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Bounds for the degrees of CM-fields of class number one

by

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1. Introduction. A number field K is said to be a *CM-field* if K is a totally imaginary quadratic extension of its maximal totally real subfield k. According to class field theory, if K is a CM-field then the class number h_k of k divides the class number h_K of K and $h_K^- = h_K/h_k$, which is a divisor of h_K, is called the *relative class number* of K. If n denotes the degree of k, then K is of even degree $2n$. Notice that an imaginary abelian number field is always a CM-field, whereas a normal imaginary number field is a CM-field if and only if the complex conjugation is in the center of its Galois group $\mathrm{Gal}(K/\mathbb{Q})$ (which implies that k also is a normal number field).

In 1974, A. M. Odlyzko proved that there are only finitely many normal CM-fields of a given class number (see [22]). In 1979, J. Hoffstein showed that normal CM-fields of degrees greater than or equal to 436 have relative class number greater than one (see [11]). In 1994, K. Yamamura completed the determination of all the abelian CM-fields of class number one: there are 172 such CM-fields and their degrees are less than or equal to 24 (see [36]). Since 1994, various authors have been working on the determination of all the non-abelian normal CM-fields with class number one (see [5], [12]–[14], [15], [16], [20], [26]). The current strategy for solving this problem is first to solve this class number one problem for the normal CM-fields of a given Galois group (see [12], [13], [16], [20]), namely the normal CM-fields of any degree with dihedral or dicyclic Galois group (the reason why these two situations are of paramount importance is that if K is a non-abelian normal CM-field of degree $4p$, p any odd prime, then since the complex conjugation must be in the center of its Galois group, this Galois group is isomorphic either to the dihedral group of order $4p$ or to the dicyclic group of order $4p$). Then one tries to solve the class number one problem for the non-abelian

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normal CM-fields of a given reasonable degree $2n$ (see [5], [14], [15], [26]). Up to now, this strategy has made it possible to determine all the non-abelian normal CM-fields of degree $2n \leq 48$. In particular, in this process examples have been found of non-abelian normal CM-fields with class number one of degrees 8, 12, 16, 20, 24, 32, 36, 40 and 48. The following theorem gives the CM-field of class number one and of highest degree known to date:

THEOREM 1 (K.-Y. Chang and S.-H. Kwon, see [5]). *Let K_{12} be the dihedral field of degree 12 cyclic over $\mathbb{Q}(\sqrt{101})$, namely $K_{12} = \mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$ with $\theta^3 - \theta^2 - 5\theta - 1 = 0$. The narrow class group of K_{12} is cyclic of order 4. Let N be the Hilbert class field in the narrow sense of K_{12}. Then N is a normal CM-field of degree 48 and of class number one. Moreover, $D_N = D_{K_{12}}^4 = 2^{32} \cdot 5^{24} \cdot 101^{24}$ and $\varrho_N = D_N^{1/48} = 2^{2/3} \cdot 5^{1/2} \cdot 101^{1/2} = 35.67\ldots$*

In order to make this strategy more reasonable it would be rather useful to have beforehand a bound for the degrees $2n$ of the non-abelian normal CM-fields with class number one which would be much smaller than J. Hoffstein's bound $2n \leq 434$. The aim of this paper is to prove that normal CM-fields with class number one are of degree $2n \leq 266$. Moreover, we will also prove that, assuming the Generalized Riemann Hypothesis, normal CM-fields with class number one are of degree $2n \leq 164$, and that CM-fields with class number one are of degree $2n \leq 174$. Not only will we improve upon J. Hoffstein's bound for the degree of the normal CM-fields with class number one, but we will also improve upon J. Hoffstein's bounds for the root discriminants of the normal CM-fields of a given degree with class number one. We emphasize that we will also prove that the Dedekind zeta function of an imaginary quadratic field F of absolute discriminant D_F has no real zero in the range $1 - 6/(\pi\sqrt{D_F}) \leq s < 1$.

If D_k denotes the absolute value of the discriminant of a number field k of degree $n \geq 1$, then its root discriminant ϱ_k is defined as $\varrho_k = D_k^{1/n}$. We will prove

THEOREM 2. *Let K be a CM-field of degree $2n$ and k its maximal totally real subfield (k is of degree n).*

1. *Assume the Generalized Riemann Hypothesis and K normal. Then $h_K^- > 1$ if $n \geq 83$, or if $n \leq 82$ but the root discriminant $\varrho_K = D_K^{1/(2n)}$ is larger than the following conditional bounds:*

Table 1. Conditional bounds, normal case

n	82	70	60	50	40	30
$D_K^{1/(2n)}$	54.76	65.76	81.93	114.5	204.0	729.3
n	24	23	20	15	10	5
$D_K^{1/(2n)}$	5252	6354	6499	6875	7653	10250

2. *Do not assume the Generalized Riemann Hypothesis anymore*, *but assume K normal. Then $h_K^- > 1$ if $n \geq 134$, or if $n \leq 133$ but the root discriminant $\varrho_K = D_K^{1/(2n)}$ is larger than the following conditional bounds*:

Table 2. Unconditional bounds, normal case

n	$D_K^{1/(2n)}$	J. Hoffstein's bounds	n	$D_K^{1/(2n)}$	J. Hoffstein's bounds
133	44.83		50	195.5	879
130	45.66		40	422.3	
120	48.84	115	30	2379	
110	52.91		27	6783	
100	58.30	154	26	8355	
90	65.76		20	9207	
80	76.71	239	15	10470	
70	94.18		10	13100	
60	125.8	494	5	22090	

3. *Assume the Generalized Riemann Hypothesis*, *but do not assume K normal. Then $h_K^- > 1$ if $n \geq 88$, or if $n \leq 87$ but the root discriminant $\varrho_K = D_K^{1/(2n)}$ is larger than the following conditional bounds*:

Table 3. Conditional bounds, non-normal case

n	87	80	70	60	50	40	30
$D_K^{1/(2n)}$	56.56	62.12	73.74	93.81	135.4	255.6	1052
n	25	24	20	15	10	5	
$D_K^{1/(2n)}$	5627	6973	7258	7823	8951	12460	

REMARKS. 1. Without assuming the Generalized Riemann Hypothesis, it is not known how to obtain lower bounds for h_K^- increasing to infinity with n in the non-normal case.

2. We have $\varrho_K = D_K^{1/(2n)} \geq D_k^{1/n} = \varrho_k$. Since the lower bounds of discriminants for the totally real fields are much better than for the totally imaginary fields, we always use the former. We denote by ϱ_n A. M. Odlyzko's conditional lower bounds for the root discriminants of the totally real number fields of degrees $\geq n$ (under the Generalized Riemann Hypothesis), and by ϱ'_n L. Tartar's unconditional lower bounds for the root discriminants of the totally real number fields of degrees $\geq n$ (without assuming the Generalized Riemann Hypothesis). We will use the values $\varrho_n = 54.8874$ for $n = 83$, $\varrho_n = 56.2325$ for $n = 88$, and $\varrho'_n = 44.6377$ for $n = 134$.

2. Outline of our method. Let K be a CM-field of degree $2n$, and k its maximal totally real subfield. The starting point is a lower bound for h_K^-. Let D_K and D_k denote the absolute values of the discriminants of K

and k, and $\operatorname{Res}\zeta_K$ and $\operatorname{Res}\zeta_k$ the residues at $s = 1$ of the Dedekind zeta functions of K and k. Let W_K be the group of complex roots of unity in K, let $w_K \geq 2$ be the order of this group, let E_K be the unit group of K, and let E_k be the unit group of k. Let $Q_K = [E_K : W_K E_k]$ be the Hasse unit index of K. We have $Q_K \in \{1, 2\}$ (see for example [35, Theorem 4.12]).

The analytic class number formula for k and K yields

$$h_K^- = \frac{Q_K w_K}{(2\pi)^n}\sqrt{\frac{D_K}{D_k}} \cdot \frac{\operatorname{Res}\zeta_K}{\operatorname{Res}\zeta_k}. \tag{1}$$

Thus, to obtain lower bounds for h_K^- which go to infinity with n, we will need lower bounds for D_K/D_k, upper bounds for $\operatorname{Res}\zeta_k$ and lower bounds for $\operatorname{Res}\zeta_K$. For all the three problems, we propose improvements over J. Hoffstein's methods. For both the discriminants and residues, we will make use of one of A. Weil's explicit formulas, which for a number field E of degree $n = r_1 + 2r_2$ reads

$$\log D_E = n\log(8\pi e^{\gamma}) + r_1\frac{\pi}{2} - nI_{n,r_1}(F) + \sum_{\varrho}\Phi(\varrho) + 2\sum_{\mathfrak{p},m}\frac{\log \mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{m/2}}\,F(m\log \mathrm{N}(\mathfrak{p})) \tag{2}$$

where F is a real-valued even function that must be chosen subject to certain conditions (see Proposition 3 below), the first sum is over the non-trivial zeros ϱ of the Dedekind zeta function of E (those of real part $\beta = \operatorname{Re}\varrho$ with $0 < \beta < 1$), the second double sum is over the prime ideals $\mathfrak{p}$ of E,

$$\Phi(s) = \int_{-\infty}^{\infty} F(x)e^{(s-1/2)x}\,dx \tag{3}$$

is the Mellin transform of F and

$$I_{n,r_1}(F) = \int_0^{\infty}\frac{1-F(x)}{2\sinh(x/2)}\,dx + \frac{r_1}{n}\int_0^{\infty}\frac{1-F(x)}{2\cosh(x/2)}\,dx + \frac{4}{n}\int_0^{\infty}F(x)\cosh(x/2)\,dx \tag{4}$$

(see for example Poitou [28] for a development of this formula). Notice that for a totally real field, $n = r_1$ and

$$I_{n,n}(F) = \frac{4}{n}\int_0^{\infty}F(x)\cosh(x/2)\,dx + \int_0^{\infty}\frac{(1-F(x))e^{x/2}}{\sinh(x)}\,dx. \tag{5}$$

If we limit ourselves to functions F for which the last two terms in A. Weil's formula (2) are non-negative, then we obtain lower bounds for discriminants (of number fields of degree n) which go to infinity with n.

It remains to choose the best possible F to get the best possible bound. We do not try to improve on the known results on the subject. J.-P. Serre, A. M. Odlyzko, G. Poitou, B. Perrin-Riou and L. Tartar have found very good functions F that suit our purposes. Using their choices, we already have a tremendous improvement over the bounds that J. Hoffstein was working with. We will briefly recall those results.

We will also use A. Weil's formula to get an upper bound for the residues at $s = 1$ of Dedekind zeta functions of number fields. This approach is new and more complex, we do not ignore the last term in A. Weil's formula and choose F accordingly. This will be detailed in Section 3.

Various ideas can be put together to further improve on J. Hoffstein's lower bounds for residues. This will be dealt with in Sections 4–6.

As a conclusion, we present the explicit bounds in Section 7.

3. Upper bounds for residues of zeta functions of totally real number fields

3.1. *Lower bounds for discriminants of number fields.* We briefly recall the methods for obtaining the currently best known lower bounds for discriminants, as we will need them to estimate residues at $s = 1$ of Dedekind zeta functions of number fields. In the following, the notations n, D_k, $\mathfrak{p}$, ϱ are associated to a totally real number field k. In particular, ϱ designates the non-trivial zeros of the Dedekind zeta function ζ_k of k, i.e. those with $0 < \operatorname{Re} \varrho < 1$.

All we ask of F is that it is even, of bounded variation and that the sums we write make sense. Furthermore, for the purpose of getting lower bounds for discriminants, we want the last two terms $\sum_\varrho$ and $\sum_{\mathfrak{p},m}$ in A. Weil's formula (2) to be both non-negative. Let us have a closer look at them.

1. The term $\sum_\varrho \Phi(\varrho)$ depends heavily on the zeros ϱ of ζ_k, and on Φ (defined in formula (3)).

Assuming the Generalized Riemann Hypothesis, the situation is fairly simple: $\sum_\varrho \Phi(\varrho)$ is non-negative if the Fourier transform $\widehat{F}$ of F is non-negative. Indeed, the zeros $\varrho = 1/2 + it$ of ζ_k all have real part $1/2$ and $\Phi(\varrho) = \Phi(1/2 + it) = \widehat{F}(t)$ where $\widehat{F}(t) := \int_{-\infty}^{\infty} F(x)e^{itx}\,dx$ is the Fourier transform of F.

Without the Generalized Riemann Hypothesis, assuming we do not have any further knowledge on the zeros, we want $\operatorname{Re}\Phi(s)$ to be non-negative on the whole region $0 < \operatorname{Re} s < 1$. Since F is real and even, we have

$$\operatorname{Re}(\Phi(\beta + it)) = \int_{-\infty}^{\infty} F(x)\cosh((\beta - 1/2)x)e^{itx}\,dx,$$

and in particular for $\beta = 0$ and $\beta = 1$,

$$\mathrm{Re}(\Phi(it)) = \mathrm{Re}(\Phi(1+it)) = \int_{-\infty}^{\infty} F(x)\cosh(x/2)e^{itx}\,dx.$$

By the Maximum-Modulus principle, if Φ has a moderate growth when t grows to infinity inside the region $0 < \mathrm{Re}\, s < 1$, then the minimum of $\mathrm{Re}\,\Phi(s)$ will be achieved on the boundary, and if we write

$$F(x) = \frac{f(x)}{\cosh(x/2)},$$

then $\sum_\varrho \Phi(\varrho)$ will be non-negative if the Fourier transform of f is non-negative.

2. The last term $\sum_{\mathfrak{p},m}$ of A. Weil's formula (2) is easier to deal with, as all we have to do to make sure it is non-negative is to choose for F and f functions that are non-negative themselves.

Finally, things seem to work out quite well if we choose $F(x) = G(x/b)$ or $f(x) = g(x/b)$. The parameter b then needs to be chosen for each degree n to give the best possible bound. Currently, the best known bounds assuming the Generalized Riemann Hypothesis are obtained with A. M. Odlyzko's choice:

$$G(x) = \begin{cases} (1-|x|)\cos(\pi x) + \frac{1}{\pi}\sin(\pi|x|) & \text{for } |x| \le 1, \\ 0 & \text{for } |x| > 1. \end{cases}$$

Without assuming the Generalized Riemann Hypothesis, the best choice is L. Tartar's:

$$g(x) = 9\left(\frac{\sin x - x\cos x}{x^3}\right)^2.$$

See A. M. Odlyzko [23] or G. Poitou [28], [29] for further information.

3.2. *Upper bound for $\zeta_k(\sigma)$, $\sigma > 1$, for totally real number fields k.* For starters, let us state precisely what we mean by A. Weil's formula for a totally real number field:

Proposition 3 (see [28, Propositions 4 and 5]). *Let F be a real-valued even function with $F(0) = 1$, for which the following conditions hold*:

(i) *The integral $\int_0^\infty F(x)\cosh(x/2)\,dx$ exists.*

(ii) *The function F is of bounded variation, the value at each point being the average of the limit to the right and the limit to the left.*

(iii) *The function $(1 - F(x))/x$ is also of bounded variation.*

(iv) *Assuming the Generalized Riemann Hypothesis, the Fourier transform of F is non-negative. Without this hypothesis, the Fourier transform of $f(x) = F(x)\cosh(x/2)$ is non-negative.*

Then

$$\log D_k \geq n\log(8\pi e^\gamma) + n\pi/2 - nI_{n,n}(F) + 2\sum_{\mathfrak{p},m} \frac{\log \mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{m/2}}\, F(m\log \mathrm{N}(\mathfrak{p}))$$

where $I_{n,n}(F)$ *was defined in* (5).

Now suppose we take $F(x) = (1/x)\exp(-(\sigma - 1/2)x)$ for $x \geq \log 2$. Then

$$(6)\qquad \sum_{\mathfrak{p},m} \frac{\log \mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{m/2}}\, F(m\log \mathrm{N}(\mathfrak{p})) = \sum_{\mathfrak{p},m} \frac{1}{m(\mathrm{N}(\mathfrak{p}))^{m\sigma}} = -\sum_{\mathfrak{p}} \log\left(1 - \frac{1}{(\mathrm{N}(\mathfrak{p}))^{\sigma}}\right) = \log \zeta_k(\sigma).$$

If we could extend F for $x < \log 2$ so that its Fourier transform is non-negative and the other conditions of Proposition 3 hold, we would obtain

$$\log D_k - n\log(8\pi e^\gamma) - n\pi/2 + nI_{n,n}(F) \geq 2\log \zeta_k(\sigma)$$

and the upper bound

$$(7)\qquad \zeta_k(\sigma) < \left(\frac{D_k}{\exp(nC_0(F))}\right)^{1/2}, \qquad C_0(F) = \log(8\pi) + \gamma + \pi/2 - I_{n,n}(F).$$

(Compare with (12).) However, we do not know how to determine whether the Fourier transform of a given function is non-negative. It is a delicate problem and no better results are known on the subject than those of R. P. Boas and M. Kac (see [3]). They gave necessary but no sufficient conditions for the Fourier transform to be non-negative.

1. We must proceed otherwise, using only functions F and f whose Fourier transforms are known to be non-negative beforehand. But if we do that, then we lose the natural relation (6) we had between the last term $\sum_{\mathfrak{p},m}$ of A. Weil's formula (2) and $\log \zeta_k$. Thus we must introduce another step, and derive an inequality of the type

$$\log \zeta_k(\sigma) < 2c_1(\sigma, F)\sum_{\mathfrak{p},m} \frac{\log \mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{m/2}}\, F(m\log \mathrm{N}(\mathfrak{p})),$$

with $c_1(\sigma, F) \leq 1/2$ (important for our purpose, see Section 3.3). This will be possible if F is chosen greater than the function $x \mapsto (1/x)\exp(-(\sigma - 1/2)x)$ for $x \geq \log 2$.

2. First, maybe the most restrictive, condition (i) in Proposition 3 states that $\int_0^\infty F(x)\cosh(x/2)\,dx$ must exist. For the record, to get lower bounds for discriminants, A. M. Odlyzko chose functions with compact support, for which the integral trivially exists. Because of the first requirement, we cannot do that here. Upon careful analysis, we actually have very little choice for the behavior at infinity of F given the first two requirements.

3. Another condition is that the discriminant bound we get should be as good as possible, i.e. we want the choice of F to make the constant $C_0(F)$ in (7) as large as possible. Numerical experimentation shows that this depends mostly on the form of F near the origin. It also shows that it is hard to do better than A. M. Odlyzko in that respect (see [23]), the best form being $F(x) = G(x/b)$ with

$$G(x) = (1-x)\cos(\pi x) + \frac{1}{\pi}\sin(\pi x).$$

4. Lastly, we want conditions (ii), (iii) and (iv) in Proposition 3 to hold.

After all that, it is indeed a wonder that we can find a function which behaves quite well for 1–4. We have the following:

Theorem 4. *Let k be a totally real number field of degree $n \geq 1$. Assume $b > 0$ is given and set*

$$F_b(x) = \frac{1}{(1+(x/b)^2)\cosh(x/2)}, \tag{8}$$

for which

$$I_{n,n}(F_b) = \frac{2\pi b}{n} + I(b) \quad \textit{with} \quad I(b) = \int_0^\infty \frac{(1-F_b(x))e^{x/2}}{\sinh(x)}\,dx \tag{9}$$

(*see formula* (5) *for the definition of $I_{n,n}(F)$*),

$$c_1(\sigma, b) = \sup_{x\geq 2} \frac{-\log(1-x^{-\sigma})}{2\sum_{m\geq 1}\frac{\log x}{x^{m/2}}F_b(m\log x)} < +\infty, \tag{10}$$

and

$$\begin{aligned} C_2(b,n) &= \log(8\pi e^{\gamma}) + \pi/2 - I_{n,n}(F_b) \\ &= \log(8\pi e^{\gamma+\pi/2}) - I(b) - 2\pi b/n. \end{aligned} \tag{11}$$

Then, *for $\sigma > 1$*,

$$\zeta_k(\sigma) < \left(\frac{D_k}{\exp(nC_2(b,n))}\right)^{c_1(\sigma,b)}. \tag{12}$$

Proof. Formula (5) gives

$$I_{n,n}(F_b) = \frac{4}{n}\int_0^\infty F_b(x)\cosh(x/2)\,dx + \int_0^\infty \frac{(1-F_b(x))e^{x/2}}{\sinh(x)}\,dx$$

and since the first integral is equal to $\pi b/2$, we obtain formula (9) easily.

We have

$$\sum_{m\geq 1}\frac{\log x}{x^{m/2}}F_b(m\log x) \geq \frac{\log x}{\sqrt{x}}F_b(\log x) = \frac{2\log x}{(1+x)(1+(\log(x/b))^2)}.$$

So for any $\sigma > 1$,

$$\frac{-\log(1-x^{-\sigma})}{2\sum_{m\ge 1}\frac{\log x}{x^{m/2}}F_b(m\log x)} = O_\sigma(x^{1-\sigma}\log x).$$

We then have, by summing over $\mathrm{N}(\mathfrak{p})$,

$$\begin{aligned}\log\zeta_k(\sigma) &= -\sum_{\mathfrak{p}}\log\left(1-\frac{1}{(\mathrm{N}(\mathfrak{p}))^\sigma}\right)\\ &< 2c_1(\sigma,b)\sum_{\mathfrak{p},m}\frac{\log \mathrm{N}(\mathfrak{p})}{(\mathrm{N}(\mathfrak{p}))^{m/2}}F_b(m\log \mathrm{N}(\mathfrak{p})).\end{aligned}$$

We now check that the conditions of Proposition 3 hold for this choice of F_b. We have $F_b(0)=1$, F_b is indeed even. The conditions (i)–(iii) trivially hold. The condition (iv) holds because the Fourier transform of $f(x) = 1/(1+(x/b)^2)$ is $\widehat{f} = \int_{-\infty}^{\infty} f(t)e^{itx}\,dt = \pi b e^{-b|t|}$, which is non-negative. We can then apply Proposition 3 to obtain

$$\begin{aligned}\log\zeta_k(\sigma) &< c_1(\sigma,b)(\log D_k - n\log(8\pi e^\gamma) - n\pi/2 + nI_{n,n}(F_b))\\ &= c_1(\sigma,b)(\log D_k - nC_2(b,n)). \ \blacksquare\end{aligned}$$

As a closing remark, it is interesting to note that this rather trivial theorem is extremely powerful for this choice of F. The constant $C_2(b,n)$ is large and when D_k happens to be close to its minimal values, the upper bound for the residue we will deduce will surpass any other known upper bound for residues. Also, numerically, the constant $c_1(\sigma,b)$ will be around 0.3, which makes it even better.

3.3. *Relationship between* $\operatorname{Res}\zeta_k$ *and* $\zeta_k(\sigma)$*.* We will use:

LEMMA 5 (see [22], [29] or [35, Lemma 11.11]). *Suppose* $\widetilde{\sigma},\sigma > 1$ *satisfy* $\widetilde{\sigma} \ge 1+\sigma/\sqrt{7+4\sqrt{2}}$ *and* $\widetilde{\sigma} \ge (5+\sqrt{12\sigma^2-5})/6$*. Then, for any complex* ϱ *in the vertical strip* $0 \le \operatorname{Re}\varrho < 1$,

$$\begin{aligned}\operatorname{Re}\left(\frac{1}{\sigma-\varrho}\right) &+ \operatorname{Re}\left(\frac{1}{\sigma-(1-\varrho)}\right)\\ &\ge \left(\sigma-\frac{1}{2}\right)\left(\operatorname{Re}\left(-\frac{1}{(\widetilde{\sigma}-\varrho)^2}\right)+\operatorname{Re}\left(-\frac{1}{(\widetilde{\sigma}-(1-\varrho))^2}\right)\right).\end{aligned} \tag{13}$$

Moreover, if $\widetilde{\sigma} \ge 1+(\sigma-1)/\sqrt{3}$, *then* (13) *holds for any* ϱ *of real part* $1/2$.

Using this lemma, we prove

LEMMA 6. *Let* k *be a totally real number field of degree* $n \ge 1$*. Set*

$$h(\sigma) = \pi^{-\sigma/2}\Gamma(\sigma/2) \quad (\sigma > 1).$$

If $\zeta_k(s)$ *has no real zero in the range* $1/2 < \beta < 1$, *set* $E_\sigma = 1$, *if* $\zeta_k(s)$ *has at least one real zero in that range, set* $E_\sigma = (1-\beta)/(\sigma-\beta)$ *where* β *is any* *of those zeros. In addition, set* $\psi(\sigma) = (\Gamma'/\Gamma)(\sigma)$, *and for* $\widetilde{\sigma}$ *satisfying the conditions of Lemma* 5, *set*

$$c_3(\widetilde{\sigma}, n) = \frac{n}{4}\,\psi'\Big(\frac{\widetilde{\sigma}}{2}\Big) - \frac{1}{\widetilde{\sigma}^2} - \frac{1}{(\widetilde{\sigma}-1)^2}. \tag{14}$$

Then, for $\sigma > 1$,

$$\operatorname{Res}\zeta_k < E_\sigma \frac{\sigma(\sigma-1)\,\zeta_k(\sigma)\,D_k^{(\sigma-1)/2}h^n(\sigma)}{\exp(\sigma(\sigma-1)c_3(\widetilde{\sigma}, n)/2)}.$$

Proof. The starting point is the following identity of H. M. Stark (which can be derived from the Weierstrass product of the entire function ξ_k of order 1):

$$\xi_k(s) := s(s-1)\left(\frac{D_k}{\pi^n}\right)^{s/2}\Gamma(s/2)^n\zeta_k(s) = e^A {\prod_{\varrho}}'\Big(1-\frac{s}{\varrho}\Big) \tag{15}$$

where ϱ runs through the non-trivial zeros of $\zeta_k(s)$ and the prime indicates that ϱ and $\overline{\varrho}$ must be grouped together. Taking logarithmic derivatives, we obtain

$$\frac{\xi_k'(\sigma)}{\xi_k(\sigma)} = {\sum}' \frac{1}{\sigma-\varrho} = \sum_{\operatorname{Im}\varrho\ge 0} \operatorname{Re}\left(\frac{1}{\sigma-\varrho}\right) \tag{16}$$

where the sum on the right side is taken over all zeros ϱ of $\zeta_k(s)$ with non-negative imaginary part. Following A. M. Odlyzko, we set

$$Z(\sigma) = -\frac{\zeta_k'(\sigma)}{\zeta_k(\sigma)} \quad \text{and} \quad Z_1(\sigma) = -\frac{d}{d\sigma}Z(\sigma)$$

and notice that $Z(\sigma) > 0$ and $Z_1(\sigma) > 0$ for $\sigma > 1$. Notice also that

$$\begin{aligned} \sum_{\operatorname{Im}\varrho\ge 0} \operatorname{Re}\Big(-\frac{1}{(\sigma-\varrho)^2}\Big) &= \left(\frac{\xi_k'}{\xi_k}\right)'(\sigma) \\ &= \frac{n}{4}\,\psi'\Big(\frac{\sigma}{2}\Big) + Z_1(\sigma) - \frac{1}{\sigma^2} - \frac{1}{(\sigma-1)^2}. \end{aligned} \tag{17}$$

Let us now prove the lemma in the case where β exists, for example. We have

$$\begin{aligned} \left(\frac{\xi_k'}{\xi_k}\right)'(\sigma) - \frac{1}{\sigma-\beta} &\ge \left(\frac{\xi_k'}{\xi_k}\right)'(\sigma) - \frac{1}{\sigma-\beta} - \frac{1}{\sigma-(1-\beta)} \\ &= \sum_{\substack{\varrho\ne\beta,1-\beta \\ \operatorname{Im}\varrho\ge 0}} \operatorname{Re}\left(\frac{1}{\sigma-\varrho}\right) = \frac{1}{2}\sum_{\substack{\varrho\ne\beta,1-\beta \\ \operatorname{Im}\varrho\ge 0}} \operatorname{Re}\left(\frac{1}{\sigma-\varrho}\right) + \operatorname{Re}\left(\frac{1}{\sigma-(1-\varrho)}\right) \end{aligned}$$

(since $\zeta_k(s) = \zeta_k(1-s)$)

$$\geq \frac{1}{2}\Big(\sigma - \frac{1}{2}\Big) \sum_{\substack{\varrho \neq \beta, 1-\beta \\ \operatorname{Im} \varrho \geq 0}} \operatorname{Re}\Big(-\frac{1}{(\widetilde{\sigma} - \varrho)^2}\Big) + \operatorname{Re}\Big(-\frac{1}{(\widetilde{\sigma} - (1-\varrho))^2}\Big)$$

(by Lemma 5)

$$\geq \Big(\sigma - \frac{1}{2}\Big) \sum_{\operatorname{Im} \varrho \geq 0} \operatorname{Re}\Big(-\frac{1}{(\widetilde{\sigma} - \varrho)^2}\Big)$$

(for $\varrho = \beta$ the right term of (13) is negative)

$$= \Big(\sigma - \frac{1}{2}\Big)\Big(\frac{n}{4}\psi'\Big(\frac{\widetilde{\sigma}}{2}\Big) + Z_1(\widetilde{\sigma}) - \frac{1}{\widetilde{\sigma}^2} - \frac{1}{(\widetilde{\sigma}-1)^2}\Big) \quad \text{(by (17))}$$

$$\geq \Big(\sigma - \frac{1}{2}\Big)\Big(\frac{n}{4}\psi'\Big(\frac{\widetilde{\sigma}}{2}\Big) - \frac{1}{\widetilde{\sigma}^2} - \frac{1}{(\widetilde{\sigma}-1)^2}\Big) = \Big(\sigma - \frac{1}{2}\Big) c_3(\widetilde{\sigma}, n).$$

If we now sum this inequality

$$\frac{\xi_k'(\sigma)}{\xi_k(\sigma)} - \frac{1}{\sigma - \beta} \geq \Big(\sigma - \frac{1}{2}\Big) c_3(\widetilde{\sigma}, n)$$

from 1 to σ, we obtain

$$\log \frac{\xi_k(\sigma)}{\xi_k(1)} + \log E_\sigma \geq \Big(\frac{\sigma^2 - \sigma}{2}\Big) c_3(\widetilde{\sigma}, n),$$

which concludes the proof. ∎

3.4. *Upper bounds for* $\operatorname{Res}\zeta_k$. In Lemma 6, we have an upper bound for $\operatorname{Res}\zeta_k$ in which $\zeta_k(\sigma)$ occurs. Combined with the upper bound for $\zeta_k(\sigma)$ we got in Theorem 4, we have an effective upper bound for the residue $\operatorname{Res}\zeta_k$. It is convenient to write this bound in the following form:

Theorem 7 (cf. [11, Theorem 1′]). *Let k be a totally real number field of degree $n \geq 1$, let $\sigma > 1$ and $b > 0$ be given, let $c_1 = c_1(\sigma, b)$ and $C_2 = C_2(b, n)$ be as in Theorem 4, let $h(\sigma) = \pi^{-\sigma/2}\Gamma(\sigma/2)$ and $c_3 = c_3(\widetilde{\sigma}, n)$ be as in Lemma 6, let*

$$c_4 = c_4(\sigma, b) = c_1(\sigma, b) + (\sigma - 1)/2, \tag{18}$$

and let $C_5 = C_5(n, b, \sigma, \widetilde{\sigma})$ be defined by

$$C_5 = \exp\Big(\frac{1}{c_4}\Big(c_1 C_2 + \frac{\sigma(\sigma-1)}{2} \cdot \frac{c_3}{n} - \frac{1}{n}\log(\sigma(\sigma-1)) - \log h(\sigma)\Big)\Big). \tag{19}$$

Finally, let E_σ be as in Lemma 6. Then

$$\operatorname{Res}\zeta_k \leq E_\sigma \Big(\frac{D_k}{C_5(n, b, \sigma, \widetilde{\sigma})^n}\Big)^{c_4(\sigma, b)}. \tag{20}$$

Possible values for $C_5(n, b, \sigma, \widetilde{\sigma})$ and $c_4(\sigma, b)$ for small degrees are given in Table 4 below.

Before we proceed any further, let us examine the behavior of $n \mapsto C_5(n, b, \sigma, \widetilde{\sigma})$ for given b, σ and $\widetilde{\sigma}$. We first notice that only $C_2 = C_2(b, n)$ and $c_3 = c_3(\widetilde{\sigma}, n)$ depend on n. As can be seen in Lemma 6, $n \mapsto c_3(\widetilde{\sigma}, n)/n$ is increasing for given $\widetilde{\sigma}$. Moreover, we have

$$C_2(b, n) = \log(8\pi e^{\gamma+\pi/2}) - 2\pi b/n - I(b).$$

Thus, if we choose $b > 0$ and $\sigma > 1$ such that

$$2\pi b\, c_1(\sigma, b) + \log(\sigma(\sigma - 1)) \geq 0, \tag{21}$$

then the function $n \mapsto C_5(n, b, \sigma, \widetilde{\sigma})$ will be increasing for given b, σ and $\widetilde{\sigma}$.

We now fix m, and we want to find b, σ (for which (21) holds) and $\widetilde{\sigma}$ (for which the conditions of Lemma 5 hold) so that $C_5(m, b, \sigma, \widetilde{\sigma})$ is as large as possible and $c_4(\sigma, b)$ is as small as possible. Notice that for such a choice of b, σ and $\widetilde{\sigma}$, we have $C_5(n, b, \sigma, \widetilde{\sigma}) \geq C_5(m) := C_5(m, b, \sigma, \widetilde{\sigma})$ and

$$\operatorname{Res} \zeta_k \leq E_\sigma \left(\frac{D_k}{C_5(m)^n} \right)^{c_4(m)}$$

for any totally real number field k of degree $n \geq m$.

To compute these values, for a given m, we first optimize b to have the largest possible $C_2(b, m)$ (b determines $I(b)$ as in Theorem 4). Then we take σ as close to 1 as possible while keeping $c_1(\sigma, b)$ fairly low. Finally, we optimize $\widetilde{\sigma}$ so that $c_3(\widetilde{\sigma}, m)$ is maximal. For further details, we refer to [2].

With or without assuming the Generalized Riemann Hypothesis, the only difference is in the optimization of $\widetilde{\sigma}$. Rather interestingly, the unconditional restrictions only come into play for degrees $n \geq 86$. We will only use degrees $n = 87$ and $n = 88$ while assuming the Generalized Riemann Hypothesis, so it is understood that those two lines in the table are under this assumption. However, we will only consider the degrees $n \geq 90$ unconditionally, and so the corresponding lines are under no particular hypothesis. The bounds in the following table could be slightly improved assuming the Generalized Riemann Hypothesis for $n \geq 90$:

Table 4. Possible constants for upper bounds of residues

m	b	σ	$\widetilde{\sigma}$	$c_4(\sigma, b)$	$C_5(m, b, \sigma, \widetilde{\sigma})$
134	5.89464	1.00755	1.28322	0.35501	43.2532
133	5.87605	1.00760	1.28324	0.35517	43.1904
130	5.81968	1.00774	1.28327	0.35565	42.9983
120	5.62499	1.00825	1.28342	0.35739	42.3126
110	5.41864	1.00884	1.28359	0.35942	41.5469
100	5.19876	1.00955	1.28379	0.36181	40.6841
90	4.96299	1.01042	1.28403	0.36468	39.7013

Table 4 (cont.)

m	b	σ	$\widetilde{\sigma}$	$c_4(\sigma, b)$	$C_5(m, b, \sigma, \widetilde{\sigma})$
88	4.91367	1.01061	1.28113	0.36532	39.4878
87	4.88871	1.01071	1.28250	0.36565	39.3786
83	4.78685	1.01114	1.28823	0.36705	38.9255
82	4.76086	1.01125	1.28972	0.36742	38.8079
80	4.70820	1.01148	1.29280	0.36819	38.5671
70	4.43018	1.01284	1.31019	0.37261	37.2380
60	4.12301	1.01463	1.33188	0.37837	35.6488
50	3.77801	1.01713	1.36007	0.38624	33.6977
40	3.38148	1.02087	1.39886	0.39776	31.2141
30	2.90979	1.02713	1.45730	0.41652	27.8811
27	2.74769	1.02993	1.48153	0.42476	26.6307
25	2.63277	1.03219	1.50033	0.43133	25.7097
24	2.57300	1.03346	1.51070	0.43503	25.2189

Any bound from this table still holds for a totally real number field of degree $n \geq m$, but of course a better bound can be obtained in that case. As we have outlined, we choose b, σ and $\widetilde{\sigma}$ depending on m only, and thus it is possible to reformulate the previous theorem:

THEOREM 8. *Let k be a totally real number field of degree $n \geq m$ and let E_σ be as in Lemma* 6. *Then*

$$\operatorname{Res} \zeta_k \leq E_\sigma \left(\frac{D_k}{C_5(m)^n} \right)^{c_4(m)}$$

where $c_4(m)$ and $C_5(m)$ are given in the following table:

Table 5. Upper bounds for residues depending on m only

m	24	25	27	30	40	50	60
$c_4(m)$	0.43503	0.43133	0.42476	0.41652	0.39776	0.38624	0.37837
$C_5(m)$	25.2189	25.7097	26.6307	27.8811	31.2141	33.6977	35.6488
m	70	80	82	83	87	88	
$c_4(m)$	0.37261	0.36819	0.36742	0.36705	0.36565	0.36532	
$C_5(m)$	37.2380	38.5671	38.8079	38.9255	39.3786	39.4878	
m	90	100	110	120	130	133	134
$c_4(m)$	0.36468	0.36181	0.35942	0.35739	0.35565	0.35517	0.35501
$C_5(m)$	39.7013	40.6841	41.5469	42.3126	42.9983	43.1904	43.2532

3.5. *Upper bound for* $\operatorname{Res} \zeta_k$ *for high discriminants.* When the discriminant is much higher than Odlyzko's bounds, the upper bound for $\operatorname{Res} \zeta_k$ we have obtained from A. Weil's formula is not good enough. We must then use another bound, for example:

THEOREM 9. *Let k be a number field of degree $n > 1$.*

1. (See [17, Theorem 1].) *We have*

$$\operatorname{Res}\zeta_k \le e\left(\frac{e\log D_k}{2n}\right)^{n-1}.$$

2. (See [18, Theorem 1].) *Moreover*, $1/2 < \beta < 1$ *and* $\zeta_k(\beta) = 0$ *imply*

$$\operatorname{Res}\zeta_k \le (1-\beta)\left(\frac{e\log D_k}{2n}\right)^{n}.$$

Regarding our objective, this upper bound for the residue is used for $n \le 23$ under the Generalized Riemann Hypothesis in the normal case, for $n \le 24$ in the non-normal case, and for $n \le 26$ without assuming the Generalized Riemann Hypothesis.

4. Explicit lower bounds for residues of zeta functions of normal number fields assuming the Generalized Riemann Hypothesis. The aim of this section is to prove the following explicit conditional lower bound for residues:

THEOREM 10. *Assume the Generalized Riemann Hypothesis. Then, for any normal number field K, we have the lower bound*

$$\operatorname{Res}\zeta_K \ge \frac{1}{(2e^{\gamma}+o(1))\log\log D_K}$$

where $o(1)$ is an effective error term decreasing towards zero as D_K increases towards infinity.

This is a particular case of a result of J. Buchmann and H. C. Williams (see [4]). They derive a similar one $(\log\log D_k)\operatorname{Res}\zeta_k \ge C$, in the non-normal case, but the effective C that would result is too large for our purpose.

We have

$$\operatorname{Res}\zeta_K = \lim_{s\to 1}(s-1)\zeta_K(s) = \lim_{s\to 1}(\zeta_K/\zeta)(s) = \prod_{p\in P} E(p)$$

where P is the set of all prime numbers $p \ge 2$ and

$$E(p) = (1-p^{-1})\prod_{\mathfrak{p}\mid p}(1-\mathrm{N}(\mathfrak{p})^{-1})^{-1}$$

is the eulerian factor associated with p and ζ_K/ζ. We write

$$\operatorname{Res}\zeta_K = M(Q)R(Q)T(Q) \tag{22}$$

with

$$M(Q) = \prod_{p \le Q} E(p), \qquad R(Q) = \prod_{\substack{p>Q \\ p\,\text{ramified}}} E(p), \qquad T(Q) = \prod_{\substack{p>Q \\ p\,\text{non-ramified}}} E(p).$$

The whole difficulty is to give a subtle estimate for the tail $T(Q)$ so that we can choose a reasonably high value for Q. Once this is done properly, we will simply use trivial lower bounds for the main term $M(Q)$ and the ramified term $R(Q)$.

4.1. *Bounds for the tail* $T(Q)$

LEMMA 11. *Let K be a normal number field of degree n. Set*

$$f_K(t) = \left(\frac{1}{\pi \log t} + \frac{5.3}{\log^2 t}\right) \log D_K + (n+1)\left(\frac{1}{2\pi} + \frac{2}{\log t}\right).$$

Then, assuming the Generalized Riemann Hypothesis, we have

$$|\log T(Q)| \le \frac{4 + 3\log Q}{\sqrt{Q}}\, f_K(Q) + \frac{n}{Q}.$$

In particular, for $a > 2$ we have $\lim_{D_K \to \infty} \log T((\log D_K)^a) = 0$.

Proof. Each eulerian factor $E(p)$ depends on the norm $\mathrm{N}(\mathfrak{p}) = p^f$ of the ideals above p and if we set

$$a_1(p) = \begin{cases} 1 & \text{if } f > 1, \\ 1 - n & \text{if } f = 1, \end{cases}$$

then

$$\frac{E(p)}{(1 - 1/p)^{a_1(p)}} = \begin{cases} 1 & \text{if } f = 1, \\ (1 - 1/p^f)^{-n/f} & \text{if } f > 1. \end{cases}$$

Hence,

$$\begin{aligned} \log T(Q) &= \sum_{\substack{p>Q \\ p\,\text{non-ram}}} a_1(p) \log\left(1 - \frac{1}{p}\right) - \sum_{\substack{p>Q \\ p\,\text{non-ram} \\ f_p>1}} \frac{n}{f_p} \log\left(1 - \frac{1}{p^{f_p}}\right) \\ &= - \sum_{\substack{p>Q \\ p\,\text{non-ram}}} \frac{a_1(p)}{p} + \varrho(Q) \end{aligned}$$

where

$$\varrho(Q) = \sum_{\substack{p>Q \\ p\,\text{non-ram}}} a_1(p)\left(\frac{1}{p} + \log\left(1 - \frac{1}{p}\right)\right) - \sum_{\substack{p>Q \\ p\,\text{non-ram} \\ f_p>1}} \frac{n}{f_p} \log\left(1 - \frac{1}{p^{f_p}}\right). \tag{23}$$

LEMMA 12. *$n \ge 2$ and $Q \ge n$ imply $|\varrho(Q)| \le n/Q$.*

Proof. We have

$$\begin{aligned}
|\varrho(Q)| &\le (n-1)\sum_{p>Q}\left(-\frac{1}{p}-\log\left(1-\frac{1}{p}\right)\right)+\frac{n}{2}\sum_{p>Q}-\log\left(1-\frac{1}{p^2}\right)\\
&\le (n-1)\sum_{p>Q}\frac{1}{2p^2}\cdot\frac{1}{1-1/p}+\frac{n}{2}\sum_{p>Q}\frac{1}{p^2}\cdot\frac{1}{1-1/p^2}\\
&\le \frac{n-1}{1-1/(n+1)}\sum_{p>Q}\frac{1}{2p^2}+\frac{n}{1-1/(n+1)^2}\sum_{p>Q}\frac{1}{2p^2}\quad (\text{for } p>Q\ge n)\\
&\le \left(\frac{n^2-1}{n}+\frac{n^3+2n^2+n}{n(n+2)}\right)\frac{1}{2Q}\\
&= \left(2n-\frac{2}{n(n+2)}\right)\frac{1}{2Q}\le\frac{n}{Q}. \ \blacksquare
\end{aligned}$$

It remains to estimate the sum $\sum_{p>Q} a_1(p)/p$. We shall use the method developed by G. Cornell and L. C. Washington [6] for cyclotomic fields. We wish to perform an Abel transform on the sum, so we set

$$A(t)=\sum_{p<t} a_1(p),\qquad \pi_1(t)=\sum_{\substack{p<t\\ f=1}} 1,$$

and we write

$$A(t)=\pi(t)-n\pi_1(t)=\pi(t)-\mathrm{li}(t)+\mathrm{li}(t)-n\pi_1(t).$$

At this point, we use J. Oesterlé's explicit form of J. C. Lagarias and A. M. Odlyzko's results, according to which, assuming the Generalized Riemann Hypothesis, we have (see [24, Théorème 3])

$$\left|\pi_1(t)-\frac{1}{n}\,\mathrm{li}(t)\right|\le\frac{1}{n}\,C_K(t)\sqrt{t}\log t$$

where

$$C_K(t)=\left(\frac{1}{\pi\log t}+\frac{5.3}{\log^2 t}\right)\log D_K+n\left(\frac{1}{2\pi}+\frac{2}{\log t}\right)$$

which in the particular case $K=\mathbb{Q}$ gives

$$|\pi(t)-\mathrm{li}(t)|\le\left(\frac{1}{2\pi}+\frac{2}{\log t}\right)\sqrt{t}\log t.$$

Hence

$$|A(t)|\le f_K(t)\sqrt{t}\log t.$$

Since $t\mapsto f_K(t)$ is a decreasing function of $t>1$, we obtain

$$\left|\sum_{p>Q}\frac{a_1(p)}{p}\right|=\left|\int_Q^\infty\frac{1}{t}\,dA(t)\right|=\left|\frac{A(Q)}{Q}+\int_Q^\infty\frac{A(t)}{t^2}\,dt\right|$$

$$\leq \frac{|A(Q)|}{Q} + f_K(Q) \int_Q^\infty \frac{\log t}{t^{3/2}}\,dt$$
$$\leq f_K(Q)\,\frac{\log Q}{\sqrt{Q}} + f_K(Q)\,\frac{4+2\log Q}{\sqrt{Q}}$$

or

$$\left|\sum_{p>Q} \frac{a_1(p)}{p}\right| \leq f_K(Q)\,\frac{4+3\log Q}{\sqrt{Q}}. \tag{24}$$

Combining (22)–(24), we get the desired result. The last assertion of the lemma comes from the fact that $n = O(\log D_K)$. ∎

4.2. *Proof of Theorem 10.* Now that the tail $T(Q)$ of the eulerian product is under control, it remains to obtain lower bounds for the main term $M(Q)$ and the ramified term $R(Q)$. For $Q \geq 285$, we have

$$M(Q) = \prod_{p\leq Q} E(p) \geq \prod_{p\leq Q}\left(1-\frac{1}{p}\right) \geq \frac{1}{e^\gamma \log Q}\left(1-\frac{1}{2\log^2 Q}\right) \tag{25}$$

(by [30, (3.25)]). For the ramified term $R(Q)$, since the ramified primes greater than Q divide D_K, there are at most $\log D_K/\log Q$ such primes and

$$R(Q) = \prod_{\substack{p>Q\\ p\,\text{ramified}}} E(p) \geq \prod_{\substack{p>Q\\ p\,\text{ramified}}}\left(1-\frac{1}{p}\right) \geq (1-1/Q)^{\log D_K/\log Q}. \tag{26}$$

Combining (25), (26) and Lemma 11, with $Q = (\log D_K)^a \geq 285$, we conclude that for $a > 2$ we have

$$\liminf_{D_K\to\infty} (\log\log D_K)\operatorname{Res}\zeta_K \geq \frac{1}{ae^\gamma},$$

which proves Theorem 10.

However, this theorem only provides an asymptotic lower bound for $\operatorname{Res}\zeta_K$ and we will need the following explicit result:

Corollary 13. *Let $m \geq 1$ and $\varrho > e$ be given. There exists an effective $c_6(m,\varrho)$ such that, for any normal number field K of degree $n \geq m$ and of root discriminant $\varrho_K = D_K^{1/n} \geq \varrho$,*

$$\operatorname{Res}\zeta_K \geq \frac{1}{c_6(m,\varrho)\log\log D_K}.$$

The following table gives the a and $c_6(m,\varrho)$ that we use to establish Theorem 2:

Table 6. Conditional lower bounds for the residues

m	166	164	140	120	100	80	60
ϱ	54.88	54.76	65.76	81.93	114.5	204.0	729.3
a	3.27897	3.28114	3.28825	3.29382	3.29534	3.28809	3.25254
$c_6(m,\varrho)$	6.43518	6.44025	6.46426	6.48259	6.49536	6.49202	6.43824
m	48	46	40	30	20	10	
ϱ	5252	6354	6499	6875	7653	10250	
a	3.20022	3.19990	3.22302	3.27400	3.35472	3.52317	
$c_6(m,\varrho)$	6.33092	6.33214	6.38811	6.51149	6.70664	7.11282	

Proof. Set

$$g_K(Q) = \frac{\exp\left(\frac{4+3\log Q}{\sqrt{Q}} f_K(Q) + \frac{n}{Q}\right)}{\left(1 - \frac{1}{2\log^2 Q}\right)\left(1 - \frac{1}{Q}\right)^{\log D_K/\log Q}}.$$

Combining (25), (26) and Lemma 11, we get

$$\operatorname{Res}\zeta_K \geq \frac{1}{ae^\gamma g(a, \varrho_K, n)\log\log D_K}$$

where $g(a,\varrho_K,n) = g_K((n\log\varrho_K)^a)$ is a function of a, ϱ_K and n only. Since the functions $\varrho_K \mapsto g(a,\varrho_K,n)$ and $n \mapsto g(a,\varrho,n)$ are decreasing, we have

$$ae^\gamma g(a,\varrho_K,n) \leq ae^\gamma g(a,\varrho,n) \leq ae^\gamma g(a,\varrho,m).$$

The function $a \mapsto g(a,\varrho,m)$ is decreasing as well. As a result, for given m and ϱ, we can find the optimal value of a which minimizes the product $ae^\gamma g(a,\varrho,m)$. This proves the existence of an absolute constant $c_6(m,\varrho)$ for which we have for any normal number field of degree $n \geq m$ the lower bound

$$\operatorname{Res}\zeta_K \geq \frac{1}{c_6(m,\varrho)\log\log D_K}. \blacksquare$$

Remark. To obtain for given m the explicit values of the constant $c_6(m,\varrho)$ in Corollary 13, we could have chosen for ϱ Odlyzko's lower bound ϱ_m for root discriminants of CM-fields of degree m, i.e. Odlyzko's lower bound ϱ_m for root discriminants of totally real fields of degree m. This would have yielded, for each m appearing in Table 6, a constant $c_6(m,\varrho) = c_6(m,\varrho_m) = c_6(m)$ depending on m only, but slightly larger than the one we give in Table 6. This would have given for the first case of Theorem 2 bounds not as good as the ones we give there. To get the best possible bounds for the first case of Theorem 2, we need, for given m, to compute $c_6(m,\varrho)$ with ϱ precisely equal to the lower bound for root discriminants of CM-fields with relative class number one of degree m, lower bound given in the first case of Theorem 2. Clearly, there is a small problem of optimization to choose those values of ϱ, but there is no problem to see that the values we give in Table 6 will prove Theorem 2.

4.3. *Non-normal case.* In the non-normal case, we can obtain a similar bound to the one we obtained in Section 4.2, but we must then work inside the normal closure of L of K, and we get bounds of the form $\operatorname{Res}\zeta_K > 1/((2e^{\gamma}+o(1))\log\log D_L)$. Unfortunately, in the general case, the only upper bound we have for D_L is $D_L \le D_K^{(2n-1)!}$, which makes those lower bounds for $\operatorname{Res}\zeta_K$ rather worthless for our purpose. We need something better. The following lower bound for residues has been communicated to us by J. Oesterlé (see the proof in Appendix):

THEOREM 14 (J. Oesterlé). *For any number field K different from $\mathbb{Q}$ for which the Riemann Hypothesis for $\zeta_K(s)$ holds true we have*

$$\operatorname{Res}\zeta_K \ge \frac{e^{-3/2}}{\sqrt{\log D_K}}\exp\left(\frac{-1}{\sqrt{\log D_K}}\right).$$

5. Unconditional explicit lower bounds for residues of zeta functions of CM-fields. If we do not assume the Generalized Riemann Hypothesis, the method is completely different, as we have no analog of the tools used in the preceding section. The important point is to control the zeros of ζ_K. For the following, K is simply a totally imaginary field of discriminant D_K and of degree $2n$.

LEMMA 15 (see [14, Lemma 15]). *Set $\kappa = (2+\sqrt{3})/4 = 0.933\ldots$ Then $\zeta_K(s)$ has at most two zeros counted with multiplicity in the range $1 - 1/(\kappa\log D_K) \le s < 1$.*

Once we have this, we use a theorem of [19], which can be slightly rewritten as:

THEOREM 16. *Let $m \ge 1$ be a positive integer. Let $c > 0$ be given. There exists an effective $\varrho(2m,c)$ such that for any totally imaginary number field K of degree $2n \ge 2m$ and root discriminant $\varrho_K := D_K^{1/(2n)} \ge \varrho(2m,c)$ we have*

$$\operatorname{Res}\zeta_K \ge \frac{1}{ce^{1/(2c)}\log D_K} \tag{27}$$

if $\zeta_K(1-1/(c\log D_K)) \le 0$, and

$$\operatorname{Res}\zeta_K \ge \frac{1-\beta}{2e^{1/(2c)}} \tag{28}$$

if $\zeta_K(\beta) \le 0$ and $1-1/(c\log D_K) \le \beta < 1$.

Moreover, for $c = \kappa = (2+\sqrt{3})/4$ and $m \ge 5$, we may take $\varrho(2m,c) = 2\pi^2$.

We remark that in the most favorable case, $\zeta_K(1-(c\log D_K)^{-1}) \le 0$ and we get

$$\operatorname{Res}\zeta_K \geq \frac{1}{ce^{1/(2c)}\log D_K}.$$

We will see in Section 7 that this “most favorable” case is in fact the worse thing that can happen. So we can already give the lower bound we will use:

$$\operatorname{Res}\zeta_K \geq \frac{1}{\kappa e^{1/(2\kappa)}\log D_K}.$$

6. A bound for Siegel's zeros of zeta functions of imaginary quadratic fields. The aim of this section is to prove the following effective upper bound on Siegel's zeros of zeta functions of imaginary quadratic fields (cf. [33, Lemma 8]):

THEOREM 17. *Let $c > 0$ be given and let F range over the imaginary quadratic number fields. There exists an effective constant D_c such that if $D_F > D_c$ then $\zeta_F(s) < 0$ in the range $1 - c/\sqrt{D_F} \leq s \leq 1$. In particular, for all imaginary quadratic fields F, we have $\zeta_F(s) < 0$ in the range $1 - 6/(\pi\sqrt{D_F}) \leq s < 1$.*

It is possible to derive explicit bounds from the theoretical bounds of Pintz [27], or of Goldfeld [8], or of Goldfeld and Schinzel [9]. However, our bounds are better and easier to obtain. Notice that J. Hoffstein used our last assertion on page 46 of [11], but the paper of his he was referring to never appeared!

From now on, we let $F = \mathbb{Q}(\sqrt{-d})$ range over the imaginary quadratic fields of discriminants $-d < -4$. We let χ_{-d} denote the primitive quadratic Dirichlet character modulo d associated with F and recall that we have the factorization $\zeta_F(s) = \zeta(s)L(s,\chi_{-d})$. We then let $h(-d)$ denote the class number of F and $Q(x,y) = ax^2 + bxy + cy^2$ will always stand for a primitive reduced binary quadratic form of discriminant $-d = b^2 - 4ac$. Recall that there are $h(-d)$ reduced binary quadratic forms of discriminant $-d$ and that if Q is reduced then $1 \leq a \leq \sqrt{d/3}$ and $-a < b \leq a$. The symbol $\sum_{a,b,c}$ will stand for sums over the $h(-d)$ reduced binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ of discriminant $-d < -4$ and for such a form we set

$$\zeta_Q(s) = \sum_{(m,n)\neq(0,0)} \left(\frac{a}{\sqrt{d}}n^2 + \frac{b}{\sqrt{d}}mn + \frac{c}{\sqrt{d}}m^2\right)^{-s}.$$

6.1. *Explicit bounds for the second derivatives of Dirichlet L-functions*

LEMMA 18. *Let $c > 0$ be given and let χ range over the primitive Dirichlet characters of conductors $f_\chi \geq 4c^2$ (which implies $1/2 \leq 1 - c/\sqrt{f_\chi} \leq 1$). Then*

$$|L''(s,\chi)| \leq \frac{1}{24}(1 + o(1))\log^3 f_\chi$$

in the range $1 - c/\sqrt{f_\chi} \leq s \leq 1$ *where* $o(1)$ *is an explicit error term which does not depend on* s *and approaches* 0 *as* f_χ *goes to infinity. In particular, in the range* $1 - 6/(\pi\sqrt{f_\chi}) \leq s \leq 1$ *we have*

$$|L''(s,\chi)| \leq \begin{cases} \frac{1}{3}\log^3 f_\chi & \text{if } f_\chi \geq 1108, \\ 0.3\log^3 f_\chi & \text{if } f_\chi \geq 1775, \\ \frac{1}{4}\log^3 f_\chi & \text{if } f_\chi \geq 4692, \\ \frac{1}{5}\log^3 f_\chi & \text{if } f_\chi \geq 23393. \end{cases}$$

The reason why we state this upper bound with such precision is because in the end it will be the main error term in the lower bound for $1-\beta$. So getting the best possible bound will have a significant impact on the end result.

Proof. Fix an integer $B \geq e^2$. Then $t \mapsto (\log^2 t)/t$ is decreasing in the range $t \geq B$. Set $A_\chi = \sqrt{f_\chi}\log f_\chi$ and assume $f_\chi \geq 128$, which implies $A_\chi \geq e^4$. This ensures that $t \mapsto (\log^2 t)/t^s$ is decreasing in the range $t \geq A_\chi$, for any s in the range $1/2 \leq s \leq 1$. Assume also that f_χ is large enough to guarantee $A_\chi \geq B$ and set $A = [A_\chi]$ = the greatest integer less than or equal to A_χ. Hence $B \leq A$. Set $X(n) = \sum_{k=A+1}^{n} \chi(k)$, and recall that $|X(n)| \leq A_\chi$ (Pólya–Vinogradov's bound). For $1 - c/\sqrt{f_\chi} \leq s \leq 1$, we have

$$\begin{aligned} L''(s,\chi) = \sum_{n=1}^{B} \frac{\chi(n)\log^2 n}{n^s} &+ \sum_{n=B+1}^{A} \frac{\chi(n)\log^2 n}{n^s} \\ &+ \sum_{n\geq A+1} X(n)\biggl(\frac{\log^2 n}{n^s} - \frac{\log^2(n+1)}{(n+1)^s}\biggr) \end{aligned}$$

and

$$\frac{1}{n^s} = \frac{n^{1-s}}{n} \leq \frac{A_\chi^{c/\sqrt{f_\chi}}}{n} \qquad \text{for } n \leq A.$$

Hence we obtain

$$\begin{aligned} |L''(s,\chi)| &\leq A_\chi^{c/\sqrt{f_\chi}}\biggl(\sum_{n=1}^{B}\frac{\log^2 n}{n} + \sum_{n=B+1}^{A}\frac{\log^2 n}{n}\biggr) \\ &\qquad + A_\chi \sum_{n\geq A+1}\biggl(\frac{\log^2 n}{n^s} - \frac{\log^2(n+1)}{(n+1)^s}\biggr) \\ &\leq A_\chi^{c/\sqrt{f_\chi}}\biggl(\sum_{n=1}^{B}\frac{\log^2 n}{n} + \int_B^A \frac{\log^2 x}{x}\,dx\biggr) + A_\chi\frac{\log^2(A+1)}{(A+1)^s} \\ &\leq A_\chi^{c/\sqrt{f_\chi}}\biggl(\frac{1}{3}\log^3 A_\chi + c_B\biggr) + A_\chi\frac{\log^2 A_\chi}{A_\chi^s} \end{aligned}$$

where

$$c_B = \sum_{n=1}^{B} \frac{\log^2 n}{n} - \frac{1}{3}\log^3 B.$$

Hence,

$$|L''(s,\chi)| \le A_\chi^{c/\sqrt{f_\chi}}\left(\frac{1}{3}\log^3 A_\chi + \log^2 A_\chi + c_B\right),$$

which yields the desired result (notice that $A_\chi^{c/\sqrt{f_\chi}} = 1 + o(1)$). ∎

REMARK 19. $B \mapsto c_B$ decreases in the range $B \ge 7$ and $c_B \le 0$ for $B \ge 3461$. To obtain the explicit bounds, we choose $B = 24$ and in that case $0.200 < c_B < 0.201$.

6.2. *Explicit bounds for $L'(1,\chi_d)$*

LEMMA 20. *Set*

$$c_7 = \pi\gamma - \frac{\pi}{2}\log 3 + 4\pi \sum_{m\ge 1} \frac{1}{m(e^{\pi m\sqrt{3}} - 1)} = 0.092\ldots$$

Then $d > 4$ implies

$$L'(1,\chi_{-d}) \le \frac{\pi^2}{6}\sum_{a,b,c}\frac{1}{a} + c_7\,\frac{h(-d)}{\sqrt{d}}.$$

Proof. We have

$$\lim_{s\to 1}\left(\zeta(s)L(s,\chi_{-d}) - \frac{L(1,\chi_{-d})}{s-1}\right) = \gamma L(1,\chi_{-d}) + L'(1,\chi_{-d}). \tag{29}$$

Since

$$\zeta_K(s) = \frac{1}{2(\sqrt{d})^s}\sum_{a,b,c}\zeta_Q(s) \qquad (d > 4)$$

and since Dirichlet's class number formula gives

$$L(1,\chi_{-d}) = \frac{\pi h(-d)}{\sqrt{d}} = \frac{\pi}{\sqrt{d}}\sum_{a,b,c} 1 \qquad (d > 4),$$

we obtain

$$\lim_{s\to 1}\left(\zeta(s)L(s,\chi_{-d}) - \frac{L(1,\chi_{-d})}{s-1}\right) = \frac{1}{2\sqrt{d}}\sum_{a,b,c}\lim_{s\to 1}\left(\zeta_Q(s) - \frac{2\pi}{s-1}\right). \tag{30}$$

Now, according to Kronecker's limit formula as given in Selberg [31, (39)], we have

$$\lim_{s\to 1}\left(\zeta_Q(s)-\frac{2\pi}{s-1}\right)=\frac{2\sqrt{d}}{a}\cdot\frac{\pi^2}{6}+4\pi\gamma+2\pi\log\frac{a}{\sqrt{d}} + 8\pi\sum_{m\ge 1}\sigma_{-1}(m)\cos\left(\frac{m\pi b}{a}\right)e^{-\pi m\sqrt{d}/a}$$

with

$$\sigma_{-1}(m)=\sum_{d\mid m}\frac{1}{d}.$$

Now recall that $a\le\sqrt{d/3}$. Therefore, we have

$$\left|\sum_{m\ge 1}\sigma_{-1}(m)\cos\left(\frac{m\pi b}{a}\right)e^{-\pi m\sqrt{d}/a}\right|\le\sum_{m\ge 1}\sigma_{-1}(m)e^{-\pi m\sqrt{3}}.$$

Also,

$$\sum_{m\ge 1}\sigma_{-1}(m)x^m=\sum_{m\ge 1}\sum_{d\mid m}\frac{1}{d}\,x^m=\sum_{d\ge 1}\sum_{n\ge 1}\frac{1}{d}\,x^{dn}=\sum_{d\ge 1}\frac{1}{d(1/x^d-1)}$$

and we get

$$\lim_{s\to 1}\left(\zeta_Q(s)-\frac{2\pi}{s-1}\right)-\frac{2\sqrt{d}}{a}\cdot\frac{\pi^2}{6}\le 4\pi\gamma-\pi\log 3+8\pi\sum_{m\ge 1}\frac{1}{m(e^{\pi m\sqrt{3}}-1)}.$$

We use this in (29) and (30), which completes the proof. ∎

LEMMA 21. *Let $N\ge 1$ denote a positive integer, and $\omega(a)$ the number of distinct prime divisors of a positive integer $a\ge 1$. Set $\lambda_1=0$,*

$$\lambda_N=\sum_{a=1}^{N-1}2^{\omega(a)}\left(\frac{1}{a}-\frac{1}{N}\right)$$

for $N\ge 2$, and

$$S(d)=\sum_{a,b,c}\frac{1}{a}.$$

Then, for any $d>4$,

$$S(d)\le\frac{1}{N}\,h(-d)+\lambda_N.$$

In particular,

$$S(d)\le h(-d),\qquad S(d)\le\tfrac{1}{3}\,h(-d)+1,$$
$$S(d)\le\tfrac{1}{2}\,h(-d)+\tfrac{1}{2},\qquad S(d)\le\tfrac{1}{4}\,h(-d)+\tfrac{17}{12}.$$

Proof. We use induction on $N\ge 1$. If $N=1$ then we must prove that $S(d)\le h(-d)$, which is clear. Assume that for some $N\ge 2$ we have proven that

$$S(d) \le \frac{1}{N-1}\, h(-d) + \lambda_{N-1} \quad \text{for all } d > 4.$$

Since for a given $a \ge 1$ there are at most $2^{\omega(a)}$ reduced forms $Q(x,y) = ax^2 + bxy + cy^2$ of discriminant $-d$, we obtain

$$S(d) \le \sum_{a=1}^{N-1} \frac{2^{\omega(a)}}{a} + \frac{1}{N}\Big(h(-d) - \sum_{a=1}^{N-1} 2^{\omega(a)}\Big) \le \frac{1}{N}\, h(-d) + \lambda_N$$

for $h(-d) \ge \sum_{a=1}^{N-1} 2^{\omega(a)}$. Suppose now that $h(-d) \le \sum_{a=1}^{N-1} 2^{\omega(a)}$. Then

$$\begin{aligned} \frac{h(-d)}{N(N-1)} &\le \Big(\sum_{a=1}^{N-2} \frac{2^{\omega(a)}}{N(N-1)}\Big) + \frac{2^{\omega(N-1)}}{N(N-1)} \\ &= \Big(\sum_{a=1}^{N-2} 2^{\omega(a)}\Big(\frac{1}{a} - \frac{1}{N}\Big)\Big) - \Big(\sum_{a=1}^{N-2} 2^{\omega(a)}\Big(\frac{1}{a} - \frac{1}{N-1}\Big)\Big) \\ &\quad + 2^{\omega(N-1)}\Big(\frac{1}{N-1} - \frac{1}{N}\Big) \\ &= \lambda_N - \lambda_{N-1} \end{aligned}$$

and

$$\frac{1}{N-1}\, h(-d) + \lambda_{N-1} \le \frac{1}{N}\, h(-d) + \lambda_N.$$

Since we have assumed that $S(d) \le \frac{1}{N-1}\, h(-d) + \lambda_{N-1}$, we deduce that $S(d) \le \frac{1}{N}\, h(-d) + \lambda_N$, which concludes the proof. ∎

6.3. *Proof of Theorem 17.* Let $c > 0$ be given and set

$$M_c(d) = \max_{1-c/\sqrt{d} \le s \le 1} |L''(s, \chi_{-d})|$$

(according to Lemma 18 we have $M_c(d) \le c_8 \log^3 d$ where the constant involved in this bound depends on c only). According to Taylor's formula, for any s in the range $1 - c/\sqrt{d} \le s \le 1$ we have

$$L(s, \chi_{-d}) \ge L(1, \chi_{-d}) - \frac{c}{\sqrt{d}}\, L'(1, \chi_{-d}) - \frac{c^2}{2d}\, M_c(d)$$

and we will have $L(s, \chi_{-d}) > 0$ for all s in the range $1 - c/\sqrt{d} \le s \le 1$ provided that d is such that

$$L(1, \chi_{-d}) > \frac{c}{\sqrt{d}}\, L'(1, \chi_{-d}) + \frac{c^2}{2d}\, M_c(d).$$

Since $L(1,\chi_{-d}) = \pi h(-d)/\sqrt{d}$ for $d > 4$ and since according to Lemmas 20 and 21 we have

$$L'(1,\chi_{-d}) \le \frac{\pi^2}{6} S(d) + c_7 \frac{h(-d)}{\sqrt{d}} \le \frac{\pi^2}{6}\left(\frac{1}{N} h(-d) + \lambda_N\right) + c_7 \frac{h(-d)}{\sqrt{d}},$$

we will have $L(s,\chi_{-d}) > 0$ for all s in the range $1 - c/\sqrt{d} \le s \le 1$ provided that d is such that

$$1 > \frac{\pi c}{6N} + \frac{\pi c \lambda_N}{6h(-d)} + \frac{cc_7}{\pi\sqrt{d}} + \frac{c^2 M_c(d)}{2\pi h(-d)\sqrt{d}} = \frac{\pi c}{6N} + o(1)$$

for we have $\lim_{d\to\infty} h(-d) = \infty$ (and this can be made effective by using Osterlé's explicit form of Gross–Zagier's bounds for $h(-d)$, see [7]). Now, for a given c, if we choose a positive integer $N > 6/(\pi c)$, then we do find that for some effective D_c we have $L(s,\chi_{-d}) > 0$ for all s in the range $1 - c/\sqrt{d} \le s \le 1$ if $d > D_c$.

To prove the last assertion of Theorem 17, we use the solutions to the class number 1, 2, 3, 4 and 5 problems (see Goldfeld [7], Arno [1, Theorem 7] and Wagner [34, Table 1]). The following table shows the various constants D_c we get for $c = 6/\pi$ depending on which class number problems we assume to be solved. For example, the first line of the table shows that we can take $D_c = 94704$ if we only use the fact that $h(-d) \ge 2$ for $d > 163$, the second line shows that we can take $D_c = 11357$ if we only use the fact that $h(-d) \ge 3$ for $d > 427$, and so forth.

Table 7

h	Last d for which $h(-d) = h$	c_8	N	λ_N	D_c
1	163	0.173	2	1/2	94704
2	427	0.219	2	1/2	11357
3	907	0.283	3	1	2375
4	1555	0.309	3	1	1556

With these values, it is easy to check that Theorem 17 holds. Indeed, it is easy to perform computer calculations and prove that there are no Siegel zeros for the Dedekind zeta functions of the imaginary quadratic fields of discriminants $-d > -1556$ (see [21, Theorem 5]). This proves the last assertion of the theorem.

7. Proof of the main theorem (Theorem 2). Let k be a totally real number field of degree n and let K be a totally imaginary quadratic extension of k. We have $D_K \ge D_k^2$ and then $\sqrt{D_K/D_k} \ge D_K^{1/4}$. Furthermore we have $Q_K w_K \ge 2$, and using both in (1), we get

$$h_K^- \geq 2 \cdot \frac{D_K^{1/4}}{(2\pi)^n} \cdot \frac{\operatorname{Res} \zeta_K}{\operatorname{Res} \zeta_k}. \tag{31}$$

7.1. *Bounds assuming the Generalized Riemann Hypothesis*

7.1.1. *Normal case.* We use (31) and the bounds on residues we obtained in Theorems 8 and 10 (notice that $E_\sigma = 1$ under the assumption of the Generalized Riemann Hypothesis) to get, for $D_K \geq \varrho^{2n}$,

$$h_K^- \geq 2\left(\frac{C_5(n)^{c_4(n)}}{2\pi}\right)^n \frac{D_K^{1/4-c_4(n)/2}}{c_6(2n,\varrho)\log\log D_K}.$$

For the values of $D_K \geq \varrho^{2n}$ we consider, this lower bound is an increasing function of D_K. Thus we are entitled to use Odlyzko's bounds $D_K^{1/(2n)} \geq D_k^{1/n} \geq \varrho_n$, and we find that $h_K^- > 1$ if $n \geq 83$ ($2n \geq 166$), and for degrees ≥ 24, we get the lower bounds for $D_K^{1/(2n)}$ as given in Theorem 2.

For degrees $n \leq 23$, we use the first assertion of Theorem 9 combined with the lower bounds of Corollary 13. We obtain

$$h_K^- \geq \frac{1}{2nc_6(2n,\varrho)}\left(\frac{1}{e\pi}\right)^n\left(\frac{\sqrt{\varrho_K}}{\log \varrho_K}\right)^n \frac{\log D_K}{\log\log D_K}.$$

Both the functions $\sqrt{\varrho_K}/\log \varrho_K$ and $\log D_K/(\log\log D_K)$ are increasing for $\varrho_K \geq \varrho$, and we get the lower bounds for $D_K^{1/(2n)}$ as given in Theorem 2.

Also, our bounds can be used to compute a lower bound for the relative class number h_K^-. For example, using the constants for the degree $n = 83$ and the Odlyzko bound $\varrho_K \geq \varrho_n = 54.8874$, we find:

PROPOSITION 22. *Assume the Generalized Riemann Hypothesis. Then the relative class numbers h_K^- of the normal CM-fields K of degree $2n \geq 166$ satisfy*

$$h_K^- \geq \frac{1}{3.2176} \cdot \frac{(1.03937)^n}{\log(8.02n)}.$$

REMARK. If we used Theorem 14 instead of Corollary 13, we would have a slightly easier proof but the resulting bounds would not be as good as the ones we give.

7.1.2. *Non-normal case.* We use (31) and the bounds on residues we obtained in Theorems 8 and 14 (again we have $E_\sigma = 1$) to get

$$h_K^- \geq 2\left(\frac{C_5(n)^{c_4(n)}}{2\pi}\right)^n \frac{D_K^{1/4-c_4(n)/2}}{e^{3/2}(\log D_K)^{1/2}} \exp\left(-\frac{1}{(\log D_K)^{1/2}}\right).$$

For the values of $D_K \geq \varrho^{2n}$ we consider, this lower bound is an increasing function of D_K. Thus we are entitled to use Odlyzko's bounds $D_K^{1/(2n)} \geq D_k^{1/n} \geq \varrho_n$, and we find that $h_K^- > 1$ if $n \geq 88$ ($2n \geq 176$), and for degrees

≥ 25, we get the lower bounds for $D_K^{1/(2n)}$ as given in Theorem 2. This solves conditionally the problem of getting reasonable bounds on degrees of CM-fields with class number one, including those that are not normal.

For degrees $n \leq 24$, we use the first assertion of Theorem 9 combined with the lower bounds of Theorem 14 to obtain

$$h_K^- \geq \frac{1}{2ne^{3/2}}\left(\frac{1}{e\pi}\right)^n\left(\frac{\sqrt{\varrho_K}}{\log \varrho_K}\right)^n \log D_K \exp\left(-\frac{1}{(\log D_K)^{1/2}}\right).$$

We thus get the lower bounds for $D_K^{1/(2n)}$ as given in Theorem 2.

7.2. *Unconditional bounds.* Using Theorems 8 and 9, Lemma 15, Theorem 16 and Theorem 17, we will now obtain the unconditional lower bound for the relative class number of normal CM-fields.

Assume that K is a normal CM-field of degree $2n$, with k its maximal totally real subfield, of degree n. By Lemma 15, $\zeta_K(s)$ has at most two zeros counted with multiplicity in the interval $[1 - 1/(\kappa \log D_K), 1[$, with $\kappa = (2+\sqrt{3})/4 = 0.93301\ldots$ One of three cases must happen:

1. $\zeta_k(s)$ has a zero $\beta \in [1 - 1/(\kappa \log D_K), 1[$. Then $\zeta_K(\beta) = 0$. Using Theorem 16 and Theorem 8 in formula (31), we get

$$h_K^- \geq 2 \cdot \frac{1-\beta}{E_\sigma}\left(\frac{C_5(n)^{c_4(n)}}{2\pi}\right)^n \frac{D_K^{1/4-c_4(n)/2}}{2e^{1/(2\kappa)}}$$

where $E_\sigma = (1-\beta)/(\sigma-\beta)$ was defined in Lemma 6, and σ is the value we computed in Section 3.4. We see that the term $1-\beta$ in the numerator of this lower bound cancels with the term $1-\beta$ in $E_\sigma \leq (1-\beta)/(\sigma-1)$. Thus we get

$$h_K^- \geq \frac{\sigma-1}{e^{1/(2\kappa)}}\left(\frac{C_5(n)^{c_4(n)}}{2\pi}\right)^n D_K^{1/4-c_4(n)/2}. \tag{32}$$

2. $\zeta_k(s)$ has no zero in the range $[1 - 1/(\kappa \log D_K), 1[$, but $\zeta_K(s)$ has a simple zero there. Then by [32, Theorem 3] (see also [10]), there exists an imaginary quadratic subfield F of K for which $\zeta_F(\beta) = 0$. By Theorem 17, we know that

$$1-\beta > \frac{6}{\pi}\cdot\frac{1}{\sqrt{D_F}} \geq \frac{6}{\pi}\cdot\frac{1}{D_K^{1/(2n)}}$$

since $D_F \leq D_K^{1/n}$.

Using Theorem 16 and Theorem 8 in formula (31) (notice that $E_\sigma = 1$ here), we get unconditionally

$$h_K^- \geq \frac{6}{\pi e^{1/(2\kappa)}}\left(\frac{C_5(n)^{c_4(n)}}{2\pi}\right)^n D_K^{1/4-c_4(n)/2-1/(2n)}. \tag{33}$$

3. $\zeta_k(s)$ has no zero in the range $[1 - 1/(\kappa \log D_K), 1[$ and $\zeta_K(s)$ has no simple zero there. In this case, either ζ_K has no zero at all there, or has a

double zero there, and in both cases we have $\zeta_K(1-1/(\kappa\log D_K))\le 0$, so that

$$h_K^- \ge \frac{2}{\kappa e^{1/(2\kappa)}}\left(\frac{C_5(n)^{c_4(n)}}{2\pi}\right)^n \frac{D_K^{1/4-c_4(n)/2}}{\log D_K}. \tag{34}$$

Now that we have taken care of all the three possible cases, we work in two ways.

• If we do not want to obtain lower bounds for the relative class number h_K^- increasing to infinity with the degree $2n$ of K but only want an upper bound for the degrees $2n$ of the CM-fields K of relative class number one, we can immediately eliminate case 2. Indeed, by [25], if F is an imaginary quadratic subfield of a CM-field K with relative class number h_K^- equal to one, then h_F divides 4, thus $h_F = 1$, 2 or 4 and $D_F \le 1555$ (see Section 6.3). But we know that the zeta functions ζ_F of such fields F have no real zeros in the range $0 < s < 1$. Only cases 1 and 3 remain, and it is enough to check that both lower bounds (32) and (34) imply $h_K^- > 1$ for $D_K \ge \varrho^{2n}$ (as in the previous sections, we check that those lower bounds in cases 1 and 3 are increasing functions of D_K for $D_K \ge \varrho^{2n}$). Using L. Tartar's bounds $D_K^{1/(2n)} \ge D_k^{1/n} \ge \varrho'_n$, we find that $h_K^- > 1$ if $n \ge 134$ (that is, $2n \ge 268$), and for degrees $n \ge 27$ we get the lower bounds for $D_K^{1/(2n)}$ as given in Theorem 2.

• If we do want to obtain lower bounds for the relative class number h_K^- increasing to infinity with the degree $2n$ of K (see for example Proposition 23 below), we must check that each of the three lower bounds (32)–(34) implies that $h_K^- > 1$ for $D_K \ge \varrho^{2n}$.

PROPOSITION 23. *If we do not assume the Generalized Riemann Hypothesis, then the relative class numbers h_K^- of the normal CM-fields K of degree $2n \ge 268$ satisfy*

$$h_K^- \ge \frac{(1.0515)^n}{6.057n}.$$

Proof. We use the constants for $n = 134$ and Tartar's bound $\varrho_K \ge \varrho'_n = 44.6377$ (see [28]). For this lower bound of the root discriminant, we see that (34) is the worst of the three lower bounds, and we get the desired result. ■

For degrees $n \le 26$, we use Theorem 9 instead of Theorem 8 and proceed as above. By Lemma 15, $\zeta_K(s)$ has at most two zeros counted with multiplicity in the interval $[1-1/(\kappa\log D_K), 1[$. One of three cases can happen:

1. ζ_k has a zero $\beta \in [1-1/(\kappa\log D_K), 1[$. Then $\zeta_K(\beta)\le 0$. We use the second assertion of Theorem 9, the $1-\beta$ terms cancel each other and we obtain

$$h_K^- \ge \frac{2}{(e\pi)^n e^{1/(2\kappa)}}\left(\frac{\sqrt{\varrho_K}}{\log \varrho_K}\right)^n.$$

2. ζ_k has no zero in the range $[1-1/(\kappa\log D_K),1[$, but ζ_K has a simple zero there. Then by [32, Theorem 3], there exists an imaginary quadratic subfield F of K for which $\zeta_F(\beta)=0$. By Theorem 17, and since h_F divides 4 if $h_K^-=1$, we obtain $D_F\le 1555$, and we know that those imaginary quadratic number fields have no Siegel zeros.

3. ζ_k has no zero in the range $[1-1/(\kappa\log D_K),1[$ and ζ_K has no simple zero there. In this case, whether or not there is a double zero, we obtain

$$h_K^-\ge\frac{1}{2n(e\pi)^n\kappa e^{1/(2\kappa)}}\left(\frac{\sqrt{\varrho_K}}{\log\varrho_K}\right)^n.$$

We check that this last case is the worst numerically and we obtain the lower bounds for $D_K^{1/(2n)}$ as given in Theorem 2 for degrees ≤ 26.

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Appendix. We give here the verbatim proof of Theorem 14 as sent to us by J. Oesterlé.

Soit k un corps de nombres. Notons n son degré, r_1 le nombre de ses places réelles, r_2 le nombre de ses places complexes, d la valeur absolue de son discriminant, ζ_k sa fonction zêta de Dedekind et κ le résidu en 1 de ζ_k.

Théorème 1. *Si ζ_k satisfait l'hypothèse de Riemann et $k\ne\mathbb{Q}$, on a*

$$\kappa\ge\frac{e^{-3/2}}{\sqrt{\log d}}\exp\left(\frac{-1}{\sqrt{\log d}}\right).$$

Posons $\xi_k(s)=s(s-1)d^{s/2}\Gamma_R(s)^{r_1+r_2}\Gamma_R(s+1)^{r_2}\zeta_k(s)$, avec $\Gamma_R(s)=\pi^{-s/2}\Gamma(s/2)$. La fonction ξ_k se prolonge en une fonction entière d'ordre 1, qui satisfait l'équation fonctionnelle $\xi_k(1-s)=\xi_k(s)$. Nous admettons dans la suite que ζ_k satisfait l'hypothèse de Riemann, i.e. que tous les zéros de ξ_k ont pour partie réelle 1/2. La fonction ξ_k possède le développement en produit de Weierstraß

$$\xi_k(s)=\xi_k(0)\prod_{\varrho}{}^{*}\left(1-\frac{s}{\varrho}\right),$$

où le produit est indexé par les zéros ϱ de ξ_k (répétés un nombre de fois égal à leur multiplicité) et où $\prod^*$ signifie que l'on effectue le produit après avoir regroupé les termes correspondant à ϱ et $1-\varrho$ (pour en assurer la convergence normale sur tout compact de $\mathbb{C}$). On en déduit, avec des conventions analogues sur les sommes,

$$\frac{\xi_k'}{\xi_k}(s) = \sum_{\varrho}{}^{*} \frac{1}{s-\varrho}.$$

Posons $\varphi(s) = s\Gamma_R(s)^{r_1+r_2}\Gamma_R(s+1)^{r_2}$, de sorte que

$$\xi_k(s) = (s-1)d^{s/2}\varphi(s)\zeta_k(s).$$

Lemme 1. *La fonction φ'/φ est croissante sur $]0,+\infty[$. La fonction $f : x \mapsto x\,\frac{\varphi'}{\varphi}(x)$ est convexe sur $]0,+\infty[$; elle est majorée par $f(1)$ sur $[1,2]$.*

On a $\varphi(s) = \varphi_0(s)\varphi_1(s)^{r_1+r_2-1}\varphi_2(s)^{r_2}$, où $\varphi_1(s) = \Gamma_R(s)$, $\varphi_2(s) = \Gamma_R(s+1)$, $\varphi_0(s) = s\Gamma_R(s) = 2\pi\Gamma_R(s+2)$. Il suffit de démontrer le lemme séparément pour chacune des fonctions φ_i. La fonction Γ est logarithmiquement convexe sur $]0,+\infty[$. Il en est donc de même de Γ_R, ce qui implique que chacune des fonctions φ_i'/φ_i est croissante sur $]0,+\infty[$.

Posons $f_i(x) = x\,\frac{\varphi_i'}{\varphi_i}(x)$. Comme

$$\frac{\Gamma'}{\Gamma}(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right),$$

on a

$$f_1(x) = -\frac{x}{2}(\log\pi+\gamma) - 1 + \sum_{n=1}^{\infty}\left(\frac{x}{2n} - \frac{x}{x+2n}\right),$$

$$f_2(x) = -\frac{x}{2}(\log\pi+\gamma) - \frac{x}{x+1} + \sum_{n=1}^{\infty}\left(\frac{x}{2n} - \frac{x}{x+2n+1}\right)$$

et $f_0(x) = 1 + f_1(x)$, de sorte que les dérivées secondes

$$f_0''(x) = f_1''(x) = 2\sum_{n=1}^{\infty}\frac{2n}{(x+2n)^3} \quad \text{et} \quad f_2''(x) = 2\sum_{n=0}^{\infty}\frac{2n+1}{(x+2n+1)^3}$$

sont positives sur $]0,+\infty[$ et que les fonctions f_i sont convexes sur $]0,+\infty[$.

Pour démontrer que f_i est majorée par $f_i(1)$ sur $[1,2]$, il suffit donc de vérifier que l'on a $f_i(2) \le f_i(1)$, ce qui résulte des égalités $f_1(1) = -\frac{1}{2}(\log\pi+\gamma) - \log 2$, $f_1(2) = -(\log\pi+\gamma)$, $f_2(1) = -\frac{1}{2}(\log\pi+\gamma)$, $f_1(2) = -(\log\pi+\gamma) + 2 - 2\log 2$, $f_0(1) = f_1(1)+1$ et $f_0(2) = f_1(2)+1$.

Lemme 2. *Soit c un nombre réel > 0. On a*

$$\frac{\xi_k(1+c)}{\xi_k(1)} \ge \frac{c}{\kappa}\, d^{c/2}\exp\left(c\,\frac{\varphi'}{\varphi}(1)\right).$$

On a $\zeta_k(1+c) \ge 1$ et φ'/φ est croissante sur $]0,+\infty[$ d'après le lemme 1, d'où

$$\frac{\xi_k(1+c)}{\xi_k(1)} = \frac{cd^{c/2}\varphi(1+c)\zeta_k(1+c)}{\kappa\varphi(1)} \ge \frac{cd^{c/2}\varphi(1+c)}{\kappa\varphi(1)} \ge \frac{c}{\kappa}\, d^{c/2}\exp\left(c\,\frac{\varphi'}{\varphi}(1)\right).$$

LEMME 3. *Soit c un nombre réel > 0. On a*

$$\frac{\xi_k(1+c)}{\xi_k(1)} \le d^{(c+c^2)/2} \exp\left(1 + c + c(1+c)\,\frac{\varphi'}{\varphi}(1+c)\right).$$

On a pour $0 \le t \le c$ et pour tout zéro ϱ de ξ_k (de partie réelle $1/2$ par hypothèse),

$$\begin{aligned}\operatorname{Re}\left(\frac{1}{1+t-\varrho}\right) &\le \frac{1+2c}{1+2t}\operatorname{Re}\left(\frac{1}{1+c-\varrho}\right)\\ &\le (1+2(c-t))\operatorname{Re}\left(\frac{1}{1+c-\varrho}\right),\end{aligned}$$

d'où, en sommant sur ϱ,

$$\frac{\xi_k'}{\xi_k}(1+t) \le (1+2(c-t))\,\frac{\xi_k'}{\xi_k}(1+c),$$

et en intégrant

$$\frac{\xi_k(1+c)}{\xi_k(1)} = \exp\left(\int_0^c \frac{\xi_k'}{\xi_k}(1+t)\,dt\right) \le \exp\left((c+c^2)\frac{\xi_k'}{\xi_k}(1+c)\right).$$

On a

$$\frac{\xi_k'}{\xi_k}(1+c) = \frac{1}{c} + \frac{1}{2}\,\log d + \frac{\varphi'}{\varphi}(1+c) + \frac{\zeta_k'}{\zeta_k}(1+c)$$

et

$$\frac{\zeta_k'}{\zeta_k}(1+c) = -\sum_{\mathfrak{p}} \frac{(\mathrm{N}(\mathfrak{p}))^{-(1+c)}}{1-(\mathrm{N}(\mathfrak{p}))^{-(1+c)}}\,\log \mathrm{N}(\mathfrak{p}) \le 0$$

(où $\mathfrak{p}$ parcourt l'ensemble des idéaux maximaux de l'anneau des entiers de K et $\mathrm{N}(\mathfrak{p})$ désigne la norme de $\mathfrak{p}$). Le lemme 3 résulte de ces trois dernières relations.

LEMME 4. *Pour tout nombre réel $c \in\,]0,1]$, on a*

$$\kappa \ge c\exp\left(-1 - c - \frac{c^2}{2}\log d\right).$$

En combinant les lemmes 2 et 3, on obtient

$$\kappa \ge c d^{-c^2/2}\exp\left(-1 - c + c\,\frac{\varphi'}{\varphi}(1) - c(1+c)\,\frac{\varphi'}{\varphi}(1+c)\right).$$

Par ailleurs, si l'on pose $f(x) = x\,\frac{\varphi'}{\varphi}(x)$, on a $f(1) \ge f(1+c)$ d'après le lemme 1, c'est-à-dire $\frac{\varphi'}{\varphi}(1) - (1+c)\frac{\varphi'}{\varphi}(1+c) \ge 0$.

Le théorème 1 se déduit du lemme 4 en prenant $c = 1/\sqrt{\log d}$, ce qui est légitime si $\log d \ge 1$, i.e. $k \ne \mathbb{Q}$.

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