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Explicit lower bounds for $\|(3/2)^k\|$

by

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1. Introduction. Let $\{x\}$ (resp. $[x]$, $\|x\|$) denote the fractional part (resp. the integer part, the distance to the nearest integer) of a real number x. It has been proved that the function $g(k)$ occurring in Waring's problem is given by the formula

$$g(k) = 2^k + [(3/2)^k] - 2$$

if the following inequality holds:

$$\{(3/2)^k\} \le 1 - (3/4)^k. \tag{1.1}$$

Moreover Mahler [7] showed that (1.1) is valid for k large enough. However his proof is ineffective and does not provide a bound from which (1.1) is satisfied. In 1990, Kubina and Wunderlich [6] checked (1.1) for $k \le 471600000$.

In 1981, Beukers [2] proved that, for $k \ge 5000$,

$$\|(3/2)^k\| > 2^{-0.9k} = (0.53588\ldots)^k. \tag{1.2}$$

This result was asymptotically improved by Dubickas [4] who showed that

$$\|(3/2)^k\| > (0.5769)^k \tag{1.3}$$

for k large enough. However he did not compute the range of validity of (1.3). We refine Dubickas's computations to prove the following theorem.

THEOREM 1. *For k large enough, we have*

$$\|(3/2)^k\| > (0.5770173776\ldots)^k. \tag{1.4}$$

We also improve on Beukers's result (1.2) by showing the following inequality.

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THEOREM 2. *For $k \geq 5$, we have*

$$\|(3/2)^k\| > 2^{-0.8k} = (0.57434\ldots)^k. \tag{1.5}$$

Our proof proceeds as those of Beukers and Dubickas. We describe diagonal Padé approximants of the function $H(a,b;t)$, the polynomial part of $(1-t)^{a+b}t^{-b}$. A precise study of the asymptotic and arithmetic behavior of these approximants leads to (1.4) and to (1.5) for $k \geq 64440000$. The range $[5, 64440000]$ is checked by using Delmer and Deshouillers's technique [3].

All the computations were performed using the system PARI.

2. Padé approximations. Let a, b be fixed nonnegative integers. Beukers [2] introduced the function

$$H(a,b;t) = t^{-b}\biggl((1-t)^{a+b} - \sum_{r=0}^{b-1}\binom{a+b}{r}(-t)^r\biggr)$$

and determined diagonal Padé approximants for this function. More precisely, he showed that, for any nonnegative integer n,

$$P_n(t) - Q_n(t)H(a,b;t) = (-1)^{n+b}t^{2n+1}E_n(t), \tag{2.1}$$

where P_n is a polynomial of degree at most n with integer coefficients, and where

$$Q_n(t) = \sum_{r=0}^{n}\binom{2n+b-r}{n+b}\binom{a-n+r-1}{r}t^r \tag{2.2}$$

$$= \frac{(a+b+n)!}{(a-n-1)!(b+n)!n!}\int_0^1 x^{a-n-1}(1-x)^{n+b}(1-x+tx)^n\,dx, \tag{2.3}$$

$$E_n(t) = \frac{(a+b+n)!}{(a-n-1)!(b+n)!n!}\int_0^1 x^n(1-x)^{n+b}(1-tx)^{a-n-1}\,dx. \tag{2.4}$$

Moreover he proved that these approximants are distinct by establishing the following relation:

$$P_n(t)Q_{n+1}(t) - P_{n+1}(t)Q_n(t) = (-1)^{n+b}\binom{a+b+n}{2n+b+1}\binom{2n+b+2}{n+b+1}t^{2n+1}. \tag{2.5}$$

We now restrict our attention to the case $(a,b) = (2m,m)$, where m is a fixed positive integer. The key point of Beukers's proof was to exhibit nontrivial divisors of the content of the polynomials P_n and Q_n. Dubickas got his improvement by refining this part of the proof. Let us show an equivalent form of Dubickas's lemma. Let $\mathcal{P}$ denote the set of all prime numbers.

LEMMA 1. *Define*

$$E_n(m)=\Big\{l\in\mathcal{P},\ l^2>\max(n+m,2m-n-1):\\ \Big\{\frac{n+m}{l}\Big\}+\Big\{\frac{2m-n-1}{l}\Big\}+\Big\{\frac{n}{l}\Big\}\ge 2\Big\}.$$

Then, for any element l *from* $E_n(m)$, *we have* $\{P_n(t),Q_n(t)\}\subset l\mathbb{Z}[t]$.

Proof. Let l be in $\mathcal{P}$, with $l^2>\max(n+m,2m-n-1)$. We first consider the content of $Q_n(t)$. Let r be an integer from $\{0,\dots,n\}$. By (2.2), we want to show that l divides $\binom{2n+m-r}{n+m}\binom{2m-n+r-1}{r}$. Put

$$\eta_1=\Big\{\frac{n+m}{l}\Big\},\quad \eta_2=\Big\{\frac{2m-n-1}{l}\Big\},\quad \eta_3=\Big\{\frac{n}{l}\Big\},\quad \theta=\Big\{\frac{j}{l}\Big\},$$

and let ω_l denote the l-adic valuation of $\binom{2n+m-r}{n+m}\binom{2m-n+r-1}{r}$. The size of l gives the following expressions for ω_l:

$$\begin{aligned}\omega_l&=\Big[\frac{2n+m-r}{l}\Big]-\Big[\frac{n+m}{l}\Big]-\Big[\frac{n-r}{l}\Big]\\&\quad+\Big[\frac{2m-n+r-1}{l}\Big]-\Big[\frac{2m-n-1}{l}\Big]-\Big[\frac{r}{l}\Big]\\&=[\eta_1+\eta_3-\theta]-[\eta_1]-[\eta_3-\theta]+[\eta_2+\theta]-[\eta_2]-[\theta]\\&=[\eta_1+\eta_3-\theta]-[\eta_3-\theta]+[\eta_2+\theta],\end{aligned}$$

which lead to the estimate

$$\omega_l\ge[\eta_1+\eta_3-\theta]+[\eta_2+\theta]\ge[\eta_1+\eta_2+\eta_3]-1.$$

When l belongs to $E_n(m)$, we know that $\eta_1+\eta_2+\eta_3$ is greater than or equal to 2, which implies that ω_l is positive. Therefore l divides the content of Q_n. Since the supports of $P_n(t)$ and $t^{2n+1}E_n(t)$ are disjoint, this also shows that l divides the content of P_n, by (2.1). ∎

The form given to this lemma was inspired by Hata's work on irrationality measures [5]. It makes it easier to compute the asymptotic behavior of the product of the elements of $E_n(m)$, as shown in the next section.

3. Asymptotic behavior. Consider $n=[\alpha(m-3/2)]+1+\eta$ with (m,α,η) belonging to the set $(\mathbb{N}\setminus\{0,1\})\times\,]0,2[\,\times\{0,1\}$. Put $\Pi_m(\alpha)=\prod_{l\in E_n(m)}l$.

Let $\delta\in\{0,1,2,3,4,5\}$ and M be an integer. By (2.5), we may choose η such that

$$P_n\Big(-\frac{1}{8}\Big)-\frac{M}{2^\delta}Q_n\Big(-\frac{1}{8}\Big)\ne 0. \tag{3.1}$$

Indeed, if not, the couple $(1, M/2^\delta)$ would be a solution of a homogeneous system of rank 2, which is impossible. Moreover the polynomial $\Pi_m(\alpha)^{-1} \times (2^\delta P_n - MQ_n)$ has integer coefficients and its degree is at most n. We thus deduce from (3.1) the estimate

$$\left|P_n\left(-\frac{1}{8}\right) - \frac{M}{2^\delta} Q_n\left(-\frac{1}{8}\right)\right| \geq 2^{-\delta-3n}\Pi_m(\alpha). \tag{3.2}$$

Let us now study what happens when m goes to infinity. Define

$$\begin{aligned} F_1(\alpha) &= \max_{x\in[0,1]} x^{2-\alpha}(1-x)^{1+\alpha}\left|1-\frac{9}{8}x\right|^{\alpha}, \\ F_2(\alpha) &= \max_{x\in[0,1]} x^{\alpha}(1-x)^{1+\alpha}\left|1+\frac{x}{8}\right|^{2-\alpha}, \\ A(\alpha) &= (\alpha+3)\log(\alpha+3) - (2-\alpha)\log(2-\alpha) \\ &\quad - (1+\alpha)\log(1+\alpha) - \alpha\log\alpha. \end{aligned}$$

PROPOSITION 1. *We have the upper bounds*

$$\left|Q_n\left(-\frac{1}{8}\right)\right| \leq \frac{(3m+n)!}{(2m-n-1)!(m+n)!n!} \cdot \frac{2F_1(\alpha)^{m-3/2}}{5}, \tag{3.3}$$

$$\left|E_n\left(-\frac{1}{8}\right)\right| \leq \frac{(3m+n)!}{(2m-n-1)!(m+n)!n!} \cdot \frac{541F_2(\alpha)^{m-3/2}}{1260}, \tag{3.4}$$

$$\log\left(\frac{(3m+n)!}{(2m-n-1)!(m+n)!n!}\right) \leq A(\alpha)m + O(1). \tag{3.5}$$

Moreover we can get a better estimate for $\alpha = 15/16$:

$$\log\left(\frac{(3m+n)!}{(2m-n-1)!(m+n)!n!}\right) \leq A\left(\frac{15}{16}\right)m - \log(2\pi) - \frac{1}{12} + \frac{1}{m}. \tag{3.6}$$

Proof. Use (2.3) and the inequalities

$$n \geq \alpha(m-3/2), \tag{3.7}$$

$$n+m \geq (1+\alpha)(m-3/2)+3/2, \tag{3.8}$$

$$2m-n-1 \geq (2-\alpha)(m-3/2), \tag{3.9}$$

to get

$$\left|Q_n\left(-\frac{1}{8}\right)\right| \leq \frac{(3m+n)!}{(2m-n-1)!(m+n)!n!} F_1(\alpha)^{m-3/2}\int_0^1 (1-x)^{3/2}\,dx,$$

which shows (3.3). Similarly, application of (2.4) together with the inequality

$2m - n - 1 \le (2-\alpha)(m - 3/2) + 2$ yields

$$\left|E_n\left(-\frac{1}{8}\right)\right| \le \frac{(3m+n)!}{(2m-n-1)!(m+n)!n!} \times F_2(\alpha)^{m-3/2}\int_0^1 (1-x)^{3/2}(1+x/8)^2\,dx,$$

and (3.4) follows.

We shall now need the following Stirling formula (cf. [8, p. 37]):

$$\log \Gamma(s) = (s-1/2)\log s - s + \log\sqrt{2\pi} + \frac{1}{2}\int_0^\infty \frac{\{x\}-\{x\}^2}{(x+s)^2}\,dx.$$

This way we get

$$\log\left(\frac{(3m+n)!}{(2m-n-1)!(m+n)!n!}\right) = \Delta + 1 - \log(2\pi) + I,$$

where $\Delta = \phi(3m+n+1) - \phi(2m-n) - \phi(n+m+1) - \phi(n+1)$, $\phi(s) = (s-1/2)\log s$ and

$$I = \int_0^\infty \frac{\{x\}-\{x\}^2}{2}\left(\frac{1}{(3m+n+1+x)^2} - \frac{1}{(2m-n+x)^2} - \frac{1}{(n+m+1+x)^2} - \frac{1}{(n+1+x)^2}\right)dx \le 0.$$

We now use the formula

$$\Delta = \left(n+m+\frac{1}{2}\right)\log\left(\frac{3m+n+1}{m+n+1}\right) + \left(2m-n-\frac{1}{2}\right)\log\left(\frac{3m+n+1}{2m-n}\right) + \left(n+\frac{1}{2}\right)\log\left(\frac{3m+n+1}{n+1}\right)$$

to complete the proof of (3.5).

Assume that $\alpha \le 1$. This implies that $-1/2 \le 1-(3/2)\alpha \le n+1-\alpha m \le 3-(3/2)\alpha$. By applying Taylor's formula to the function ϕ, we get

$$\begin{aligned}\Delta \le\ & \phi((3+\alpha)m) + (n+1-\alpha m)\phi'((3+\alpha)m)\\ &+ \frac{(n+1-\alpha m)^2}{2}\phi''((3+\alpha)m)\\ &- \phi((2-\alpha)m) + (n+1-\alpha m)\phi'((2-\alpha)m) - \phi'((2-\alpha)m)\\ &- \phi((1+\alpha)m) - (n+1-\alpha m)\phi'((1+\alpha)m)\\ &- \phi(\alpha m) - (n+1-\alpha m)\phi'(\alpha m)\end{aligned}$$

$$= A(\alpha)m + \frac{1}{2}\log\left(\frac{\alpha(1+\alpha)}{(2-\alpha)(3+\alpha)}\right) - 1 + \frac{1}{2(2-\alpha)m}$$
$$+ (n+1-\alpha m)\log\left(\frac{\alpha(1+\alpha)}{(2-\alpha)(3+\alpha)}\right)$$
$$- \frac{n+1-\alpha m}{2m}\left(\frac{1}{3+\alpha} + \frac{1}{2-\alpha} - \frac{1}{1+\alpha} - \frac{1}{\alpha}\right)$$
$$+ \frac{(n+1-\alpha m)^2}{2}\left(\frac{1}{(3+\alpha)m} + \frac{1}{2(3+\alpha)^2m^2}\right)$$
$$\le A(\alpha)m - 1 - \frac{3}{2}(1-\alpha)\log\left(\frac{(2-\alpha)(3+\alpha)}{\alpha(1+\alpha)}\right) + \frac{3-(3/2)\alpha}{2m}$$
$$\times\left(\frac{1+2\alpha}{\alpha(1+\alpha)} + \frac{2-(3/2)\alpha}{3+\alpha} + \frac{1/4}{(3+\alpha)^2} - \frac{2-(3/2)\alpha}{(2-\alpha)(3-(3/2)\alpha)}\right)$$
$$\le A(\alpha)m - 1 - \frac{3}{2}(1-\alpha)\log\left(\frac{(2-\alpha)(3+\alpha)}{\alpha(1+\alpha)}\right) + \frac{505}{512m}.$$

For $\alpha = 15/16$, we get (3.6). ■

We still have to determine the asymptotic behavior of $\Pi_m(\alpha)$. Put

$$E_\alpha = \{x > 0 : \{(1+\alpha)x\} + \{(2-\alpha)x\} + \{\alpha x\} \ge 2\} \quad \text{and} \quad I(\alpha) = \int_{E_\alpha} \frac{dx}{x^2}.$$

Note that, when $\alpha = u/v$ is a rational, the function $x \to \{(1+\alpha)x\} + \{(2-\alpha)x\} + \{\alpha x\}$ is v-periodic and the set E_α may be written as

$$E_\alpha = \bigcup_{1\le i\le j_\alpha} ([a_i, b_i[+ v\mathbb{N})$$

with $0 < a_1 < b_1 < a_2 < \ldots < b_{j_\alpha} \le v$. Moreover the functions $x \mapsto \{(1+\alpha)x\}$, $x \mapsto \{(2-\alpha)x\}$ and $x \mapsto \{\alpha x\}$ are constant on any of the intervals $[a_i, b_i[$ (otherwise there will be a jump by 1 and there would exist a point x_0 such that $\{(1+\alpha)x_0\} + \{(2-\alpha)x_0\} + \{\alpha x_0\} < 2$). This in turn implies that the fractional part is a nondecreasing function on any of the intervals $[(1+\alpha)(a_i+vq), (1+\alpha)(b_i+vq)[$, $[(2-\alpha)(a_i+vq), (2-\alpha)(b_i+vq)[$ and $[\alpha(a_i+vq), \alpha(b_i+vq)[$.

Proposition 2. *When m goes to infinity, we have*

$$\log \Pi_m(\alpha) \ge I(\alpha)m + O\left(\frac{m}{\log m}\right). \tag{3.10}$$

Moreover, for $m \ge 10740000$, the following inequality holds:

$$\log \Pi_m(15/16) \ge 0.3945m + 9. \tag{3.11}$$

Proof. There exist absolute constants $C_1, C_2 > 0$ such that

$$-C_1 \le \max(n - \alpha m, -n - 1 + \alpha m) \le C_2.$$

Put $C_3 = C_2 \max(1/\alpha, 1/(2-\alpha))$, $C_4 = C_1 \max(1/\alpha, 1/(2-\alpha))$. Assume $m \ge C_3 + C_4$ and introduce

$$q_0(m) = \min\left(\frac{a_1 + b_1}{v}\left(\frac{m}{C_3 + C_4} - 1\right), \frac{m + C_3}{(v+1)\sqrt{3m}}\right) = O(\sqrt{m}).$$

Let us prove that, for $0 \le q \le q_0(m)$ and $1 \le i \le j_\alpha$, any prime number from the interval $\left]\frac{m+C_3}{b_i+vq}, \frac{m-C_4}{a_i+vq}\right]$ belongs to $E_n(m)$. The definition of $q_0(m)$ implies the inequality $\frac{m+C_3}{b_i+vq} \le \frac{m-C_4}{a_i+vq}$ and shows that

$$\frac{m + C_3}{b_i + vq} \ge \sqrt{3m} \ge \sqrt{\max(n+m, 2m-n-1)}.$$

Thus any prime number from the interval $\left]\frac{m+C_3}{b_i+vq}, \frac{m-C_4}{a_i+vq}\right]$ satisfies the condition $l^2 > \max(n+m, 2m-n-1)$. Moreover we have the following inequalities:

$$\begin{aligned}
\frac{m+n}{l} &\ge \frac{(1+\alpha)m - C_1}{l} \ge (1+\alpha)(a_i + vq) + \frac{C_4(1+\alpha) - C_1}{l} \\
&\ge (1+\alpha)(a_i + vq), \\
\frac{m+n}{l} &\le \frac{(1+\alpha)m + C_2}{l} < (1+\alpha)(b_i + vq) - \frac{C_3(1+\alpha) - C_2}{l} \\
&\le (1+\alpha)(b_i + vq), \\
\frac{2m-n-1}{l} &\ge \frac{(2-\alpha)m - C_1}{l} \ge (2-\alpha)(a_i + vq) + \frac{C_4(2-\alpha) - C_1}{l} \\
&\ge (2-\alpha)(a_i + vq), \\
\frac{2m-n-1}{l} &\le \frac{(2-\alpha)m + C_2}{l} < (2-\alpha)(b_i + vq) - \frac{C_3(2-\alpha) - C_2}{l} \\
&\le (2-\alpha)(b_i + vq), \\
\frac{n}{l} &\ge \frac{\alpha m - C_1}{l} \ge \alpha(a_i + vq) + \frac{C_4\alpha - C_1}{l} \ge \alpha(a_i + vq), \\
\frac{n}{l} &\le \frac{\alpha m + C_2}{l} < \alpha(b_i + vq) + \frac{C_3\alpha - C_2}{l} \le \alpha(b_i + vq),
\end{aligned}$$

which lead to

$$\begin{aligned}
&\left\{\frac{m+n}{l}\right\} + \left\{\frac{2m-n-1}{l}\right\} + \left\{\frac{n}{l}\right\} \\
&\qquad \ge \{(1+\alpha)(a_i + vq)\} + \{(2-\alpha)(a_i + vq)\} + \{\alpha(a_i + vq)\} \ge 2.
\end{aligned}$$

Therefore we get the inclusion

$$E_n(m) \supseteq \bigcup_{0 \le q \le q_0(m)} \bigcup_{1 \le i \le j_\alpha} \left(\left]\frac{m + C_3}{b_i + vq}, \frac{m - C_4}{a_i + vq}\right] \cap \mathcal{P}\right).$$

This implies the estimate

$$\log \mathit{\Pi}_m(\alpha) \ge \sum_{0\le q\le q_0(m)} \sum_{1\le i\le j_\alpha} \left(\Theta\left(\frac{m-C_4}{a_i+vq}\right) - \Theta\left(\frac{m+C_3}{b_i+vq}\right)\right), \tag{3.12}$$

where $\Theta(x) = \sum_{p\in\mathcal{P},\, p\le x} \log p$. We now use Schoenfeld's estimate for the function $\Theta(x)$ [9, Theorem 8*]: $|\Theta(x) - x| \le 8.072x/\log^2 x$ for $x > 1$, to get

$$\Theta\left(\frac{m-C_4}{a_i+vq}\right) - \Theta\left(\frac{m+C_3}{b_i+vq}\right) \ge \frac{m-C_4}{a_i+vq} - \frac{m+C_3}{b_i+vq} - \frac{m-C_4}{a_i+vq}\,\frac{8.072}{(\log\sqrt{3m})^2} - \frac{m+C_3}{b_i+vq}\,\frac{8.072}{(\log\sqrt{3m})^2},$$

for $q \le q_0(m)$. We deduce from (3.12) the lower bound

$$\begin{aligned} \log \mathit{\Pi}_m(\alpha) &\ge m(1+O(1/m)) \int_{E_\alpha\cap[0,(1+v)q_0(m)]} \frac{dx}{x^2} + O\left(\frac{m}{\log^2 m} \sum_{1\le q\le q_0(m)} \frac{1}{q}\right) \\ &= I(\alpha)m + O(\sqrt{m}) + O(m/\log m) \end{aligned} \tag{3.13}$$

and the first part of the proposition is proved.

For $\alpha = 15/16$, we have $C_3 = 19/30$ and $C_4 = 17/10$. The a_i's and b_i's are given below:

i	(a_i, b_i)	i	(a_i, b_i)	i	(a_i, b_i)
1	$(32/63, 16/31)$	2	$(16/21, 16/17)$	3	$(64/63, 32/31)$
4	$(32/21, 48/31)$	5	$(16/9, 32/17)$	6	$(128/63, 64/31)$
7	$(160/63, 80/31)$	8	$(176/63, 48/17)$	9	$(64/21, 96/31)$
10	$(32/9, 112/31)$	11	$(256/63, 128/31)$	12	$(32/7, 144/31)$
13	$(320/63, 160/31)$	14	$(352/63, 96/17)$	15	$(128/21, 192/31)$
16	$(400/63, 32/5)$	17	$(64/9, 224/31)$	18	$(464/63, 112/15)$
19	$(512/63, 256/31)$	20	$(176/21, 144/17)$	21	$(64/7, 288/31)$
22	$(592/63, 160/17)$	23	$(640/63, 320/31)$	24	$(704/63, 192/17)$
25	$(736/63, 176/15)$	26	$(256/21, 208/17)$	27	$(800/63, 64/5)$
28	$(96/7, 208/15)$	29	$(928/63, 224/15)$	30	$(992/63, 16)$

To prove the second part of the proposition, we shall need the bound

$$\log \mathit{\Pi}_m(\alpha) \ge \sum_{0\le q\le 10} \sum_{1\le i\le 30} \left(\Theta\left(\frac{m-17/10}{a_i+16q}\right) - \Theta\left(\frac{m+19/30}{b_i+16q}\right)\right). \tag{3.14}$$

For $m > 5\cdot 10^{10}$, we use the following estimates from [9]:

$$-0.0077629\,\frac{x}{\log x} < \Theta(x) - x < 0.000081x \quad \text{for } x \ge 1.04\cdot 10^7.$$

We find $\log \mathit{\Pi}_m(\alpha) \ge 0.40127m - 32 > 0.3945m + 9$.

For $5 \cdot 10^{10} \geq m > 5 \cdot 10^7$, we use the additional estimates from [9]: $0.998697x < \Theta(x) < x$ for $1155901 \leq x < 10^{11}$. We find $\log \Pi_m(\alpha) \geq 0.39572m - 27 > 0.3945m + 9$.

For $5 \cdot 10^7 \geq m > 1.074 \cdot 10^7$, we use other estimates from [1]:

$$\frac{\Theta(x) - x}{\sqrt{x}} \begin{cases} < -0.344 & \text{if } 0 < x < 10^8, \\ > -1.833 & \text{if } 19801 < x < 10^8, \end{cases}$$

together with Theorem 6* and Corollary 2 of [9], which give pairs (c, d) such that $\Theta(x) > x - x/(c \log x)$ for $x \geq d$. We find $\log \Pi_m(\alpha) \geq 0.39454m - 26 > 0.3945m + 9$. ■

4. Proof of Theorems 1 and 2. We shall use the notations from the previous section.

PROPOSITION 3. *For any positive number* ε *and for any integer* $k > k_0(\varepsilon)$ ($k_0(\varepsilon)$ *effective*), *we have*

$$\|(3/2)^k\| \geq e^{(C_1(\alpha)-\varepsilon)k} - e^{(C_2(\alpha)-\varepsilon)k}, \tag{4.1}$$

where

$$C_1(\alpha) = (-3\alpha \log 2 + I(\alpha) - A(\alpha) - \log F_1(\alpha))/6,$$
$$C_2(\alpha) = (-6\alpha \log 2 + \log F_2(\alpha) - \log F_1(\alpha))/6.$$

Proof. Take $k = 6m - \delta$ with $\delta \in \{0, 1, 2, 3, 4, 5\}$ and choose the integer M_0 for which the distance from $(3/2)^k$ to $\mathbb{Z}$ is attained. Then we have

$$\begin{aligned}\|(3/2)^k\| &= (3/2)^{-\delta}((3/2)^{6m} - (3/2)^{\delta} M_0) \\ &= (2/3)^{\delta}(-1)^m (H(2m, m; -1/8) - M2^{-\delta})\end{aligned}$$

for some integer M, by the definition of $H(a, b; t)$. By (2.1) we know that

$$\begin{aligned}&H(2m, m; -1/8) - M2^{-\delta} \\ &\quad = \frac{P_n(-1/8) - M2^{-\delta} Q_n(-1/8)}{Q_n(-1/8)} + (-1)^{m+n} 2^{-3(2n+1)} \frac{E_n(-1/8)}{Q_n(-1/8)}.\end{aligned}$$

We use (3.2) to get the inequality

$$\|(3/2)^k\| \geq \frac{2^{-3n} \Pi_m(\alpha) - 2^{\delta - 3(2n+1)} |E_n(-1/8)|}{3^{\delta} |Q_n(-1/8)|}. \tag{4.2}$$

The estimates (3.3)–(3.5) and (3.7) then complete the proof of (4.1). ■

In order to get the best lower bound for $\|(3/2)^k\|$, we have to find for which value of α the first exponent in (4.1) is maximal, under the condition $C_1(\alpha) > C_2(\alpha)$. It appears that the difference between C_1 and C_2 is negative for low values of values of α; moreover, once this difference becomes positive, the value of $C_1(\alpha)$ decreases. Therefore we are looking for good upper bounds for the solution α_0 of $C_1(\alpha) = C_2(\alpha)$. The computations show that α_0 is smaller than 1, and more precisely that α_0 belongs to the range $[0.9, 0.95]$.

Computing $(C_1 - C_2)(1 - 1/p)$ for $p = 10, \ldots, 20$ gives the better estimate $\alpha_0 \in [13/14, 14/15]$. We can get more precise estimates for α_0 by determining the continued fraction expansion of α_0. We find this way

$$\frac{198478}{212871} = [0,1,13,1,3,1,3,6,3,1,2,1,7] < \alpha_0 < [0,1,13,1,3,1,3,6,3,1,2,1,8] = \frac{224141}{240395}.$$

Since PARI gives

$$(C_1 - C_2)(224141/240395) = 1.0057378 \cdot 10^{-11}$$

and $e^{C_1(224141/240395)} = 0.57701737767006\ldots$, the proof of Theorem 1 is complete. Note that Dubickas's result was obtained by choosing $\alpha = 1/1.0723 = 0.93257483\ldots$, which was pretty close to our better choice $\alpha = 224141/240395 = 0.93238628\ldots$ To prove Theorem 2, we shall give an explicit version of Proposition 3 for $\alpha = 15/16$.

PARI gives the numerical values

$$F_1(15/16) = 0.0964204654\ldots,$$
$$(F_2/F_1)(15/16) = 1.7628240038\ldots,$$
$$A(15/16) = 4.1111565348\ldots$$

From (3.3), (3.4) and (3.6) we deduce

$$\left|Q_n\left(-\frac{1}{8}\right)\right|^{-1} \geq \exp(-1.7721197321m + 0.6711),$$
$$\left|E_n\left(-\frac{1}{8}\right)\right| \leq \exp(2.3390368029m - 0.1084).$$

Since $(15/16)m - 45/32 \leq n \leq (15/16)m + 19/32$, from (4.2) and (3.8) we get

$$\begin{aligned}\|(3/2)^k\| &\geq \exp(-3.327097m + 8.43 - 1.1\delta) \\ &\quad - \exp(-3.332035m + 4.34 - 0.4\delta) \\ &\geq 2^{-0.8k}(\exp(0.17) - \exp(-0.005m)) \geq 2^{-0.8k}\end{aligned}$$

for $m \geq 10740000$. Therefore (1.5) is proved for $k \geq 64440000$. For $k < 64440000$, we shall use the following lemma, inspired by Delmer and Deshouillers [3].

LEMMA 2. *For a positive integer n, let $l(n)$ denote the maximal number of identical consecutive digits in the binary expansion of n. Then, if $l(3^p) \leq 0.8p - 2$, we have*

$$\|(3/2)^k\| \geq 2^{-0.8k} \quad \textit{for} \quad \frac{\frac{\log 3}{\log 2}p + l(3^p) + 2}{\frac{\log 3}{\log 2} + 0.8} \leq k \leq p.$$

Proof. Follow exactly the proof of [3, Proposition 1]. ■

Define now the finite sequence $(k_0, \ldots, k_r)$ by the initial value $k_0 = 64440000$ and the recursion relation

$$k_{i+1} = \left\lceil \frac{\frac{\log 3}{\log 2} k_i + l(3^{k_i}) + 2}{\frac{\log 3}{\log 2} + 0.8} \right\rceil \quad \text{if} \quad l(3^{k_i}) < 0.8k_i - 2.$$

This sequence is decreasing and terminates when the condition $l(3^{k_i}) < 0.8k_i - 2$ is not satisfied. PARI gives $r = 41$ and $k_r = 11$. Since formula (1.5) is true for $k = 5, \ldots, 11$ and $k \in \{k_r + 1, \ldots, k_0\}$ by Lemma 2, the proof of Theorem 2 is complete.

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