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Rademacher–Carlitz polynomials

by

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1. Introduction. While studying the transformation properties of what we now call the *Dedekind η-function* $\eta(z) := e^{\pi i z/12} \prod_{n\ge 1} \left(1 - e^{2\pi i n z}\right)$ under $\mathrm{SL}_2(\mathbb{Z})$, Richard Dedekind, in the 1880s [10], naturally arrived at what we today call the *Dedekind sum*

$$\mathrm{s}(a,b) := \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right),$$

where a and b are positive integers and

$$((x)) := \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

The Dedekind sum and its generalizations have since intrigued mathematicians from various areas such as analytic (see, e.g., [1, 3]) and algebraic number theory (see, e.g., [9, 17, 22]), topology (see, e.g., [13, 15]), algebraic (see, e.g., [7, 12, 19]) and combinatorial geometry (see, e.g., [6, 16]), and algorithmic complexity (see, e.g., [14]).

Almost a century after the appearance of Dedekind sums, Leonard Carlitz introduced a polynomial analogue, the *Dedekind–Carlitz polynomial*

$$\mathrm{c}(u,v,a,b) := \sum_{k=1}^{b-1} u^{\lfloor ka/b \rfloor} v^{k-1}.$$

Here u and v are indeterminates and a and b are positive integers. Undoubtedly the most important basic property of any Dedekind-like sum is *reciprocity*. For the Dedekind–Carlitz polynomials, it says that if a and b are relatively prime then [8]

$$(v-1)\,\mathrm{c}(u,v,a,b) + (u-1)\,\mathrm{c}(v,u,b,a) = u^{a-1}v^{b-1} - 1. \tag{1}$$

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Carlitz's reciprocity theorem generalizes that of Dedekind [10], which states that for relatively prime positive integers a and b,

$$\mathrm{s}(a,b)+\mathrm{s}(b,a)=-\frac{1}{4}+\frac{1}{12}\left(\frac{a}{b}+\frac{1}{ab}+\frac{b}{a}\right). \tag{2}$$

Dedekind reciprocity follows from (1) by applying the operators $u\partial u$ twice and $v\partial v$ once to Carlitz's reciprocity identity. As a historical aside, we note that (2) is equivalent to quadratic reciprocity (see, e.g., [21]).

Dedekind sums have many generalizations. One of the earliest will play a central role in this paper: for $a,b\in\mathbb{Z}_{>0}$, and $t\in\mathbb{R}$, we define the *Dedekind–Rademacher sum* [20]

$$\mathrm{r}_t(a,b):=\sum_{k=0}^{b-1}\left(\left(\frac{ka+t}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right). \tag{3}$$

Our goal is to introduce and study an analogue of this sum in the world of polynomials: for $a,b\in\mathbb{Z}_{>0}$, $s,t\in\mathbb{R}$, and variables u and v, we define the *Rademacher–Carlitz polynomial*

$$\mathrm{R}(u,v,s,t,a,b):=\sum_{k=\lceil s\rceil}^{\lceil s\rceil+b-1}u^{\lfloor ka+t/b\rfloor}v^k.$$

Naturally, Dedekind–Carlitz polynomials are special cases of Rademacher–Carlitz polynomials, in the sense that $v\ \mathrm{c}(u,v,a,b)=\mathrm{R}(u,v,0,0,a,b)-1$. It will be handy to abbreviate the linear function $(ax+t)/b=:f(x)$ which appears in the exponent of u, and so we will typically use the notation

$$\mathrm{R}(u,v,s,f):=\sum_{k=\lceil s\rceil}^{\lceil s\rceil+b-1}u^{\lfloor f(k)\rfloor}v^k$$

with the understanding that b equals the denominator in the linear function f.

Our motivation to study Rademacher–Carlitz polynomials is twofold: first, they seem natural generalizations of Dedekind–Carlitz polynomials and, as we will see below, they give rise not only to new reciprocity theorems but also to new results on old constructs, such as Dedekind–Rademacher sums. Our second motivation stems from the fact that Rademacher–Carlitz polynomials appear naturally—as we will also show below—in the *integer-point transforms*

$$\sigma_{\mathcal{P}}(x,y):=\sum_{(m,n)\in\mathcal{P}\cap\mathbb{Z}^2}x^my^n$$

of 2-dimensional rational polyhedra $\mathcal{P}$, in particular, 2-dimensional cones/polygons with rational vertices. In fact, our paper extends some of the methods introduced in [4], which showed that Dedekind–Carlitz polynomials are natural ingredients for 2-dimensional *lattice* polyhedra, i.e., those with integral vertices. Carlitz's reciprocity theorem (1) was a natural by-product of the geometric approach of [4], and our first result, which mirrors the geometric setup of [4], is a reciprocity theorem for Rademacher–Carlitz polynomials.

Theorem 1. *Let $f(x) := (ax+t)/b$ be a linear function with relatively prime $a, b \in \mathbb{Z}_{>0}$, $t \in \mathbb{R}$, and let $(p,q) \in \mathbb{R}^2$ be a point on the graph of f. Then*

$$v(1-u)\,\mathrm{R}(v,u,p,f) + u(1-v)\,\mathrm{R}(u,v,q,f^{-1}) = u^{\lceil p\rceil} v^{\lceil q\rceil}(1-u^b v^a) - u^c v^d (1-u)(1-v),$$

where $(c,d) \in \mathbb{Z}^2$ is the unique lattice point on the half-open line segment $[(p,q),(p+b,q+a))$; if there are no integer points on the graph of f (and so (c,d) does not exist), the last term on the right-hand side should be omitted.

We give a proof in Section 2, where we will also show how (1) follows as a corollary. One can phrase the conditions in Theorem 1 in purely number-theoretic terms as follows.

Corollary 2. *Let $a, b \in \mathbb{Z}_{>0}$ be relatively prime and $p, q \in \mathbb{R}$. Then*

$$v(1-u)\,\mathrm{R}(v,u,p,bq-ap,a,b) + u(1-v)\,\mathrm{R}(u,v,q,ap-bq,b,a) = u^{\lceil p\rceil} v^{\lceil q\rceil}(1-u^b v^a) - u^c v^d (1-u)(1-v),$$

where $c \in \mathbb{Z}$ is (uniquely) determined by the conditions

$$ac \equiv ap - bq \bmod b \quad \text{and} \quad p \le c < p+b,$$

and $d := (ac+bq-ap)/b$. If $ap-bq \notin \mathbb{Z}$ then the last term on the right-hand side should be omitted.

Returning to our second motivation, we remark that the evaluation $\sigma_{\mathcal{P}}(1,1)$ of an integer-point transform yields the number of integer lattice points in $\mathcal{P}$. Ehrhart [11] famously proved in the 1960s that the counting function

$$\mathrm{ehr}_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$$

is a polynomial in the positive integer variable t when $\mathcal{P}$ is a lattice polytope, and a quasipolynomial when $\mathcal{P}$ is a rational polytope (see, e.g., [6] for more on Ehrhart quasipolynomials). It is a natural question how to compute Ehrhart (quasi)polynomials and integer-point transforms, both in the computational complexity aspect and in terms of ingredients for possible formulas. We will only briefly touch on the computational aspect, which is governed by Barvinok's theorem [2]. The ingredients of degree-2 Ehrhart polynomials are easy; they essentially follow from Pick's theorem [18] (of which Ehrhart's theorem can be viewed as a far-reaching generalization).

The classification question for degree-2 Ehrhart *quasi*polynomials, i.e., stemming from rational polygons, was answered much more recently [5]; here Dedekind–Rademacher sums play a crucial role as the only nontrivial ingredients. The analogous classification question for integer-point transforms of lattice polygons was answered in [4], and Dedekind–Carlitz polynomials played there the role of the nontrivial ingredients. Our next result provides formulas for the integer-point transforms of *rational* polygons; it can be viewed as a common generalization (and combination) of the classification results in [4] and [5], and indeed, from this point of view, it should come as no surprise that Rademacher–Carlitz polynomials make an appearance.

Theorem 3. *Let* $a, b, c, d, e, f, g, h \in \mathbb{Z}_{>0}$, *and let* Δ *denote the triangle with vertices* $(e/f, g/h)$, $(a/b, g/h)$ *and* $(e/f, c/d)$. *Moreover, we define* $\alpha := dh(be - af)$, $\beta := bf(ch - dg)$, *and* $l(x) := (\beta/\alpha)x + c/d - e\alpha/(f\beta)$. *Then the integer-point transform of* Δ *equals*

$$\sigma_\Delta(x, y) = \frac{x^{\lceil e/f \rceil} y^{\lceil g/h \rceil}}{(1-x)(1-y)} + \frac{\mathrm{R}(x, y, g/h, l^{-1})}{(1-x^{-1})(1-x^\alpha y^\beta)} + \frac{\mathrm{R}(y, x, e/f, l)}{(1-y^{-1})(1-x^{-\alpha} y^{-\beta})}.$$

We give a proof in Section 3. Theorem 3 suffices to provide formulas for the integer-point transform of *any* rational polygon: we can triangulate a given rational polygon, hence we only have to treat the case of rational triangles and rational line segments, whose integer-point transforms are relatively straightforward to compute. Using a simple geometric argument (see Section 3), we can reduce the case of rational triangles to rational *right* triangles with edges parallel to the x- and y-axis, which are the content of Theorem 3.

Our final result is a pleasant by-product of the geometric treatment of Dedekind-like sums; it turns out that we obtain the following reciprocity theorem for Dedekind–Rademacher sums.

Corollary 4. *Let* a *and* b *be relatively prime positive integers with* $a < b$, *and let* $t \in \mathbb{R}$ *with* $0 \le t < b$. *Then*

$$\mathrm{r}_{-t}(a, b) + \mathrm{r}_t(b, a) = \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) - \frac{1}{4} + \frac{1}{2ab}\lfloor t \rfloor(\lfloor t \rfloor + 1) - \frac{1}{2}\left\lfloor \frac{t}{a} \right\rfloor - \frac{\chi}{2}\left(\left(\left(\frac{a^{-1}t}{b}\right)\right) + \left(\left(\frac{b^{-1}t}{a}\right)\right)\right),$$

where χ *equals* 1 *or* 0 *depending on whether or not* $t \in \mathbb{Z}$, $aa^{-1} \equiv 1 \bmod b$, *and* $bb^{-1} \equiv 1 \bmod a$.

Note that the conditions on a, b, and t do not constitute a restriction for practical purposes, as

$$r_t(a, b) = r_{t \bmod b}(a \bmod b, b).$$

At any rate, our proof of Corollary 4, which we give in Section 4, contains reformulations without the conditions $a < b$ and $0 \le t < b$.

Dedekind's reciprocity theorem (2) follows naturally from Corollary 4 by setting $t = 0$. However, a more interesting comparison is with Rademacher's reciprocity theorem, which he stated as follows [20]: For $a, b \in \mathbb{Z}$ and $x, y \in \mathbb{R}$, let

$$\mathrm{s}(a, b; x, y) := \sum_{k=0}^{b-1} \left(\left(\frac{(k+y)a}{b} + x \right)\right) \left(\left(\frac{k+y}{b} \right)\right). \tag{4}$$

Then, if a and b are relatively prime and x and y are not both integers,

$$\begin{aligned} &\mathrm{s}(a, b; x, y) + \mathrm{s}(b, a; y, x) \\ &\qquad = ((x))((y)) + \frac{1}{2}\left(\frac{a}{b} B_2(y) + \frac{1}{ab} B_2(ay + bx) + \frac{b}{a} B_2(x) \right), \end{aligned}$$

where $B_2(x) := \{x\}^2 - \{x\} + 1/6$ is the periodized second Bernoulli polynomial. A moment's thought reveals that any sum of the form (3) can be expressed in the form (4) and vice versa. Indeed, setting $y = 0$ and $x = t/b$ gives

$$\begin{aligned} \mathrm{s}\left(a, b; \frac{t}{b}, 0\right) &= \sum_{k=0}^{b-1} \left(\left(\frac{ka+t}{b} \right)\right) \left(\left(\frac{k}{b} \right)\right), \\ \mathrm{s}\left(b, a; 0, \frac{t}{b}\right) &= \sum_{k=0}^{a-1} \left(\left(\frac{kb+t}{a} \right)\right) \left(\left(\frac{k + \frac{t}{b}}{b} \right)\right). \end{aligned}$$

The latter sum equals $\sum_{k=0}^{a-1} \left(\left(\frac{kb+t}{a}\right)\right)\left(\left(\frac{k}{b}\right)\right)$ plus some trivial terms. So Rademacher's reciprocity theorem expressed in terms of $\mathrm{r}_t(a, b)$ says that

$$\mathrm{r}_t(a, b) + \mathrm{r}_t(b, a)$$

equals a simple expression in terms of a, b, and t. It is, on the other hand, not hard to see that $r_{-t}(a, b) = r_t(a, b)$, and so Corollary 4 is equivalent to Rademacher reciprocity. As we will briefly outline below, our geometric approach to Theorems 1 and 3 could, in principle, easily generalize to higher-dimensional settings; the analogue of Corollary 4 will then be a Dedekind–Rademacher sum with more than two factors.

2. The reciprocity theorem for Rademacher–Carlitz polynomials

Proof of Theorem 1. As mentioned in the introduction, we follow the ideas of [4] which gave a novel geometric proof of (1). Let $f(x) := (ax + t)/b$

with $a, b \in \mathbb{Z}_{>0}$, where $\gcd(a, b) = 1$, and $t \in \mathbb{R}$, and let $(p, q) \in \mathbb{R}^2$ be a point on the graph of f. Consider the half-open cones

$$\mathcal{K}_1 := \{(p, q) + \lambda_1(1, 0) + \lambda_2(b, a) : \lambda_1 > 0,\ \lambda_2 \geq 0\},$$
$$\mathcal{K}_2 := \{(p, q) + \lambda_1(0, 1) + \lambda_2(b, a) : \lambda_1 > 0,\ \lambda_2 \geq 0\}$$

and the ray

$$\mathcal{K}_3 := \{(p, q) + \lambda(b, a) : \lambda \geq 0\}.$$

These three objects give a disjoint conic decomposition of the shifted first

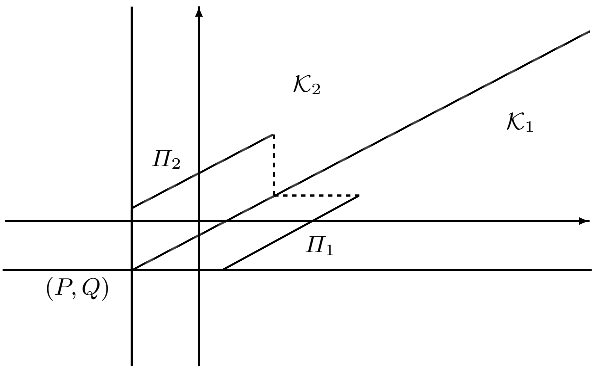

Fig. 1. The shifted first quadrant split into two pointed cones

quadrant, shown in Figure 1:

$$\{(p, q) + \lambda_1(1, 0) + \lambda_2(0, 1) : \lambda_1, \lambda_2 \geq 0\} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3, \tag{5}$$

and our goal is to compute the integer-point transforms on both sides.

For the shifted first quadrant, this integer-point transform is

$$\frac{u^{\lceil p \rceil} v^{\lceil q \rceil}}{(1 - u)(1 - v)}.$$

By a simple tiling argument (see, for example, [6, Chapter 3]), the integer-point transform $\sigma_{\mathcal{K}_1}(u, v)$ of the half-open cone $\mathcal{K}_1$ is

$$\sigma_{\mathcal{K}_1}(u, v) = \frac{\sigma_{\mathit{\Pi}_1}(u, v)}{(1 - u)\,(1 - u^b v^a)}$$

where

$$\mathit{\Pi}_1 := \{(p, q) + \lambda_1(1, 0) + \lambda_2(b, a) : 0 < \lambda_1 \leq 1,\ 0 \leq \lambda_2 < 1\},$$

the *fundamental parallelogram* of $\mathcal{K}_1$. Since it has width 1, there is exactly one integer point in $\mathit{\Pi}_1$ for each y running from $\lceil q \rceil$ to $\lceil q \rceil + a - 1$. The

x-coordinate of this integer point is $\lfloor f^{-1}(y) \rfloor + 1$. Thus

$$\sigma_{\Pi_1}(u,v) = \sum_{k=\lceil q \rceil}^{\lceil q \rceil + a - 1} u^{\lfloor f^{-1}(k) \rfloor + 1} v^k = u\,\mathrm{R}(u,v,q,f^{-1}).$$

With a similar argument, changing the roles of the axes, we obtain our second integer-point transform:

$$\sigma_{\mathcal{K}_2}(u,v) = \frac{\sigma_{\Pi_2}(u,v)}{(1-v)(1-u^b v^a)}$$

where

$$\sigma_{\Pi_2}(u,v) = \sum_{k=\lceil p \rceil}^{\lceil p \rceil + b - 1} u^k v^{\lfloor f(k) \rfloor + 1} = v\,\mathrm{R}(v,u,p,f).$$

It remains to compute the integer-point transform of the ray $\mathcal{K}_3$. It is clear that any two lattice points on $\mathcal{K}_3$ differ by a multiple of (b,a) and so

$$\sigma_{\mathcal{K}_3}(u,v) = \frac{u^c v^d}{1-u^b v^a}$$

where (c,d) is the lattice point on $\mathcal{K}_3$ with the smallest coordinates, if there is a lattice point on $\mathcal{K}_3$ at all—otherwise $\sigma_{\mathcal{K}_3}(u,v)$ will simply not appear in our formulas.

Thus (5) translates into the identity of rational generating functions

$$\frac{u^{\lceil p \rceil} v^{\lceil q \rceil}}{(1-u)(1-v)} = \frac{u\,\mathrm{R}(u,v,q,f^{-1})}{(1-u)(1-u^b v^a)} + \frac{v\,\mathrm{R}(v,u,p,f)}{(1-v)(1-u^b v^a)} + \frac{u^c v^d}{1-u^b v^a},$$

where the last term only appears if $\mathcal{K}_3$ contains lattice points. Clearing denominators gives Theorem 1. ∎

We remark that, in principle, we could derive an analogue of Theorem 1 in higher dimension, similarly to the proof of [4, Theorem 3], involving an arbitrary number of variables.

Carlitz's reciprocity theorem (1) follows as an immediate corollary by choosing $t = p = q = 0$: note that then $c = d = 0$, and so Theorem 1 gives in this special case

$$v(1-u)\,\mathrm{R}(v,u,0,f) + u(1-v)\,\mathrm{R}(u,v,0,f^{-1}) = 1 - u^b v^a - (1-u)(1-v).$$

We rewrite the expression on the left to see Dedekind–Carlitz polynomials appear:

$$\begin{aligned} &v(1-u)(\mathrm{R}(v,u,0,f)-1) + u(1-v)(\mathrm{R}(u,v,0,f^{-1})-1) \\ &\quad = 1 - u^b v^a - (1-u)(1-v) - v(1-u) - u(1-v) = -u^b v^a + uv. \end{aligned}$$

Dividing by $-uv$ gives (1).

We finish this section with a remark about computational complexity. In the introduction we hinted at Barvinok's theorem [2], which says that in fixed dimensions, the integer-point transform $\sigma_{\mathcal{P}}(x_1, \ldots, x_d)$ of a rational polyhedron $\mathcal{P}$ can be computed as a sum of short rational functions in $x_1, \ldots, x_d$ in time polynomial in the input size of $\mathcal{P}$. Thus (say) $\sigma_{\Pi_2}(u, v)$ can be computed efficiently, which means we can compute Rademacher–Carlitz sums efficiently. (This is a nontrivial statement, since Rademacher–Carlitz sums have exponentially many terms when measured in the input size of their parameters.)

3. Integer-point transforms of rational polygons. In this section, we give the details behind our claim that Theorem 3 suffices to characterize the integer-point transform of any rational polygon, and we will prove Theorem 3.

As mentioned in the introduction, any rational polygon can be triangulated, and so we can compute its integer-point transform in an inclusion-exclusion fashion from integer-point transforms of rational line segments and rational triangles. Furthermore, we can embed an arbitrary triangle in a rectangle in such a way that we can express the triangle as a set union/subtraction of rectangles and right triangles with edges parallel to the x- and y-axis, as suggested by Figure 2; if the triangle was rational to begin with, so will be the rectangles and right triangles.

The integer-point transforms of rectangles are easy, and thus it remains to compute integer-point transforms of right triangles with edges parallel to the x- and y-axis, which (by a harmless lattice transformation) we may assume to be in the first quadrant with its right angle in the southwestern vertex. That is, it remains to prove Theorem 3.

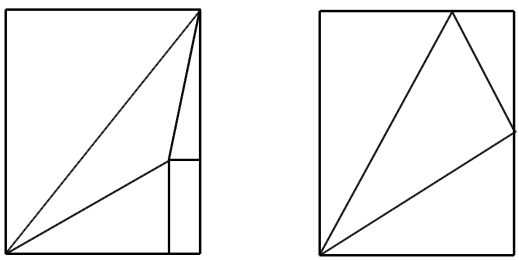

Fig. 2. Triangles embedded in a rectangle and right triangles

Proof of Theorem 3. As stated in the conditions, we assume that Δ looks like in Figure 2. To compute the integer-point transform of Δ, we use Brion's theorem [7], which says that $\sigma_{\Delta}(x, y)$ equals the sum of the integer-point transforms of the three vertex cones of Δ. (The *vertex cone* of a polytope

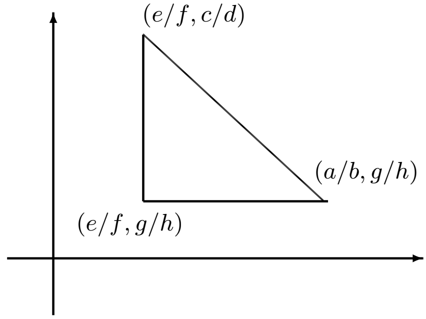

Fig. 3. The rational right triangle from Theorem 3

$\mathcal{P}$ at a vertex $\boldsymbol{v}$ is the smallest cone with apex $\boldsymbol{v}$ that contains $\mathcal{P}$.) Thus we need to compute the integer-point transforms of the vertex cones

$$V_1 := \{(e/f, g/h) + \lambda_1(1,0) + \lambda_2(0,1) : \lambda_1, \lambda_2 \geq 0\},$$
$$V_2 := \{(a/b, g/h) + \lambda_1(-1,0) + \lambda_2\big(dh(be-af), bf(ch-dg)\big) : \lambda_1, \lambda_2 \geq 0\},$$
$$V_3 := \{(e/f, c/d) + \lambda_1(0,-1) + \lambda_2\big(-dh(be-af), -bf(ch-dg)\big) : \lambda_1, \lambda_2 \geq 0\}.$$

To shorten notation, we define, as in the statement of Theorem 3, $\alpha := dh(be-af)$ and $\beta := bf(ch-dg)$. The integer-point transform of V_1 is straightforward:

$$\sigma_{V_1}(x,y) = \sum_{k \geq \lceil e/f \rceil,\ j \geq \lceil g/h \rceil} x^k y^j = \frac{x^{\lceil e/f \rceil} y^{\lceil g/h \rceil}}{(1-x)(1-y)}. \tag{6}$$

For the other two vertex cones, we use a tiling argument similar to the one in the proof of Theorem 1. This gives

$$\sigma_{V_2}(x,y) = \frac{\sigma_{\Pi_2}(x,y)}{(1-x^{-1})(1-x^{\alpha}y^{\beta})}, \tag{7}$$

$$\sigma_{V_3}(x,y) = \frac{\sigma_{\Pi_3}(x,y)}{(1-y^{-1})(1-x^{-\alpha}y^{-\beta})}, \tag{8}$$

where

$$\Pi_2 := \{(a/b, g/h) + \lambda_1(-1,0) + \lambda_2(\alpha,\beta) : 0 \leq \lambda_1, \lambda_2 < 1\},$$
$$\Pi_3 := \{(e/f, c/d) + \lambda_1(0,-1) + \lambda_2(-\alpha,-\beta) : 0 \leq \lambda_1, \lambda_2 < 1\}$$

are the fundamental parallelograms of V_2 and V_3, respectively. To compute the integer-point transform of Π_2, we note that the linear function $l(x) := (\beta/\alpha)x + c/d - e\alpha/(f\beta)$ given in the statement of Theorem 3 describes the line that contains the hypotenuse of Δ. Since Π_2 has height 1 and is half-open, for every integral y-coordinate between $\lceil g/h \rceil$ and $\lceil g/h \rceil + \beta - 1$ there is exactly one point with integral x-coordinate, namely $\lfloor l^{-1}(y) \rfloor$, and so

$$\sigma_{\Pi_2}(x,y) = \sum_{k=\lceil g/h\rceil}^{\lceil g/h\rceil+\beta-1} x^{\lfloor l^{-1}(k)\rfloor} y^k = \mathrm{R}\Big(x,y,\frac{g}{h},l^{-1}\Big). \tag{9}$$

A parallel argument yields

$$\sigma_{\Pi_3}(x,y) = \sum_{k=\lceil e/f\rceil}^{\lceil e/f\rceil+\alpha-1} x^k y^{\lfloor l(k)\rfloor} = \mathrm{R}\Big(y,x,\frac{e}{f},l\Big). \tag{10}$$

Brion's theorem says

$$\sigma_\Delta(x,y) = \sigma_{V_1}(x,y) + \sigma_{V_2}(x,y) + \sigma_{V_3}(x,y),$$

which, using (6)–(10), yields Theorem 3. ■

4. The reciprocity theorem for Dedekind–Rademacher sums. Our goal in this section is to prove Corollary 4. We will need a few identities that are slightly technical but straightforward. For $x \in \mathbb{R}$ and $m \in \mathbb{Z}_{>0}$, we denote by $[x]_m$ the smallest nonnegative real number congruent to x mod m.

LEMMA 5. *Let a and b be positive relatively prime integers, and let $t \in \mathbb{R}$. Then*

(a) $\displaystyle\sum_{k=0}^{b-1} \left\{\frac{ak+t}{b}\right\} = \frac{b-1}{2} + \{t\},$

(b) $\displaystyle\sum_{k=0}^{b-1} k\left\{\frac{ak+t}{b}\right\} = b\,\mathrm{r}_t(a,b) + \tfrac14 b(b-1) + \tfrac12 b\{t\} - \tfrac12 [t]_b + \tfrac12 \chi\, b\bigg(\bigg(\frac{ta^{-1}}{b}\bigg)\bigg)$

where χ equals 1 or 0 depending on whether or not t is an integer.

Proof. (a) is essentially *Raabe's formula* (see, e.g., [21, Lemma 1]).

(b) We compute

$$\begin{aligned}\frac1b \sum_{k=0}^{b-1} k\left\{\frac{ak+t}{b}\right\} &= \sum_{k=1}^{b-1}\left\{\frac{k}{b}\right\}\left\{\frac{ak+t}{b}\right\}\\ &= \sum_{k=1}^{b-1}\bigg(\bigg(\frac{k}{b}\bigg)\bigg)\bigg(\bigg(\frac{ak+t}{b}\bigg)\bigg) + \frac12\sum_{k=1}^{b-1}\left\{\frac{ak+t}{b}\right\} + \frac12\sum_{k=1}^{b-1}\left\{\frac{k}{b}\right\}\\ &\quad - \frac{b-1}{4} + \frac{\chi}{2}\bigg(\bigg(\frac{ta^{-1}}{b}\bigg)\bigg).\end{aligned}$$

The last correction term comes from a case-by-case analysis of $\big(\big(\frac{ak+t}{b}\big)\big)$: the argument is an integer if and only if t is an integer congruent to $-ak$ for some integer k between 1 and $b-1$. With part (a) and the definition of the

Dedekind–Rademacher sum, this becomes

$$\frac{1}{b}\sum_{k=0}^{b-1} k\left\{\frac{ak+t}{b}\right\} = \mathrm{r}_t(a,b) + \frac{b-1}{4} + \frac{1}{2}\{t\} - \frac{1}{2}\left\{\frac{t}{b}\right\} + \frac{\chi}{2}\left(\left(\frac{ta^{-1}}{b}\right)\right).$$

With $b\{x/b\} = [x]_b$, this gives (b). ∎

Proof of Corollary 4. We start by applying the operators $u\partial u$ twice and $v\partial v$ once to the identity in Theorem 1, which yields

$$\begin{aligned}(11)\quad 2\sum_{k=\lceil p\rceil}^{\lceil p\rceil+b-1} k\left\lfloor\frac{ak+t}{b}\right\rfloor + 2\sum_{k=\lceil p\rceil}^{\lceil p\rceil+b-1} k + \sum_{k=\lceil p\rceil}^{\lceil p\rceil+b-1}\left\lfloor\frac{ak+t}{b}\right\rfloor + b \\ + \sum_{k=\lceil q\rceil}^{\lceil q\rceil+a-1}\left\lfloor\frac{bk-t}{a}\right\rfloor^2 + 2\sum_{k=\lceil q\rceil}^{\lceil q\rceil+a-1}\left\lfloor\frac{bk-t}{a}\right\rfloor + a \\ = (\lceil p\rceil + 2b)a\lceil p\rceil + (2\lceil p\rceil + b)b\lceil q\rceil + ab^2 + \chi(2c+1).\end{aligned}$$

Here χ equals 1 or 0 depending on whether or not there are integer points on the graph of $f(x) = (ax+t)/b$; since a and b are relatively prime, there will be integer points if and only if $t \in \mathbb{Z}$, and thus χ has the same meaning as in Lemma 5. Recall also from the statement of Theorem 1 that c is the x-coordinate of the unique lattice point on the half-open line segment $[(p,q),(p+b,q+a))$. Thus $c \in \mathbb{Z}$ is uniquely determined by the conditions

$$c \equiv a^{-1}(ap-bq) \bmod b \quad\text{and}\quad p \le c < p+b,$$

where $aa^{-1} \equiv 1 \bmod b$.

There are four nontrivial sums in (11), which we will uncover now one by one, with the help of Lemma 5. First,

$$\begin{aligned}\sum_{k=\lceil p\rceil}^{\lceil p\rceil+b-1} k\left\lfloor\frac{ak+t}{b}\right\rfloor &= \sum_{k=\lceil p\rceil}^{\lceil p\rceil+b-1} k\,\frac{ak+t}{b} - \sum_{k=\lceil p\rceil}^{\lceil p\rceil+b-1} k\left\{\frac{ak+t}{b}\right\} \\ &= \frac{1}{3}ab^2 + ab\lceil p\rceil + a\lceil p\rceil^2 - \frac{1}{2}ab - a\lceil p\rceil + \frac{1}{2}bt + \lceil p\rceil t + \frac{1}{6}a - \frac{1}{2}t \\ &\quad - \sum_{k=0}^{b-1}(k+\lceil p\rceil)\left\{\frac{a(k+\lceil p\rceil)+t}{b}\right\} \\ &= \frac{1}{3}ab^2 + ab\lceil p\rceil + a\lceil p\rceil^2 - \frac{1}{2}ab - a\lceil p\rceil + \frac{1}{2}b\lfloor t\rfloor + \lceil p\rceil\lfloor t\rfloor + \frac{1}{6}a - \frac{1}{2}t \\ &\quad + \frac{1}{2}[a\lceil p\rceil + t]_b - \frac{1}{4a}b^2 - \frac{1}{2}b\lceil p\rceil + \frac{1}{4}b + \frac{1}{2}\lceil p\rceil - b\,\mathrm{r}_{a\lceil p\rceil+t}(a,b) \\ &\quad - \frac{1}{2}\chi\, b\left(\left(\frac{\lceil p\rceil + ta^{-1}}{b}\right)\right),\end{aligned}$$

where again $aa^{-1} \equiv 1 \bmod b$. (Note that $a\lceil p\rceil + t \in \mathbb{Z}$ if and only if $t \in \mathbb{Z}$.) Next,

$$\begin{aligned}\sum_{k=\lceil p\rceil}^{\lceil p\rceil+b-1} \left\lfloor \frac{ak+t}{b} \right\rfloor &= \sum_{k=\lceil p\rceil}^{\lceil p\rceil+b-1} \frac{ak+t}{b} - \sum_{k=\lceil p\rceil}^{\lceil p\rceil+b-1} \left\{ \frac{ak+t}{b} \right\} \\ &= \frac{1}{2}a(b-1) + a\lceil p\rceil + t - \sum_{k=0}^{b-1} \left\{ \frac{k+t}{b} \right\} \\ &= \frac{1}{2}(a-1)(b-1) + a\lceil p\rceil + \lfloor t\rfloor .\end{aligned}$$

Analogously,

$$\sum_{k=\lceil q\rceil}^{\lceil q\rceil+a-1} \left\lfloor \frac{bk-t}{a} \right\rfloor = \frac{1}{2}(a-1)(b-1) + b\lceil q\rceil + \lfloor -t\rfloor .$$

Finally,

$$\begin{aligned}&\sum_{k=\lceil q\rceil}^{\lceil q\rceil+a-1} \left\lfloor \frac{bk-t}{a} \right\rfloor^2 \\ &= \sum_{k=\lceil q\rceil}^{\lceil q\rceil+a-1} \left(\frac{bk-t}{a}\right)^2 - 2\sum_{k=\lceil q\rceil}^{\lceil q\rceil+a-1} \frac{bk-t}{a}\left\{\frac{bk-t}{a}\right\} + \sum_{k=\lceil q\rceil}^{\lceil q\rceil+a-1} \left\{\frac{bk-t}{a}\right\}^2 \\ &= \frac{1}{3}ab^2 + b^2\lceil q\rceil - \frac{1}{2}b^2 - bt + \frac{1}{a}\left(b^2\lceil q\rceil^2 - b^2\lceil q\rceil - 2b\lceil q\rceil t + \frac{1}{6}b^2 + bt + t^2\right) \\ &\quad - \frac{2b}{a}\sum_{k=\lceil q\rceil}^{\lceil q\rceil+a-1} k\left\{\frac{bk-t}{a}\right\} + \frac{2t}{a}\sum_{k=\lceil q\rceil}^{\lceil q\rceil+a-1} \left\{\frac{bk-t}{a}\right\} + \sum_{k=0}^{a-1}\left\{\frac{k+\{-t\}}{a}\right\}^2 \\ &= \frac{1}{3}ab^2 + b^2\lceil q\rceil - \frac{1}{2}b^2 - bt + \frac{1}{a}\left(b^2\lceil q\rceil^2 - b^2\lceil q\rceil - 2b\lceil q\rceil t + \frac{1}{6}b^2 + bt + t^2\right) \\ &\quad - \frac{2b}{a}\sum_{k=0}^{a-1}(k+\lceil q\rceil)\left\{\frac{b(k+\lceil q\rceil)-t}{a}\right\} + \frac{2t}{a}\sum_{k=0}^{a-1}\left\{\frac{k-t}{a}\right\} + \sum_{k=0}^{a-1}\left(\frac{k+\{-t\}}{a}\right)^2 \\ &= \frac{1}{3}ab^2 - \frac{1}{2}ab + \frac{1}{3}a + b^2\lceil q\rceil - bt - \frac{1}{2}b^2 + \frac{1}{2}b - \frac{1}{2} - \lfloor -t\rfloor - b\lceil q\rceil \\ &\quad + \frac{1}{a}\left(b^2\lceil q\rceil^2 - b^2\lceil q\rceil + 2b\lceil q\rceil\lfloor -t\rfloor + \frac{1}{6}b^2 + bt + \frac{1}{6} + \lfloor -t\rfloor^2 + \lfloor -t\rfloor + b\lceil q\rceil\right) \\ &\quad - 2b\,\mathrm{r}_{b\lceil q\rceil - t}(b,a) - \frac{b}{a}[-at]_a + \frac{b}{a}[b\lceil q\rceil - t]_a - \chi\, b\left(\left(\frac{q-tb^{-1}}{a}\right)\right),\end{aligned}$$

where $bb^{-1} \equiv 1 \bmod a$. (Note that $b\lceil q\rceil - t \in \mathbb{Z}$ if and only if $t \in \mathbb{Z}$.)

We are all set to substitute the expressions we found back into (11). Simplifying terms such as $\{t\}+\{-t\}$ (which equals 1 if $t \notin \mathbb{Z}$ and 0 if $t \in \mathbb{Z}$) and $[x]_a/a = \{x/a\}$ gives

$$\begin{aligned}
&\mathrm{r}_{a\lceil p\rceil+t}(a,b) + \mathrm{r}_{b\lceil q\rceil-t}(b,a) \\
&\quad= \frac{a\lceil p\rceil^2}{2b} - \frac{a\lceil p\rceil}{2b} + \frac{b\lceil q\rceil^2}{2a} - \frac{b\lceil q\rceil}{2a} + \frac{b}{12a} + \frac{a}{12b} + \frac{1}{12ab} + \frac{\lceil q\rceil\lfloor -t\rfloor}{a} + \frac{\lceil q\rceil}{2a} \\
&\qquad + \frac{\lceil p\rceil\lfloor t\rfloor}{b} + \frac{\lceil p\rceil}{2b} + \frac{t}{2a} - \frac{t}{2b} - \lceil p\rceil\lceil q\rceil + \frac{\lceil p\rceil}{2} + \frac{\lceil q\rceil}{2} - \frac{3}{4} + \frac{\lfloor -t\rfloor^2}{2ab} + \frac{\lfloor -t\rfloor}{2ab} \\
&\qquad + \frac{1}{2}\left\{\frac{a\lceil p\rceil+t}{b}\right\} + \frac{1}{2}\left\{\frac{b\lceil q\rceil-t}{a}\right\} \\
&\qquad + \chi\left(-\frac{1}{2}\left(\left(\frac{a^{-1}(a\lceil p\rceil+t)}{b}\right)\right) - \frac{1}{2}\left(\left(\frac{b^{-1}(b\lceil q\rceil-t)}{a}\right)\right) + \frac{1}{2} - \frac{c}{b}\right).
\end{aligned}$$

Now we use the relation $bq = ap + t$, which simplifies the left-hand side to

$$\mathrm{r}_{a\{-p\}-b\{-q\}}(a,b) + \mathrm{r}_{b\{-q\}-a\{-p\}}(b,a).$$

But this means we might as well choose p and q in some interval of length 1; it is easiest to assume $-1 < p, q \le 0$, since this will simplify the right-hand side most easily:

$$\begin{aligned}
&\mathrm{r}_{bq-ap}(a,b) + \mathrm{r}_{ap-bq}(b,a) \\
&\quad= \frac{a}{12b} + \frac{b}{12a} + \frac{1}{12ab} - \frac{3}{4} + \frac{\lfloor ap-bq\rfloor^2}{2ab} + \frac{\lfloor ap-bq\rfloor}{2ab} \\
&\qquad - \frac{1}{2}\left\lfloor\frac{ap-bq}{a}\right\rfloor - \frac{1}{2}\left\lfloor\frac{bq-ap}{b}\right\rfloor \\
&\qquad + \chi\left(\frac{1}{2} - \frac{c}{b} - \frac{1}{2}\left(\left(\frac{a^{-1}(bq-ap)}{b}\right)\right) - \frac{1}{2}\left(\left(\frac{b^{-1}(ap-bq)}{a}\right)\right)\right).
\end{aligned}$$

Recall that c is the unique integer satisfying

$$c \equiv a^{-1}(ap-bq) \bmod b \quad \text{and} \quad p \le c < p+b.$$

Since $-1 < p \le 0$, this condition simply says that c is the smallest nonnegative integer congruent to $a^{-1}(ap-bq)$ modulo b, that is,

$$c = b\left\{\frac{a^{-1}(ap-bq)}{b}\right\} = -b\left(\left(\frac{a^{-1}(bq-ap)}{b}\right)\right) + (1-\mu)\,\frac{b}{2},$$

where $\mu = 1$ if $b \mid bq - ap$ and $\mu = 0$ otherwise. This yields

$$\begin{aligned}
&\mathrm{r}_{bq-ap}(a,b) + \mathrm{r}_{ap-bq}(b,a) \\
&\quad= \frac{a}{12b} + \frac{b}{12a} + \frac{1}{12ab} - \frac{3}{4} + \frac{\lfloor ap-bq\rfloor^2}{2ab} + \frac{\lfloor ap-bq\rfloor}{2ab} \\
&\qquad - \frac{1}{2}\left\lfloor\frac{ap-bq}{a}\right\rfloor - \frac{1}{2}\left\lfloor\frac{bq-ap}{b}\right\rfloor \\
&\qquad + \chi\left(\frac{\mu}{2} + \frac{1}{2}\left(\left(\frac{a^{-1}(bq-ap)}{b}\right)\right) - \frac{1}{2}\left(\left(\frac{b^{-1}(ap-bq)}{a}\right)\right)\right).
\end{aligned}$$

We now set $q = 0$ and assume that $a < b$, for which the above identity simplifies to

$$\begin{aligned}
&\mathrm{r}_{bq}(a,b) + \mathrm{r}_{-bq}(b,a) \\
&\quad = \frac{a}{12b} + \frac{b}{12a} + \frac{1}{12ab} - \frac{3}{4} + \frac{\lfloor -bq \rfloor^2}{2ab} + \frac{\lfloor -bq \rfloor}{2ab} - \frac{1}{2}\left\lfloor \frac{-bq}{a} \right\rfloor - \frac{1}{2}\lfloor q \rfloor \\
&\qquad + \chi\left(\frac{\mu}{2} - \frac{1}{2}\left(\left(\frac{a^{-1}(-bq)}{b}\right)\right) - \frac{1}{2}\left(\left(\frac{b^{-1}(-bq)}{a}\right)\right)\right),
\end{aligned}$$

where χ equals 1 or 0 depending on whether or not bq is an integer, μ equals 1 or 0 depending on whether or not $q = 0$, $aa^{-1} \equiv 1 \bmod b$, and $bb^{-1} \equiv 1 \bmod a$. Noticing that $\lfloor q \rfloor = -1$ unless $q = 0$, and setting $t = -bq$ (which is a real number in the interval $[0, b)$) yields Corollary 4. ∎

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