N(mkl)N(p_2)\\ &= \int_0^1 \sum_{\substack{n/10<mkl\le n\\ m\sim M,\, k\sim K\\ (m,n)=(k,n)=1}} a(m)b(k)N(mkl)e(\alpha mkl)\cdot \sum_{n/10<p\le n} N(p)e(\alpha p)e(-\alpha n)\, d\alpha.\end{aligned}$$

Let

$$g(\alpha) = \sum_{\substack{n/10<mkl\le n\\ m\sim M,\, k\sim K\\ (m,n)=(k,n)=1}} a(m)b(k)N(mkl)e(\alpha mkl),$$

$$f(\alpha) = \gamma \sum_{\substack{n/10<mkl\le n\\ m\sim M,\, k\sim K\\ (m,n)=(k,n)=1}} a(m)b(k)(mkl)^{\gamma-1}e(\alpha mkl).$$

By the discussion in [1], the asymptotic formula

$$g(\alpha) = f(\alpha) + O(N^{3\gamma/2-1/2-5\varepsilon}) \tag{4.2}$$

depends on the fact that for $J \le H_1 = N^{3(1-\gamma)/2+8\varepsilon}$ and $0 \le u \le 1$,

$$\begin{aligned}&\min\bigg(1, \frac{N^{1-\gamma}}{J}\bigg)\\ &\quad\times \sum_{h\sim J}\Big|\sum_{m\sim M}\sum_{k\sim K}\sum_{mkl\sim N} a(m)b(k)e(\alpha kml + h(kml+u)^{\gamma})\Big|\\ &\qquad\ll N^{1/2+\gamma/2-6\varepsilon}.\end{aligned} \tag{4.3}$$

If either M or K is larger than $N^{4/23}$, then by Lemma 1 with $\Delta = \frac{1}{2}(1-\gamma)$, (4.3) holds. If $M, K \le N^{4/23}$, then $MK \ll N^{8/23} \ll N^{13\gamma/2-11/2-50\varepsilon}$. By Lemma 2 with $\Delta = \frac{1}{2}(1-\gamma)$, (4.3) also holds. Hence (4.2) holds.

Let

$$D(\alpha) = \sum_{n/10<p\le n} N(p)e(\alpha p), \qquad S(\alpha) = \gamma \sum_{n/10<p\le n} p^{\gamma-1}e(\alpha p).$$

By (2.5) and (4.2) we have

$$\begin{aligned} g(\alpha)D(\alpha) - f(\alpha)S(\alpha) &= (g(\alpha)-f(\alpha))D(\alpha) + f(\alpha)(D(\alpha)-S(\alpha)) \\ &\ll N^{3\gamma/2-1/2-5\varepsilon}|D(\alpha)| + N^{\gamma-5\varepsilon}|f(\alpha)|. \end{aligned}$$

Thus

$$\Sigma_1 = \int_0^1 g(\alpha)D(\alpha)e(-\alpha n)\,d\alpha = \int_0^1 f(\alpha)S(\alpha)e(-\alpha n)\,d\alpha + \Psi.$$

We note that $N(p) = 0$ or 1 and that $p \in P_\gamma$ is equivalent to $N(p) = 1$; we also have the estimate $\sum_{p\le n} N(p) \le \sum_{l\le n} N(l) \ll N^\gamma$. Hence by the Parseval theorem,

$$\begin{aligned} \sum_{N<n\le 2N} |\Psi|^2 &\le \int_0^1 |g(\alpha)D(\alpha) - f(\alpha)S(\alpha)|^2\,d\alpha \\ &\ll N^{3\gamma-1-10\varepsilon}\int_0^1 |D(\alpha)|^2\,d\alpha + N^{2\gamma-10\varepsilon}\int_0^1 |f(\alpha)|^2\,d\alpha \ll N^{4\gamma-1-9\varepsilon}. \end{aligned} \tag{4.4}$$

In the following we investigate

$$\Sigma_2 = \int_0^1 f(\alpha)S(\alpha)e(-\alpha n)\,d\alpha.$$

Let $Q = N\log^{-80} N$ and

$$E_1 = \bigcup_{1\le q\le \log^{80} N} \bigcup_{\substack{1\le a\le q\\(a,q)=1}} I(a,q), \qquad E_2 = (-1/Q, 1-1/Q]\backslash E_1, \tag{4.5}$$

where

$$I(a,q) = [a/q - q^{-1}Q^{-1}, a/q + q^{-1}Q^{-1}]. \tag{4.6}$$

Then E_1 is the major arcs, and E_2 is the minor arcs. Thus

$$\Sigma_2 = \Big(\int_{E_1} + \int_{E_2}\Big) f(\alpha)S(\alpha)e(-\alpha n)\,d\alpha.$$

For any $\alpha \in E_2$, there is one q ($\log^{80} N < q \le Q$) such that $|\alpha - a/q| < 1/(qQ)$. By (2) in Section 25 of [2] we have $S(\alpha) \ll N^\gamma \log^{-35} N$. Hence by the Parseval theorem and Lemma 6 of [5] we have

$$\sum_{N<n\le 2N}\Big|\int_{E_2} f(\alpha)S(\alpha)e(-\alpha n)\,d\alpha\Big|^2 \tag{4.7}$$
$$\le \int_{E_2}|f(\alpha)S(\alpha)|^2\,d\alpha \ll N^{2\gamma}\log^{-70}N\int_0^1|f(\alpha)|^2\,d\alpha \ll N^{4\gamma-1}\log^{-60}N.$$

If $\alpha = a/q+\beta\in E_1$, let $R=MK$ and

$$j(r)=\gamma\sum_{\substack{mk=r\\ m\sim M,\,k\sim K}} a(m)b(k). \tag{4.8}$$

As in (25) of [5] we have

$$f(\alpha)=\sum_{\substack{r\sim R\\ (r,n)=1,\,q\mid r}}\frac{j(r)}{r}\sum_{n/10<s\le n}s^{\gamma-1}e(\beta s)+O(N^{\gamma-\varepsilon}). \tag{4.9}$$

By (27) of [5] we obtain

$$S(\alpha)=\gamma\frac{\mu(q)}{\phi(q)}\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log s}+O(N^\gamma\exp(-c_2\sqrt{\log N})). \tag{4.10}$$

Hence

$$\begin{aligned}
\Sigma_3 &= \int_{E_1} f(\alpha)S(\alpha)e(-\alpha n)\,d\alpha\\
&= \sum_{q\le\log^{80}N}\sum_{\substack{a=0\\ (a,q)=1}}^{q-1} e\Big(-\frac{an}{q}\Big)\int_{-1/(qQ)}^{1/(qQ)} f\Big(\frac{a}{q}+\beta\Big)S\Big(\frac{a}{q}+\beta\Big)e(-\beta n)\,d\beta\\
&= \gamma\sum_{q\le\log^{80}N}\frac{\mu(q)C(q,-n)}{\phi(q)}\sum_{\substack{r\sim R\\ (r,n)=1,\,q\mid r}}\frac{j(r)}{r}\int_{-1/(qQ)}^{1/(qQ)}\Big(\sum_{n/10<s\le n}s^{\gamma-1}e(\beta s)\Big)\\
&\quad\times\Big(\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log s}\Big)e(-\beta n)\,d\beta+O\Big(\frac{N^{2\gamma-1}}{\log^{20}N}\Big),
\end{aligned}$$

where

$$C(q,m)=\sum_{\substack{a=0\\ (a,q)=1}}^{q-1} e\Big(\frac{am}{q}\Big).$$

Since

$$\int_{1/(qQ)}^{1/2}\Big(\sum_{n/10<s\le n} s^{\gamma-1}e(\beta s)\Big)\Big(\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log s}\Big)e(-\beta n)\,d\beta \ll \int_{1/(qQ)}^{1/2} n^{2(\gamma-1)}\,\frac{d\beta}{\beta^2} \ll \frac{qN^{2\gamma-1}}{\log^{80}N},$$

we obtain

$$\varSigma_3 = \frac{1}{\gamma}\,I(n)\sum_{q\le\log^{80}N}\frac{\mu(q)C(q,-n)}{\phi(q)}\sum_{\substack{r\sim R\\(r,n)=1,\,q\mid r}}\frac{j(r)}{r} + O\Big(\frac{N^{2\gamma-1}}{\log^{20}N}\Big),$$

where

$$I(n) = \sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}}\frac{\gamma^2(n_1n_2)^{\gamma-1}}{\log n_2}.$$

Let

$$\begin{aligned}\varOmega &= \sum_{q\le\log^{80}N}\frac{\mu(q)C(q,-n)}{\phi(q)}\sum_{\substack{r\sim R\\(r,n)=1,\,q\mid r}}\frac{j(r)}{r}\\ &= \sum_{\substack{r\sim R\\(r,n)=1}}\frac{j(r)}{r}\sum_{\substack{q\le\log^{80}N\\ q\mid r}}\frac{\mu(q)C(q,-n)}{\phi(q)}.\end{aligned}$$

Now

$$\sum_{\substack{r\sim R\\(r,n)=1}}\frac{j(r)}{r}\sum_{\substack{q>\log^{80}N\\ q\mid r}}\frac{\mu(q)C(q,-n)}{\phi(q)} \ll \frac{1}{\log^{60}N}\sum_{r\sim R}\frac{d^2(r)}{r} \ll \frac{1}{\log^{50}N},$$

so

$$\begin{aligned}\varOmega &= \sum_{\substack{r\sim R\\(r,n)=1}}\frac{j(r)}{r}\sum_{q\mid r}\frac{\mu(q)C(q,-n)}{\phi(q)} + O\Big(\frac{1}{\log^{50}N}\Big)\\ &= \sum_{\substack{r\sim R\\(r,n)=1}}\frac{j(r)}{r}\sum_{q\mid r}\frac{\mu^2(q)}{\phi(q)} + O\Big(\frac{1}{\log^{50}N}\Big)\\ &= \sum_{\substack{r\sim R\\(r,n)=1}}\frac{j(r)}{\phi(r)} + O\Big(\frac{1}{\log^{50}N}\Big) = \gamma\sum_{\substack{m\sim M,\,k\sim K\\(m,n)=(k,n)=1}}\frac{a(m)b(k)}{\phi(mk)} + O\Big(\frac{1}{\log^{50}N}\Big).\end{aligned}$$

Hence

$$\varSigma_3 = I(n)\sum_{\substack{m\sim M,\,k\sim K\\(m,n)=(k,n)=1}}\frac{a(m)b(k)}{\phi(mk)} + O\Big(\frac{N^{2\gamma-1}}{\log^{20}N}\Big). \tag{4.11}$$

By (4.4), (4.7) and (4.11), the lemma follows.

LEMMA 7. *Assume that* $M, K \ll N^{7/23}$ *and that* $a(m), b(k) = O(1)$. *Let*

$$J_1(n) = \sum_{n^{7/23} < p_1 \le n^{1/2}} \frac{1}{p_1} \sum_{\substack{n = n_1 + n_2 \\ n/10 < n_1, n_2 \le n}} \frac{\gamma^2 (n_1 n_2)^{\gamma - 1}}{\log \frac{n_2}{p_1}}, \tag{4.12}$$

$$J_2(n) = \sum_{n^{7/23} < p_1 \le n^{1/3}} \sum_{p_1 < p_2 \le \sqrt{n/p_1}} \frac{1}{p_1 p_2} \sum_{\substack{n = n_1 + n_2 \\ n/10 < n_1, n_2 \le n}} \frac{\gamma^2 (n_1 n_2)^{\gamma - 1}}{\log \frac{n_2}{p_1 p_2}}. \tag{4.13}$$

Then for $N < n \le 2N$, *except for* $O(N \log^{-2} N)$ *values, we have*

$$\sum_{\substack{m \sim M,\, k \sim K \\ (m,n) = (k,n) = 1}} a(m) b(k) \times \Big(\sum_{\substack{n = mkl + p_1 p_2 \\ n/10 < mkl,\, p_1 p_2 \le n \\ n^{7/23} < p_1 \le n^{1/2} \\ p_1 < p_2}} N(mkl) N(p_1 p_2) - \frac{1}{\phi(mk)} J_1(n) \Big) = O\Big(\frac{N^{2\gamma - 1}}{\log^{20} N} \Big), \tag{4.14}$$

and

$$\sum_{\substack{m \sim M,\, k \sim K \\ (m,n) = (k,n) = 1}} a(m) b(k) \times \Big(\sum_{\substack{n = mkl + p_1 p_2 p_3 \\ n/10 < mkl,\, p_1 p_2 p_3 \le n \\ n^{7/23} < p_1 \le n^{1/3} \\ p_1 < p_2 < p_3}} N(mkl) N(p_1 p_2 p_3) - \frac{1}{\phi(mk)} J_2(n) \Big) = O\Big(\frac{N^{2\gamma - 1}}{\log^{20} N} \Big). \tag{4.15}$$

Proof. This can be proved in almost the same way as Lemma 6; we only give the outline of the proof of (4.14). Let

$$D(\alpha) = \sum_{\substack{n/10 < p_1 p_2 \le n \\ n^{7/23} < p_1 \le n^{1/3} \\ p_1 < p_2}} N(p_1 p_2) e(\alpha p_1 p_2),$$

$$S(\alpha) = \gamma \sum_{\substack{n/10 < p_1 p_2 \le n \\ n^{7/23} < p_1 \le n^{1/3} \\ p_1 < p_2}} (p_1 p_2)^{\gamma - 1} e(\alpha p_1 p_2).$$

For any $\alpha \in E_2$, there is one q ($\log^{80} N < q \le Q$) such that $|\alpha - a/q| < 1/(qQ)$. By Lemma 5.7 of [7] we have $S(\alpha) \ll N^\gamma \log^{-35} N$.

If $\alpha = a/q + \beta \in E_1$, then just as for $g(\alpha)$ in the proof of Lemma 18 in [5], we obtain

$$S(\alpha)=\gamma\,\frac{\mu(q)}{\phi(q)}\sum_{n^{7/23}<p_1\le n^{1/2}}\frac{1}{p_1}\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log\frac{s}{p_1}}+O(N^{\gamma}\exp(-c_2\sqrt{\log N})).$$

Then we can prove (4.14) in the same way used in Lemma 6.

5. Asymptotic formulas

LEMMA 8. *Assume that* $N^{16/23}\ll M\ll N^{19/23}$, $0\le a(m)=O(1)$ *and that* $a(m)=0$ *if* m *has a prime factor* $<N^{\varepsilon}$. *Then for* $N<n\le 2N$, *except for* $O(N\log^{-2}N)$ *values, we have*

$$\begin{aligned}\Sigma_4&=\sum_{\substack{n=mp_1+p_2\\ n/10<mp_1,\,p_2\le n\\ m\sim M}}a(m)N(mp_1)N(p_2)\\ &=(1+O(\varepsilon))Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{n\log n}\sum_{m\sim M}a(m)\sum_{n/m<p\le 2n/m}1+O\bigg(\frac{N^{2\gamma-1}}{\log^{10}N}\bigg),\end{aligned}$$

where

$$Z(\gamma)=\gamma^2\int_{1/10}^{9/10}u^{\gamma-1}(1-u)^{\gamma-1}\,du. \tag{5.1}$$

Proof. We have

$$\Sigma_4=\int_0^1\sum_{\substack{n/10<mp_1\le n\\ m\sim M}}a(m)N(mp_1)e(\alpha mp_1)\cdot\sum_{n/10<p\le n}N(p)e(\alpha p)e(-\alpha n)\,d\alpha.$$

As in Lemma 6, for $N<n\le 2N$, except for $O(N\log^{-2}N)$ values, we have

$$\Sigma_4=\int_{E_1}g(\alpha)S(\alpha)e(-\alpha n)\,d\alpha+O\bigg(\frac{N^{2\gamma-1}}{\log^{10}N}\bigg),$$

where E_1 is defined in Lemma 6,

$$g(\alpha)=\sum_{\substack{n/10<mp_1\le n\\ m\sim M}}a(m)(mp_1)^{\gamma-1}e(\alpha mp_1),\qquad S(\alpha)=\gamma\sum_{n/10<p\le n}p^{\gamma-1}e(\alpha p).$$

By page 22 of [5] we have

$$g(\alpha)=\gamma\,\frac{\mu(q)}{\phi(q)}\sum_{m\sim M}\frac{a(m)}{m}\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log\frac{s}{m}}+O(N^{\gamma}\exp(-c_1\sqrt{\log N})),$$

$$S(\alpha)=\gamma\,\frac{\mu(q)}{\phi(q)}\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log s}+O(N^{\gamma}\exp(-c_2\sqrt{\log N})).$$

Hence

$$\begin{aligned}\Sigma_4 &= \sum_{q\le \log^{80} N}\ \sum_{\substack{a=0\\(a,q)=1}}^{q-1} e\Big(-\frac{an}{q}\Big)\int_{-1/(qQ)}^{1/(qQ)} g\Big(\frac{a}{q}+\beta\Big)S\Big(\frac{a}{q}+\beta\Big)e(-\beta n)\,d\beta\\ &\quad + O\Big(\frac{N^{2\gamma-1}}{\log^{10} N}\Big)\\ &= \gamma^2\sum_{q\le \log^{80} N}\frac{\mu^2(q)C(q,-n)}{\phi^2(q)}\int_{-1/(qQ)}^{1/(qQ)}\sum_{m\sim M}\frac{a(m)}{m}\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log\frac{s}{m}}\\ &\quad\times\Big(\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log s}\Big)e(-\beta n)\,d\beta + O\Big(\frac{N^{2\gamma-1}}{\log^{20} N}\Big)\\ &= \sum_{q\le\log^{80} N}\frac{\mu^2(q)C(q,-n)}{\phi^2(q)}\,K(n) + O\Big(\frac{N^{2\gamma-1}}{\log^{20} N}\Big),\end{aligned}$$

where

$$\begin{aligned}K(n) &= \gamma^2\sum_{m\sim M}\frac{a(m)}{m}\sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}}\frac{(n_1n_2)^{\gamma-1}}{\log\frac{n_1}{m}\log n_2}\\ &= (1+O(\varepsilon))Z(\gamma)\,\frac{n^{2\gamma-1}}{\log n}\sum_{m\sim M}\frac{a(m)}{m\log\frac{n}{m}}\\ &= (1+O(\varepsilon))Z(\gamma)\,\frac{n^{2\gamma-1}}{n\log n}\sum_{m\sim M}a(m)\sum_{n/m<p\le 2n/m}1.\end{aligned}$$

Moreover,

$$\begin{aligned}\sum_{q\le\log^{80} N}\frac{\mu^2(q)C(q,-n)}{\phi^2(q)} &= \sum_{q=1}^{\infty}\frac{\mu^2(q)C(q,-n)}{\phi^2(q)} + O\Big(\frac{1}{\log^{30} N}\Big)\\ &= C(n) + O\Big(\frac{1}{\log^{30} N}\Big).\end{aligned}$$

Hence the lemma follows.

Lemma 9. *Assume that* $N^{16/23} \ll M \ll N^{19/23}$, $0 \le a(m) = O(1)$ *and that* $a(m) = 0$ *if* m *has a prime factor* $< N^{\varepsilon}$. *Let*

$$\Sigma_5 = \sum_{\substack{n=mp+d\\ n/10<mp,\,d\le n\\ m\sim M}} a(m)N(mp)N(d),$$

where $d=p_1p_2$ $(n^{7/23}<p_1\le n^{1/2},\ p_1<p_2)$ *or* $d=p_1p_2p_3$ $(n^{7/23}<p_1\le n^{1/3},\ p_1<p_2<p_3)$. *Then for* $N<n\le 2N$, *except for* $O(N\log^{-2} N)$ *values, we*

have

$$\Sigma_5 = (1+O(\varepsilon))Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{n\log n}\sum_{m\sim M} a(m)\sum_{n/m<p\le 2n/m} 1$$
$$\times\bigg(\int_{7/23}^{1/2}\frac{dt}{t(1-t)}+\int_{7/23}^{1/3}\frac{dt}{t}\int_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\bigg)+O\bigg(\frac{N^{2\gamma-1}}{\log^8 N}\bigg).$$

Proof. In almost the same way as in Lemma 8, referring to Lemma 7, for $N<n\le 2N$, except for $O(N\log^{-2}N)$ values, we obtain

$$\Sigma_5 = (1+O(\varepsilon))C(n)\sum_{m\sim M}\frac{a(m)}{m}\bigg(\sum_{n^{7/23}<p_1\le n^{1/2}}\frac{1}{p_1}\sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}}\frac{\gamma^2(n_1n_2)^{\gamma-1}}{\log\frac{n_1}{m}\log\frac{n_2}{p_1}}$$
$$+\sum_{n^{7/23}<p_1\le n^{1/3}}\sum_{p_1<p_2<\sqrt{n/p_1}}\frac{1}{p_1p_2}\sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}}\frac{\gamma^2(n_1n_2)^{\gamma-1}}{\log\frac{n_1}{m}\log\frac{n_2}{p_1p_2}}\bigg)$$
$$+O\bigg(\frac{N^{2\gamma-1}}{\log^8 N}\bigg)$$
$$=(1+O(\varepsilon))C(n)\gamma^2\sum_{m\sim M}\frac{a(m)}{m\log\frac{n}{m}}\bigg(\sum_{n^{7/23}<p_1\le n^{1/2}}\frac{1}{p_1\log\frac{n}{p_1}}$$
$$+\sum_{n^{7/23}<p_1\le n^{1/3}}\sum_{p_1<p_2<\sqrt{n/p_1}}\frac{1}{p_1p_2\log\frac{n}{p_1p_2}}\bigg)\sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}}(n_1n_2)^{\gamma-1}$$
$$+O\bigg(\frac{N^{2\gamma-1}}{\log^8 N}\bigg)$$
$$=(1+O(\varepsilon))Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{n\log n}\sum_{m\sim M} a(m)\sum_{n/m<p\le 2n/m} 1$$
$$\times\bigg(\int_{7/23}^{1/2}\frac{dt}{t(1-t)}+\int_{7/23}^{1/3}\frac{dt}{t}\int_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\bigg)+O\bigg(\frac{N^{2\gamma-1}}{\log^8 N}\bigg).$$

6. Sieve method. Set

$$\mathcal{A}=\{a: a=n-p,\ N(a)=N(p)=1,\ n/10<p\le n\},$$
$$\mathcal{B}=\{b: b=n-d,\ N(b)=N(d)=1,\ 0<d\le 9n/10,$$
$$d=p_1p_2\ (n^{7/23}<p_1\le n^{1/2},\ p_1<p_2)\ \text{or}$$
$$d=p_1p_2p_3\ (n^{7/23}<p_1\le n^{1/3},\ p_1<p_2<p_3)\},$$

$$P(z) = \prod_{p<z,\, p\nmid n} p, \qquad \mathcal{S}(\mathcal{A}, z) = \sum_{\substack{a\in\mathcal{A}\\ (a,P(z))=1}} 1, \qquad \mathcal{S}(\mathcal{B}, w) = \sum_{\substack{b\in\mathcal{B}\\ (b,P(w))=1}} 1.$$

Note once again that $p \in P_\gamma$ is equivalent to $N(p) = 1$. Applying Buchstab's identity, we get

$$\begin{aligned}(6.1)\qquad T(n) &\ge \mathcal{S}(\mathcal{A}, n^{1/2})\\ &= \mathcal{S}(\mathcal{A}, n^{4/23}) - \sum_{n^{4/23}<p\le n^{7/23}} \mathcal{S}(\mathcal{A}_p, p) - \sum_{n^{7/23}<p\le n^{1/2}} \mathcal{S}(\mathcal{A}_p, p)\\ &= \mathcal{S}_1 - \mathcal{S}_2 - \mathcal{S}_3.\end{aligned}$$

Using Buchstab's identity again, we get

$$\begin{aligned}(6.2)\qquad \mathcal{S}_1 &= \mathcal{S}(\mathcal{A}, n^{7/46}) - \sum_{n^{7/46}<p\le n^{4/23}} \mathcal{S}(\mathcal{A}_p, p)\\ &= \mathcal{S}(\mathcal{A}, n^{7/46}) - \sum_{n^{7/46}<p\le n^{4/23}} \mathcal{S}(\mathcal{A}_p, (n^{14/23}/p)^{1/5})\\ &\quad + \sum_{n^{7/46}<p\le n^{4/23}} \ \sum_{(n^{14/23}/p)^{1/5}<q\le n^{7/23}/p} \mathcal{S}(\mathcal{A}_{pq}, q)\\ &\quad + \sum_{n^{7/46}<p\le n^{4/23}} \ \sum_{n^{7/23}/p<q\le p} \mathcal{S}(\mathcal{A}_{pq}, q) = \Phi_1 - \Phi_2 + \Phi_3 + \Phi_4.\end{aligned}$$

Next,

$$\begin{aligned}(6.3)\qquad \mathcal{S}_3 &= \sum_{n^{7/23}<p\le n^{1/2}} \mathcal{S}(\mathcal{A}_p, p)\\ &= \sharp\{d : d = n - p_4,\ N(d) = N(p_4) = 1,\ n/10 < p_4 \le n,\\ &\qquad d = p_1p_2\ (n^{7/23} < p_1 \le n^{1/2},\ p_1 < p_2) \text{ or}\\ &\qquad d = p_1p_2p_3\ (n^{7/23} < p_1 \le n^{1/3},\ p_1 < p_2 < p_3)\}\\ &= \sharp\{p_4 : p_4 = n - d,\ N(p_4) = N(d) = 1,\ 0 < d \le 9n/10,\\ &\qquad d = p_1p_2\ (n^{7/23} < p_1 \le n^{1/2},\ p_1 < p_2) \text{ or}\\ &\qquad d = p_1p_2p_3\ (n^{7/23} < p_1 \le n^{1/3},\ p_1 < p_2 < p_3)\}\\ &= \mathcal{S}(\mathcal{B}, n^{1/2}).\end{aligned}$$

Using Buchstab's identity again, we have

$$\begin{aligned}(6.4)\qquad \mathcal{S}(\mathcal{B}, n^{1/2}) &= \mathcal{S}(\mathcal{B}, n^{7/46}) - \sum_{n^{7/46}<p\le n^{4/23}} \mathcal{S}(\mathcal{B}_p, p)\\ &\quad - \sum_{n^{4/23}<p\le n^{7/23}} \mathcal{S}(\mathcal{B}_p, p) - \sum_{n^{7/23}<p\le n^{1/2}} \mathcal{S}(\mathcal{B}_p, p)\end{aligned}$$

$$\le \mathcal{S}(\mathcal{B}, n^{7/46}) - \sum_{n^{7/46}<p\le n^{4/23}} \mathcal{S}(\mathcal{B}_p, (n^{14/23}/p)^{1/5})$$

$$+ \sum_{n^{7/46}<p\le n^{4/23}} \sum_{(n^{14/23}/p)^{1/5}<q\le n^{7/23}/p} \mathcal{S}(\mathcal{B}_{pq}, q)$$

$$+ \sum_{n^{7/46}<p\le n^{4/23}} \sum_{n^{7/23}/p<q\le p} \mathcal{S}(\mathcal{B}_{pq}, q) - \sum_{n^{4/23}<p\le n^{7/23}} \mathcal{S}(\mathcal{B}_p, p)$$

$$= \Gamma_1 - \Gamma_2 + \Gamma_3 + \Gamma_4 - \Gamma_5.$$

LEMMA 10.

$$\Phi_1 = \mathcal{S}(\mathcal{A}, n^{4/23}) \ge 3.60972 Z(\gamma) C(n) \frac{n^{2\gamma-1}}{\log^2 n}.$$

Proof. Take

$$X = I(n) = \sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}} \frac{\gamma^2(n_1n_2)^{\gamma-1}}{\log n_2}$$

and

$$\omega(d) = \begin{cases} d/\phi(d), & (d,n)=1,\\ 0, & (d,n)>1, \end{cases} \qquad r(d) = \sharp\mathcal{A}_d - \frac{X}{\phi(d)}.$$

By Theorem 7.11 and (7.40) of [7], we have

$$W(z) = \prod_{p<z}\left(1 - \frac{\omega(p)}{p}\right) = C(n)\frac{e^{-\gamma}}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right),$$

where γ is Euler's constant.

Let $z = n^{7/46}$, $D = n^{14/23}$. By Iwaniec's bilinear sieve method (see [4, Theorem 1]), we obtain

$$\Phi_1 \ge \frac{C(n)X}{\log z}\left(f\left(\frac{\log D}{\log z}\right) - O(\varepsilon)\right) - \sum_{\substack{m<n^{7/23},\, k<n^{7/23}\\ (m,n)=(k,n)=1}} a(m)b(k)r(mk),$$

where $f(u)$ is a standard function. In particular

$$f(u) = \begin{cases} \frac{2}{u}\log(u-1), & 2\le u\le 4,\\ \frac{2}{u}\left(\log(u-1) + \int_3^{u-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(s-1)}{s}\,ds\right), & 4\le u\le 6. \end{cases} \tag{6.5}$$

By Lemma 6, we have

$$\sum_{\substack{m<n^{7/23},\, k<n^{7/23}\\ (m,n)=(k,n)=1}} a(m)b(k)r(mk) = O\left(\frac{N^{2\gamma-1}}{\log^{10} N}\right).$$

On the other hand,

$$X=\frac{(1+O(\varepsilon))\gamma^2}{\log n}\sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}}(n_1n_2)^{\gamma-1}=(1+O(\varepsilon))Z(\gamma)\,\frac{n^{2\gamma-1}}{\log n}.$$

Hence,

$$\Phi_1\ge 3.60972Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}.$$

LEMMA 11.

$$\Phi_2=\sum_{n^{7/46}<p\le n^{4/23}}\mathcal{S}(\mathcal{A}_p,(n^{14/23}/p)^{1/5})\le 0.84233Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}.$$

Proof. Take

$$z(p)=\left(\frac{n^{14/23}}{p}\right)^{1/5},\qquad D(p)=\frac{n^{14/23}}{p}.$$

By Iwaniec's bilinear sieve method we obtain

$$\Phi_2\le(1+O(\varepsilon))C(n)X\sum_{n^{7/46}<p\le n^{4/23}}\frac{1}{p\log z(p)}\,F\left(\frac{\log D(p)}{\log z(p)}\right)+R^+,$$

where

$$R^+=\sum_{\substack{n^{7/46}<p\le n^{4/23}\\ (p,n)=1}}\ \sum_{\substack{h<n^{7/23}/p,\,k<n^{7/23}\\ (h,n)=(k,n)=1}}c(h)b(k)r(phk),$$

and $F(u)$ is a standard function. In particular,

$$F(u)=\begin{cases}2/u, & 2\le u\le 3,\\ \dfrac{2}{u}\left(1+\displaystyle\int_2^{u-1}\frac{\log(t-1)}{t}\,dt\right), & 3\le u\le 5.\end{cases}\tag{6.6}$$

In R^+, let $ph=m$. By Lemma 6 we have

$$R^+=O\left(\frac{n^{2\gamma-1}}{\log^{10} n}\right).$$

From the above discussion and the prime number theorem, we have

$$\begin{aligned}\Phi_2&\le Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log n}\sum_{n^{7/46}<p\le n^{4/23}}\frac{5F(5)}{p\log\frac{n^{14/23}}{p}}+O\left(\frac{\varepsilon n^{2\gamma-1}}{\log^2 n}\right)\\ &=Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log n}\int_{7/46}^{4/23}\frac{2\,dt}{t\left(\frac{14}{23}-t\right)}\left(1+\int_2^4\frac{\log(u-1)}{u}\,du\right)+O\left(\frac{\varepsilon n^{2\gamma-1}}{\log^2 n}\right)\\ &\le 0.84233Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}.\end{aligned}$$

LEMMA 12.

$$\Gamma_1 = \mathcal{S}(\mathcal{B}, n^{7/46}) \le 3.18061 Z(\gamma) C(n) \frac{n^{2\gamma-1}}{\log^2 n}.$$

Proof. We take $Y = J_1(n) + J_2(n)$, where $J_1(n)$, $J_2(n)$ are defined in (4.12) and (4.13) respectively, and

$$r(d) = \sharp\mathcal{B}_d - \frac{Y}{\phi(d)}.$$

By Iwaniec's bilinear sieve method, we have

$$\Gamma_1 \le \frac{C(n)Y}{\log n} \cdot \frac{46}{7}(F(4) + O(\varepsilon)) + \sum_{\substack{m<n^{7/23},\, k<n^{7/23} \\ (m,n)=(k,n)=1}} a(m)b(k)r(mk).$$

Applying Lemma 7, we have

$$\sum_{\substack{m<n^{7/23},\, k<n^{7/23} \\ (m,n)=(k,n)=1}} a(m)b(k)r(mk) = O\left(\frac{N^{2\gamma-1}}{\log^{10} N}\right).$$

On the other hand,

$$Y = (1+O(\varepsilon))Z(\gamma)\frac{n^{2\gamma-1}}{\log n} \times \left(\int_{7/23}^{1/2} \frac{dt}{t(1-t)} + \int_{7/23}^{1/3} \frac{dt}{t} \int_{t}^{(1-t)/2} \frac{dw}{w(1-t-w)}\right). \tag{6.7}$$

Hence,

$$\Gamma_1 \le 3.18061 Z(\gamma) C(n) \frac{n^{2\gamma-1}}{\log^2 n}.$$

LEMMA 13.

$$\Gamma_2 = \sum_{n^{7/46}<p\le n^{4/23}} \mathcal{S}(\mathcal{B}_p, (n^{14/23}/p)^{1/5}) \ge 0.70826 Z(\gamma) C(n) \frac{n^{2\gamma-1}}{\log^2 n}.$$

Proof. Using Lemma 7, in almost the same way as in Lemma 11, we obtain

$$\Gamma_2 \ge (1+O(\varepsilon))C(n)Y \sum_{n^{7/46}<p\le n^{4/23}} \frac{5f(5)}{p\log\frac{n^{14/23}}{p}} \ge 0.70826 Z(\gamma)C(n)\frac{n^{2\gamma-1}}{\log^2 n}.$$

LEMMA 14.

$$\Gamma_4 = \sum_{n^{7/46}<p\le n^{4/23}} \sum_{n^{7/23}/p<q\le p} \mathcal{S}(\mathcal{B}_{pq}, q) \le 0.10885 Z(\gamma)C(n)\frac{n^{2\gamma-1}}{\log^2 n}.$$

Proof. We have

$$\Gamma_4 \le \sum_{n^{7/46}<p\le n^{4/23}} \sum_{n^{7/23}/p<q\le p} \mathcal{S}\Big(\mathcal{B}_{pq}, \Big(\frac{n^{14/23}}{pq}\Big)^{1/3}\Big).$$

Take

$$D(p,q) = \frac{n^{14/23}}{pq}.$$

By Iwaniec's bilinear sieve method, we have

$$\Gamma_4 \le (1+O(\varepsilon))C(n)Y \sum_{n^{7/46}<p\le n^{4/23}} \sum_{n^{7/23}/p<q\le p} \frac{3F(3)}{pq\log\frac{n^{14/23}}{pq}} - R^-,$$

where

$$R^- = \sum_{n^{7/46}<p\le n^{4/23}} \sum_{n^{7/23}/p<q\le p} \sum_{h<n^{7/23}/p} \sum_{g<n^{7/23}/q} c(h)v(g)r(pqhg).$$

In R^-, let $ph=m$, $qg=k$. By Lemma 7, we have

$$R^- = O\Big(\frac{N^{2\gamma-1}}{\log^{10} N}\Big).$$

Hence,

$$\Gamma_4 \le 0.10885Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}.$$

7. The estimation of $\mathcal{S}_2$, Φ_3, Γ_3 and Γ_5

LEMMA 15.

$$\mathcal{S}_2 = \sum_{n^{4/23}<p\le n^{7/23}} \mathcal{S}(\mathcal{A}_p, p) \le 1.38679Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}.$$

Proof. By Lemmas 4, 5 and 8, it follows that

$$\begin{aligned}\mathcal{S}_2 &= \sum_{\substack{n=rp+p_2\\ n/10<rp,\,p_2\le n\\ n^{4/23}<p\le n^{7/23},\,(r,P(p))=1}} N(rp)N(p_2)\\
&= (1+O(\varepsilon))Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log n} \sum_{n^{4/23}<p\le n^{7/23}} \sum_{\substack{n/p<r\le 2n/p\\ (r,p)=1}} 1 + O\Big(\frac{n^{2\gamma-1}}{\log^8 n}\Big)\\
&= Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log n} \sum_{n^{4/23}<p\le n^{7/23}} \frac{1}{p\log p}\, w\Big(\frac{\log\frac{n}{p}}{\log p}\Big) + O\Big(\frac{\varepsilon n^{2\gamma-1}}{\log^2 n}\Big)\end{aligned}$$

$$= Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}\int\limits_{4/23}^{7/23}\frac{1}{u^2}\,w\left(\frac{1-u}{u}\right)du + O\left(\frac{\varepsilon n^{2\gamma-1}}{\log^2 n}\right)$$

$$= Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}\int\limits_{16/7}^{19/4} w(t)\,dt + O\left(\frac{\varepsilon n^{2\gamma-1}}{\log^2 n}\right) \le 1.38679 Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}.$$

LEMMA 16.

$$\varPhi_3 = \sum_{n^{7/46}<p\le n^{4/23}}\ \sum_{(n^{14/23}/p)^{1/5}<q\le n^{7/23}/p} \mathcal{S}(\mathcal{A}_{pq}, q) \ge 0.30967 Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}.$$

Proof. We have

$$\varPhi_3 = \sum_{\substack{n=rpq+p_2\\ n/10<rpq,\,p_2\le n\\ n^{7/46}<p\le n^{4/23}\\ (n^{14/23}/p)^{1/5}<q\le n^{7/23}/p\\ (r,P(q))=1}} N(rpq)N(p_2).$$

Note that $n^{4/23} \ll pq \ll n^{7/23}$ and $n^{16/23} \ll r \ll n^{19/23}$. By Lemma 8 with a small modification, and Lemmas 4 and 5, we have

$$\varPhi_3 = (1+O(\varepsilon))Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{n\log n}$$
$$\times \sum_{n^{7/46}<p\le n^{4/23}}\ \sum_{(n^{14/23}/p)^{1/5}<q\le n^{7/23}/p}\ \sum_{\substack{n/(pq)<r\le 2n/(pq)\\ (r,P(q))=1}} 1 + O\left(\frac{n^{2\gamma-1}}{\log^8 n}\right)$$
$$= Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log n}$$
$$\times \sum_{n^{7/46}<p\le n^{4/23}}\ \sum_{(n^{14/23}/p)^{1/5}<q\le n^{7/23}/p} \frac{1}{pq\log q}\,w\left(\frac{\log\frac{n}{pq}}{\log q}\right) + O\left(\frac{\varepsilon n^{2\gamma-1}}{\log^2 n}\right)$$
$$\ge 0.5607 Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}\int\limits_{7/46}^{4/23}\frac{dt}{t}\int\limits_{(14/23-t)/5}^{7/23-t}\frac{dw}{w^2}$$
$$\ge 0.30967 Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}.$$

LEMMA 17.

$$\varGamma_5 = \sum_{n^{4/23}<p\le n^{7/23}} \mathcal{S}(\mathcal{B}_p, p) \ge 1.16780 Z(\gamma)C(n)\,\frac{n^{2\gamma-1}}{\log^2 n}.$$

Proof. We have

$$\varGamma_5 = \sum_{\substack{n=rp+d\\ n/10<rp,\, d\le n\\ n^{4/23}<p\le n^{7/23},\,(r,P(p))=1}} N(rp)N(d),$$

where $d=p_1p_2$ ($n^{7/23}<p_1\le n^{1/2}$, $p_1<p_2$) or $d=p_1p_2p_3$ ($n^{7/23}<p_1\le n^{1/3}$, $p_1<p_2<p_3$). By Lemmas 4, 5 and 9, in almost the same way as in Lemma 15, we have

$$\begin{aligned}\varGamma_5 &= Z(\gamma)C(n)\frac{n^{2\gamma-1}}{\log^2 n}\int_{16/7}^{19/4} w(t)\,dt\\ &\quad\times\Bigg(\int_{7/23}^{1/2}\frac{dt}{t(1-t)}+\int_{7/23}^{1/3}\frac{dt}{t}\int_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\Bigg)+O\bigg(\frac{\varepsilon n^{2\gamma-1}}{\log^2 n}\bigg)\\ &\ge 1.16780Z(\gamma)C(n)\frac{n^{2\gamma-1}}{\log^2 n}.\end{aligned}$$

LEMMA 18.

$$\varGamma_3 = \sum_{n^{7/46}<p\le n^{4/23}}\ \sum_{(n^{14/23}/p)^{1/5}<q\le n^{7/23}/p} \mathcal{S}(\mathcal{B}_{pq},q) \le 0.26308Z(\gamma)C(n)\frac{n^{2\gamma-1}}{\log^2 n}.$$

Proof. We have

$$\varGamma_3 = \sum_{\substack{n=rpq+d\\ n/10<rpq,\, d\le n\\ n^{7/46}<p\le n^{4/23}\\ (n^{14/23}/p)^{1/5}<q\le n^{7/23}/p\\ (r,P(q))=1}} N(rpq)N(d),$$

where $d=p_1p_2$ ($n^{7/23}<p_1\le n^{1/2}$, $p_1<p_2$) or $d=p_1p_2p_3$ ($n^{7/23}<p_1\le n^{1/3}$, $p_1<p_2<p_3$). By Lemma 9 and the deduction in Lemma 16, we get

$$\begin{aligned}\varGamma_3 &\le 0.5644Z(\gamma)C(n)\frac{n^{2\gamma-1}}{\log^2 n}\int_{7/46}^{4/23}\frac{dt}{t}\int_{(14/23-t)/5}^{7/23-t}\frac{dw}{w^2}\\ &\quad\times\Bigg(\int_{7/23}^{1/2}\frac{dt}{t(1-t)}+\int_{7/23}^{1/3}\frac{dt}{t}\int_{t}^{(1-t)/2}\frac{dw}{w(1-t-w)}\Bigg)+O\bigg(\frac{\varepsilon n^{2\gamma-1}}{\log^2 n}\bigg)\\ &\le 0.26308Z(\gamma)C(n)\frac{n^{2\gamma-1}}{\log^2 n}.\end{aligned}$$

8. The proof of Theorem 3. Applying Lemmas 10, 11 and 16 to the expression in (6.2), we obtain

$$\mathcal{S}_1 \geq 3.07706 Z(\gamma) C(n) \frac{n^{2\gamma-1}}{\log^2 n}.$$

Applying Lemmas 12, 13, 14, 17 and 18 to the expression in (6.4), we get

$$\mathcal{S}_3 \leq 1.67648 Z(\gamma) C(n) \frac{n^{2\gamma-1}}{\log^2 n}.$$

In (6.1), the above two inequalities and Lemma 15 yield

$$T(n) \geq 0.01379 Z(\gamma) C(n) \frac{n^{2\gamma-1}}{\log^2 n} \geq \frac{\varrho_0 C(n) n^{2\gamma-1}}{\log^2 n}.$$

Hence, Theorem 3 follows.

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