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The divisor function on residue classes I

by

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1. Introduction. In this memoir we are concerned with the distribution of the divisor function $d(n)$ for n lying in a given residue class. Here $d(n)$ is the number of positive divisors of n,

$$d(n) = \sum_{m|n} 1.$$

Suppose that $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ and define, for each $x \in \mathbb{R}$ with $x \geq 1$,

$$S(x;q,a) = \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} d(n).$$

Various estimates are known for $S(x;q,a)$. Many of the estimates in the literature fall into two categories. In the first kind the error term is commendably small, but the acceptable range for q is small. In the second kind the range for q is greater, but the relative error is generally not very small. Our interest here is in estimates which are uniform in q for as large a range as possible, yet the error term is as small as possible.

Fouvry and Iwaniec [FI] assume $(q,a)=1$ and state that by using Fourier series techniques and Weil's estimate for Kloosterman's sum $K(q;a,b)$ one can show that

$$S(x;q,a) = \frac{1}{\phi(q)} \sum_{\substack{n \leq x \\ (n,q)=1}} d(n) + O((x^{1/3} + q^{1/2})x^{\varepsilon}).$$

Here

$$K(q;a,b) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e((ar + b[q,r])/q)$$

where $[q,r]$ denotes an integer u such that $ur \equiv 1 \bmod q$.

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As far as we are aware, there is no proof in print. Our interest in this is that we have an application in mind. Indeed, we need more than is stated above, as we require the corresponding result when $(q,a)>1$. In this paper we remedy this by adumbrating a proof in this general case. We also make a novel contribution to the subject by providing a main term that is both useful and uniform in q and a in all cases.

Theorem 1.1. *Suppose that $q\in\mathbb{N}$, $a\in\mathbb{Z}$, $x\in\mathbb{R}$ and $x\ge 1$. Then*

$$S(x;q,a)=\frac{x}{q}\sum_{r\mid q}\frac{c_r(a)}{r}\left(\log\frac{x}{r^2}+2\gamma-1\right)+O((x^{1/3}+q^{1/2})x^{\varepsilon}) \tag{1.1}$$

where γ is Euler's constant and $c_r(a)$ is Ramanujan's sum

$$c_r(a)=\sum_{\substack{m=1\\(m,r)=1}}^{r} e(am/r).$$

The implicit constant in (1.1) depends at most on ε.

2. Lemmata

Lemma 2.1. *Suppose that $q\in\mathbb{N}$ and $a,b\in\mathbb{Z}$. Then*

$$K(q;a,b)\ll d(q)q^{1/2}(q,a,b)^{1/2}$$

where (q,a,b) is the greatest common divisor of q,a,b.

This is sometimes called "Weil's estimate for $K(q;a,b)$", although as far as we are aware the first complete proof for all q,a,b is due to Estermann [E].

Lemma 2.2. *Suppose that $X<Y$, F'' exists and is continuous on $[X,Y]$ and F' is monotonic on $[X,Y]$. Let H_1 and H_2 be integers such that $H_1\le F'(\alpha)\le H_2$ for every $\alpha\in[X,Y]$. Then*

$$\sum_{X<n\le Y} e(F(n))=\sum_{h=H_1}^{H_2}\int_X^Y e(F(\alpha)-h\alpha)\,d\alpha+O(\log(2+H))$$

where $H=\max(|H_1|,|H_2|)$.

This is a standard finite form of the Poisson summation formula, and proofs can be found in Vaughan [V, Lemma 4.2] or Titchmarsh [T, Lemma 4.7].

Lemma 2.3 ([T, Lemma 4.2]). *Suppose that $X<Y$, F'' exists and is continuous on $[X,Y]$ and F' is monotonic on $[X,Y]$. Let J be a positive*

real number such that $|F'(\alpha)| \geq J$ *for all* $\alpha \in [X, Y]$. *Then*

$$\Big|\int_X^Y e^{iF(\alpha)}\, d\alpha\Big| \leq \frac{4}{J}.$$

LEMMA 2.4 ([T, Lemma 4.4]). *Suppose that* $X < Y$, F'' *exists and is continuous on* $[X, Y]$. *Let* K *be a positive real number such that* $|F''(\alpha)| \geq K$ *for all* $\alpha \in [X, Y]$. *Then*

$$\Big|\int_X^Y e^{iF(\alpha)}\, d\alpha\Big| \leq \frac{8}{\sqrt{K}}.$$

It is useful to establish first of all a weak version of our main theorem.

LEMMA 2.5. *Suppose that* $q \in \mathbb{N}$, $a \in \mathbb{Z}$, $x \in \mathbb{R}$ *and* $x \geq 1$. *Then*

$$S(x; q, a) = \frac{x}{q} \sum_{r|q} \frac{c_r(a)}{r} \Big(\log \frac{x}{r^2} + 2\gamma - 1\Big) + O((x^{1/2} + q) \log 2q).$$

Proof. Let $r \in \mathbb{N}$, $b \in \mathbb{Z}$ with $(r, b) = 1$. Dirichlet's method of the hyperbola gives

$$\sum_{n \leq x} d(n) e(bn/r) = \sum_{u \leq \sqrt{x}} \Big(\sum_{v \leq x/u} 2 - \sum_{v \leq \sqrt{x}} 1\Big) e(buv/r).$$

When $r \nmid u$ the inner sums are $\ll \|bu/r\|^{-1}$, and so such terms contribute

$$\ll (\sqrt{x} + r) \log 2r$$

to the double sums. The remaining terms, with $r \mid u$, are easily seen to contribute

$$\frac{x}{r}\Big(\log \frac{x}{r^2} + 2\gamma - 1\Big) + O(\sqrt{x}).$$

The lemma then follows by considering

$$S(x; q, a) = \frac{1}{q} \sum_{j=1}^{q} e(-ja/q) \sum_{n \leq x} d(n) e(jn/q). \ \blacksquare$$

It is useful to define the periodic polynomials

$$B_1(\alpha) = \alpha - \lfloor \alpha \rfloor - \tfrac{1}{2}, \qquad B_2(\alpha) = \tfrac{1}{2}(\alpha - \lfloor \alpha \rfloor)^2 - \tfrac{1}{2}(\alpha - \lfloor \alpha \rfloor) + \tfrac{1}{12}.$$

They are of period 1, $B_1(x)$ is continuous on $\mathbb{R} \setminus \mathbb{Z}$, $B_2(x)$ is differentiable on $\mathbb{R} \setminus \mathbb{Z}$ and continuous on $\mathbb{R}$, and they are related by

$$\int_0^x B_1(y)\, dy = B_2(x) - \tfrac{1}{12}.$$

LEMMA 2.6. *Suppose that $q \in \mathbb{N}$, $a \in \mathbb{Z}$, $(q,a) = 1$. Then there is a real number $C(q)$ such that whenever $x \in \mathbb{R}$ and $x \geq 1$ we have*

$$S(x;q,a) = \frac{\phi(q)}{q^2} x(\log x + C(q)) - S_1 - S_2 + S_3 - S_4$$

where

$$S_1 = \sum_{\substack{n \leq \sqrt{x} \\ (n,q)=1}} 2B_1\left(\frac{x - an[q,n]}{nq}\right),$$

$$S_2 = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} 2B_2\left(\frac{\sqrt{x} - r}{q}\right),$$

$$S_3 = 4x \int_{\sqrt{x}}^{\infty} \frac{1}{\beta^3} \sum_{\substack{r=1 \\ (r,q)=1}}^{q} B_2\left(\frac{\beta - r}{q}\right) d\beta,$$

$$S_4 = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \left(B_1\left(\frac{\sqrt{x} - r}{q}\right) + B_1\left(\frac{-r}{q}\right)\right)\left(B_1\left(\frac{\sqrt{x} - a[q,r]}{q}\right) - B_1\left(\frac{-a[q,r]}{q}\right)\right).$$

Proof. Multiple use is made of the observation that

$$\sum_{\substack{n \leq y \\ n \equiv b \bmod q}} 1 = \frac{y}{q} - B_1\left(\frac{y-b}{q}\right) + B_1\left(\frac{-b}{q}\right).$$

Again we start with Dirichlet's method of the hyperbola. Thus

$$S(x;q,a) = \sum_{\substack{\ell \leq \sqrt{x} \\ (\ell,q)=1}} \sum_{\substack{r=1 \\ \ell r \equiv a \bmod q}}^{q} \left(\frac{2x}{\ell q} - \frac{\sqrt{x}}{q} - 2A + B\right) \tag{2.1}$$

where

$$A = B_1\left(\frac{x - r\ell}{\ell q}\right) - B_1\left(\frac{-r}{q}\right), \qquad B = B_1\left(\frac{\sqrt{x} - r}{q}\right) - B_1\left(\frac{-r}{q}\right).$$

The expression B contributes

$$\sum_{\substack{r=1 \\ (r,q)=1}}^{q} \sum_{\substack{\ell \leq \sqrt{x} \\ \ell r \equiv a \bmod q}} B = \frac{\sqrt{x}}{q} \sum_{\substack{r=1 \\ (r,q)=1}}^{q} B - C \tag{2.2}$$

where

$$C = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \left(B_1\left(\frac{\sqrt{x} - r}{q}\right) - B_1\left(\frac{-r}{q}\right)\right)\left(B_1\left(\frac{\sqrt{x} - a[q,r]}{q}\right) - B_1\left(\frac{-a[q,r]}{q}\right)\right).$$

The contribution to (2.1) of

$$-\sum_{\substack{\ell\le\sqrt{x}\\(\ell,q)=1}}\ \sum_{\substack{r=1\\ \ell r\equiv a \bmod q}}^{q}\frac{\sqrt{x}}{q}$$

is

$$-\frac{\sqrt{x}}{q}\sum_{\substack{\ell\le\sqrt{x}\\(\ell,q)=1}}1=-\frac{x\phi(q)}{q^2}+\frac{\sqrt{x}}{q}\sum_{\substack{r=1\\(r,q)=1}}^{q}B. \tag{2.3}$$

The contribution to (2.1) of

$$-\sum_{\substack{\ell\le\sqrt{x}\\(\ell,q)=1}}\ \sum_{\substack{r=1\\ \ell r\equiv a \bmod q}}^{q}2A$$

is

$$-S_1+\sum_{\substack{r=1\\(r,q)=1}}^{q}\ \sum_{\substack{\ell\le\sqrt{x}\\ \ell r\equiv a \bmod q}}2B_1\Big(\frac{-r}{q}\Big)$$

and the sum here is

$$\sum_{\substack{r=1\\(r,q)=1}}^{q}2B_1\Big(\frac{-r}{q}\Big)\bigg(\frac{\sqrt{x}}{q}-B_1\Big(\frac{\sqrt{x}-a[q,r]}{q}\Big)+B_1\Big(\frac{-a[q,r]}{q}\Big)\bigg).$$

Combining this with (2.2) and (2.3) gives

$$-\frac{x\phi(q)}{q^2}+\frac{2\sqrt{x}}{q}\sum_{\substack{r=1\\(r,q)=1}}^{q}B_1\Big(\frac{\sqrt{x}-r}{q}\Big)-S_1-S_4. \tag{2.4}$$

It remains to consider the contribution to (2.1) of

$$\sum_{\substack{\ell\le\sqrt{x}\\(\ell,q)=1}}\ \sum_{\substack{r=1\\ \ell r\equiv a \bmod q}}^{q}\frac{2x}{\ell q}=\frac{2x}{q}\sum_{\substack{r=1\\(r,q)=1}}^{q}\ \sum_{\substack{\ell\le\sqrt{x}\\ \ell\equiv r \bmod q}}\frac{1}{\ell}.$$

By integration by parts,

$$\sum_{\substack{\ell\le\sqrt{x}\\ \ell\equiv r \bmod q}}\frac{1}{\ell}=\frac{1}{q}\log\sqrt{x}+\frac{1}{q}+B_1\Big(\frac{-r}{q}\Big)-\frac{1}{\sqrt{x}}B_1\Big(\frac{\sqrt{x}-r}{q}\Big)-\int_1^{\sqrt{x}}\beta^{-2}B_1\Big(\frac{\beta-r}{q}\Big)\,d\beta.$$

The integral here is

$$C_1(q,r)-\int_{\sqrt{x}}^{\infty}\beta^{-2}B_1\Big(\frac{\beta-r}{q}\Big)\,d\beta$$

where

$$C_1(q,r) = \int_1^\infty \beta^{-2} B_1\Big(\frac{\beta - r}{q}\Big)\,d\beta.$$

We have

$$\int_{\sqrt{x}}^\infty \beta^{-2} B_1\Big(\frac{\beta - r}{q}\Big)\,d\beta = -\frac{q}{x} B_2\Big(\frac{\sqrt{x} - r}{q}\Big) + \int_{\sqrt{x}}^\infty 2q\beta^{-3} B_2\Big(\frac{\beta - r}{q}\Big)\,d\beta.$$

Thus the total contribution to our sum from this term is

$$\frac{x\phi(q)}{q^2}(\log x + C_2(q)) - \frac{2\sqrt{x}}{q} \sum_{\substack{r=1\\(r,q)=1}}^{q} B_1\Big(\frac{\sqrt{x}-r}{q}\Big) - S_2 + S_3 \tag{2.5}$$

where

$$C_2(q) = 2 + \frac{2q}{\varphi(q)} \sum_{\substack{r=1\\(r,q)=1}}^{q} B_1\Big(-\frac{r}{q}\Big) - \frac{2q}{\varphi(q)} \sum_{\substack{r=1\\(r,q)=1}}^{q} C_1(q,r).$$

Combining (2.4) and (2.5), we obtain

$$S(x;q,a) = \frac{\varphi(q)x}{q^2}(\log x + C(q)) - S_1 - S_2 + S_3 - S_4$$

where $C(q) = C_2(q) - 1$ is independent of a. One can also see this by reference to Lemma 2.5. When $(q,a) = 1$, we have $c_r(a) = \mu(r)$, and then subtracting main terms, dividing by x and letting $x \to \infty$ gives another formula for $C(q)$,

$$C(q) = \frac{q}{\varphi(q)} \sum_{r\mid q} \frac{\mu(r)}{r}(-2\log r + 2\gamma - 1), \tag{2.6}$$

which is independent of a. ∎

Lemma 2.7. *Suppose that $q \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then*

$$\sum_{\substack{r=1\\(r,q)=1}}^{q} B_2\Big(\alpha - \frac{r}{q}\Big) \ll \frac{q}{\phi(q)}. \tag{2.7}$$

Proof. $B_2(\beta)$ has the Fourier expansion (Montgomery and Vaughan [MV, Theorem B.2])

$$\sum_{h\neq 0} \frac{e(\beta h)}{4\pi^2 h^2},$$

which converges to $B_2(\beta)$ for every β. Thus the sum in question is

$$\sum_{h\neq 0} \frac{e(\alpha h)}{4\pi^2 h^2} c_q(-h).$$

By Ramanujan's formula $c_q(-h) = \sum_{m|(q,h)} m\mu(q/m)$, the above becomes

$$\sum_{m|q} \frac{1}{m}\mu(q/m) \sum_{j\neq 0} \frac{e(\alpha mj)}{4\pi^2 j^2} = \frac{1}{q}\sum_{m|q} m\mu(m)B_2(\alpha q/m) \ll \frac{1}{q}\sigma(q) \ll \frac{q}{\varphi(q)}. \blacksquare$$

LEMMA 2.8. *Suppose that $q \in \mathbb{N}$, $a \in \mathbb{Z}$, $(q,a) = 1$, and $\alpha, \beta \in \mathbb{R}$. Then*

$$\sum_{\substack{r=1\\(r,q)=1}}^{q} B_1(\alpha - r/q)B_1(\beta - a[q,r]/q) \ll d(q)q^{1/2}(\log 2q)^2.$$

Proof. The result obviously holds for $q = 1, 2$. So we assume that $q \geq 3$ and let $H \geq 3$ be a number to be specified later. By Lemma D.1 of Montgomery and Vaughan [MV] we have

$$B_1(\alpha) = -\sum_{0<|h|\leq H} \frac{e(\alpha h)}{2\pi i h} + O(G(\alpha))$$

where

$$G(\alpha) = G(\alpha, H) = \min\left(1, \frac{1}{H\|\alpha\|}\right)$$

and $\|\alpha\| = \min_{n\in\mathbb{Z}} |\alpha - n|$. In particular, this implies that

$$\sum_{0<|h|\leq H} \frac{e(\alpha h)}{2\pi i h} \ll 1$$

uniformly in H and α. Thus the sum in question is

$$-\sum_{0<|h|\leq H}\sum_{0<|j|\leq H} \frac{e(\alpha h)}{2\pi h}\frac{e(\beta j)}{2\pi j} K(q; -h, -ja) + O(E) \tag{2.8}$$

where

$$E = \sum_{\substack{r=1\\(r,q)=1}}^{q} (G(\alpha - r/q) + G(\beta - a[q,r]/q)). \tag{2.9}$$

We also record that the function $G(\alpha)$ has the Fourier expansion

$$G(\alpha) = \sum_{h=-\infty}^{\infty} g(h)e(\alpha h) \quad \text{where} \quad g(h) = \int_{-1/2}^{1/2} G(\alpha)e(-\alpha h)\,d\alpha. \tag{2.10}$$

Thus

$$g(0) = \int_{-1/2}^{-1/H} \frac{d\alpha}{H(-\alpha)} + \int_{-1/H}^{1/H} d\alpha + \int_{1/H}^{1/2} \frac{d\alpha}{H\alpha} = \frac{2}{H}\log\frac{He}{2}.$$

Also, when $h \neq 0$,

$$\begin{aligned}g(h) &= \int_{-1/2}^{1/2} G(\alpha)e(-\alpha h)\,d\alpha\\ &= \int_{-1/H}^{1/H} e(-\alpha h)\,d\alpha + \int_{-1/2}^{-1/H} \frac{1}{H(-\alpha)}e(-\alpha h)\,d\alpha + \int_{1/H}^{1/2} \frac{1}{H\alpha}e(-\alpha h)\,d\alpha\\ &= \frac{\sin(2\pi h/H)}{\pi h} + \int_{1/H}^{1/2} \frac{2\cos 2\pi\alpha h}{H\alpha}\,d\alpha.\end{aligned}$$

By integration by parts this is

$$\int_{1/H}^{1/2} \frac{\sin 2\pi\alpha h}{\pi h H\alpha^2}\,d\alpha.$$

It is easy to check that for $h \neq 0$, we have $|g(h)| \leq g(0) \ll \frac{\log 2H}{H}$, and $|g(h)| \leq \frac{1}{\pi|h|}$, and by integration by parts, $g(h) \ll \frac{H}{h^2}$. Thus

$$g(h) \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{h^2}\right) \quad (h \neq 0). \tag{2.11}$$

Now let $H = q$. Then by (2.9)–(2.11), we obtain

$$\begin{aligned}E &= \sum_{h=-\infty}^{\infty} g(h)\big(e(\alpha h) + e(\beta h)\big)c_q(h) \ll \sum_{h=-\infty}^{\infty} |g(h)| \sum_{r\mid(q,h)} r\\ &\ll \sum_{r\mid q} r\Bigg(\sum_{\substack{h=1\\ r\mid h}}^{q} \frac{1}{h} + \sum_{\substack{h=q+1\\ r\mid h}}^{\infty} \frac{q}{h^2}\Bigg) + g(0)qd(q) \ll d(q)\log 2q.\end{aligned}$$

So it remains to consider the main term in (2.8). By Lemma 2.1,

$$K(q; -h, -ja) \ll d(q)q^{1/2}(q, h, ja)^{1/2} = d(q)q^{1/2}(q; h, j)^{1/2},$$

and therefore the main term in (2.8) is

$$\begin{aligned}&\ll d(q)q^{1/2} \sum_{h=1}^{q}\sum_{j=1}^{q} \frac{1}{hj}(q, h, j)^{1/2} = d(q)q^{1/2} \sum_{d\mid q} d^{1/2} \sum_{\substack{h=1\\ (q,h,j)=d}}^{q}\sum_{j=1}^{q} \frac{1}{hj}\\ &= d(q)q^{1/2} \sum_{d\mid q} d^{1/2} \sum_{\substack{u=1\\ (q/d,u,v)=1}}^{q/d}\sum_{v=1}^{q/d} \frac{1}{d^2uv} \leq d(q)q^{1/2} \sum_{d\mid q} \frac{1}{d^{3/2}}\Bigg(\sum_{u=1}^{q/d} \frac{1}{u}\Bigg)^2\\ &\ll d(q)q^{1/2}(\log 2q)^2.\end{aligned}$$

This completes the proof of Lemma 2.8. ∎

3. Proof of Theorem 1.1 when $(q, a) = 1$**.** Since the conclusion is trivial when $q > x^{2/3}$, we may suppose that $q \le x^{2/3}$.

It follows by Lemma 2.7 that

$$S_2 \ll \frac{q}{\phi(q)},$$

and likewise for S_3. Also, by several applications of Lemma 2.8 we have

$$S_4 \ll d(q)q^{1/2}(\log 2q)^2.$$

Our main aim is to prove that

$$S_1 \ll (x^{1/3} + q^{1/2})x^{\varepsilon},$$

for then the desired conclusion will follow from Lemma 2.6, the formula for $C(q)$ in (2.6), and the fact that $c_r(a) = \mu(r)$ when $(q, a) = 1$ and $r \mid q$.

Consider

$$T(Y) = \sum_{\substack{Y<n\le Y'\\(n,q)=1}} 2B_1\Big(\frac{x - an[q,n]}{nq}\Big)$$

where $Y < Y' \le 2Y$ and $Y \le \sqrt{x}$. We are only concerned with Y satisfying

$$x^{1/3} \le Y \le x^{1/2}.$$

We estimate B_1 as in Lemma 2.8. Thus

$$B_1\Big(\frac{x - an[q,n]}{nq}\Big) = -\sum_{0<|h|\le H} \frac{e\big(\frac{hx-ahn[q,n]}{nq}\big)}{2\pi i h} + O\Big(G\Big(\frac{x - an[q,n]}{nq}\Big)\Big), \tag{3.1}$$

and in the error term here

$$G\Big(\frac{x - an[q,n]}{nq}\Big) = \sum_{h=-\infty}^{\infty} g(h) e\Big(\frac{hx - ahn[q,n]}{nq}\Big). \tag{3.2}$$

Therefore

$$\begin{aligned} T(Y) = -2 \sum_{0<|h|\le H} \frac{1}{2\pi i h} \sum_{\substack{Y<n\le Y'\\(n,q)=1}} e\Big(\frac{hx - ahn[q,n]}{nq}\Big) \\ + O\Big(\sum_{h=-\infty}^{\infty} g(h) \sum_{\substack{Y<n\le Y'\\(n,q)=1}} e\Big(\frac{hx - ahn[q,n]}{nq}\Big)\Big). \end{aligned} \tag{3.3}$$

The term $h = 0$ will contribute in total

$$\ll Y g(0) \ll \frac{Y \log 2H}{H},$$

and we shall see that for a suitable choice of H this is adequate.

When $h \neq 0$ we are interested in

$$\sum_{\substack{Y<n\le Y'\\(n,q)=1}} e\left(\frac{hx-ahn[q,n]}{nq}\right) = \sum_{\substack{r=1\\(r,q)=1}}^{q} e\left(\frac{-ah[q,r]}{q}\right) \sum_{\substack{Y<n\le Y'\\n\equiv r \bmod q}} e\left(\frac{hx}{nq}\right),$$

which is equal to

$$\frac{1}{q}\sum_{k=1}^{q} K(q;k,-ah) \sum_{Y<n\le Y'} e\left(\frac{hx}{nq}-\frac{kn}{q}\right).$$

For $\alpha \in [Y,Y']$, let $F(\alpha) = \frac{hx}{\alpha q} - \frac{k\alpha}{q}$, so that $F'(\alpha) = \frac{-hx}{\alpha^2 q} - \frac{k}{q}$, F' is monotonic on $[Y,Y']$, and $|F'(\alpha)| \le \frac{|h|x}{qY^2} + 1$. Thus, by Lemma 2.2 we have

$$\sum_{Y<n\le Y'} e\left(\frac{hx}{nq}-\frac{kn}{q}\right) = \sum_{J_k^-\le j\le J_k^+} \int_Y^{Y'} e(F(\alpha)-j\alpha)\,d\alpha + O(\log 2xH)$$

where $J_k^{\pm}$ are integers such that $J_k^- \le F'(\alpha) \le J_k^+$. Here we take $J_k^+ = \left\lfloor \frac{|h|x}{qY^2}\right\rfloor + 2$ and $J_k^- = -\left\lfloor \frac{|h|x}{qY^2}\right\rfloor - 2$. The total contribution from the error term here is readily estimated by reference to Lemma 2.1. Concentrating on the main term we have to consider

$$\frac{1}{q}\sum_{k=1}^{q}\sum_{J_k^-\le j\le J_k^+} K(q;k,-ah)\int_Y^{Y'} e(F(\alpha)-j\alpha)\,d\alpha. \tag{3.4}$$

Let $\ell = \ell(k,j) = k + jq$ for $k = 1,\dots,q$ and $J_k^- \le j \le J_k^+$. If $k_1 + j_1 q = k_2 + j_2 q$, then $|j_1 - j_2| = |k_1 - k_2|/q \le (q-1)/q < 1$, which implies $j_1 = j_2$ and $k_1 = k_2$. Thus $\ell(k,j)$ are all distinct and

$$|\ell(k,j)| \le \max\{q + J_k^+ q, |1 + J_k^- q|\} \le |h|xY^{-2} + 3q.$$

We divide the multiple summation in (3.4) into two cases.

If $\frac{|h|x}{Y^2 q} \le \frac{|k+jq|}{2q}$, then $k + jq \neq 0$ and

$$|F'(\alpha) - j| = \left|\frac{hx}{\alpha^2 q} + \frac{k+jq}{q}\right| \ge \frac{|k+jq|}{q} - \frac{|h|x}{Y^2 q} \ge \frac{|k+jq|}{2q}.$$

Hence by Lemma 2.3 the integral appearing in (3.4) is $\ll q/|k+jq|$, and the contribution from these terms to (3.4) is

$$\begin{aligned}\frac{1}{q}\sum_{1\le|\ell|\le|h|xY^{-2}+3q} |K(q;\ell,-ah)|\frac{q}{|\ell|} &\ll d(q)q^{1/2} \sum_{1\le\ell\le|h|xY^{-2}+3q} \frac{(q,h,\ell)^{1/2}}{\ell}\\ &\ll d(q)q^{1/2}\sum_{m|(q,h)} m^{-1/2} \sum_{1\le k\le(|h|xY^{-2}+3q)m^{-1}} \frac{1}{k} \ll q^{1/2}x^{\varepsilon}.\end{aligned}$$

If $\frac{|k+jq|}{2q} \le \frac{|h|x}{Y^2 q}$, then $|\ell| \le 2|h|xY^{-2}$, and by using Lemma 2.4 we obtain the bound

$$\begin{aligned}
&\frac{1}{q} \sum_{|\ell| \le 2|h|xY^{-2}} |K(q;\ell,-ah)| \left(\frac{qY^3}{|h|x}\right)^{1/2} \\
&\qquad \ll d(q)\left(\frac{Y^3}{|h|x}\right)^{1/2} \sum_{|\ell| \le 2|h|xY^{-2}} (q,h,\ell)^{1/2} \\
&\qquad \ll d(q)\left(\frac{Y^3}{|h|x}\right)^{1/2} \Big((q,h)^{1/2} + \sum_{m|(q,h)} m^{1/2} \sum_{1 \le k \le 2|h|xY^{-2}m^{-1}} 1\Big) \\
&\qquad \ll d(q)\left(\frac{Y^3}{|h|x}\right)^{1/2} \Big((q,h)^{1/2} + \sum_{m|(q,h)} m^{-1/2} \frac{2|h|x}{Y^2}\Big) \\
&\qquad \ll d(q)(q,h)^{1/2}\left(\frac{Y^3}{|h|x}\right)^{1/2} + d(q)^2\left(\frac{|h|x}{Y}\right)^{1/2}.
\end{aligned}$$

We now insert this estimate in (3.3). The main term is

$$\begin{aligned}
&\ll \sum_{0<|h|\le H} \frac{1}{|h|}\left(q^{1/2}x^{\varepsilon} + d(q)(q,h)^{1/2}\left(\frac{Y^3}{|h|x}\right)^{1/2} + d(q)^2\left(\frac{|h|x}{Y}\right)^{1/2}\right) \\
&\ll q^{1/2}x^{\varepsilon}\log 2H + d(q)\left(\frac{Y^3}{x}\right)^{1/2}\log 2H + d(q)^2 H^{1/2}\left(\frac{x}{Y}\right)^{1/2}.
\end{aligned}$$

For the error term in (3.3), we recall the bound on $g(h)$ in (2.11). The term $h = 0$ is $\ll (Y\log 2H)/H$. If $0 < |h| \le H$, then we use the bound $g(h) \ll 1/|h|$, which leads to the same estimate as in the main term. If $|h| \ge H$, we apply the bound $g(h) \ll H/|h|^2$, which gives a total estimate

$$\ll q^{1/2}x^{\varepsilon} + d(q)\left(\frac{Y^3}{x}\right)^{1/2} + d(q)^2 H^{1/2}\left(\frac{x}{Y}\right)^{1/2}.$$

Hence

$$T(Y) \ll q^{1/2}x^{\varepsilon}\log 2H + q^{\varepsilon}\left(\frac{Y^3}{x}\right)^{1/2}\log 2H + q^{\varepsilon}H^{1/2}\left(\frac{x}{Y}\right)^{1/2} + \frac{Y\log 2H}{H}.$$

When $x^{1/3} \le Y \le x^{1/2}$, a good choice for H is $Yx^{-1/3}$, and it follows that

$$T(Y) \ll x^{1/3+\varepsilon} + q^{1/2}x^{\varepsilon}.$$

By dyadic summing we obtain $S_1 \ll x^{1/2+\varepsilon} + q^{1/2}x^{\varepsilon}$ as desired.

4. Proof of Theorem 1.1 when $(q,a)>1$**.** Let $k=(q,a)$, $r=q/k$, $b=a/k$, so that $(r,b)=1$ and

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} d(n) = \sum_{\substack{m\le x/k\\ m\equiv b \bmod r}} d(km).$$

Define

$$k_1=\prod_{\substack{p^t\|k\\ p|r}} p^t, \qquad k_2=\prod_{\substack{p^t\|k\\ p\nmid r}} p^t.$$

Thus

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} d(n) = d(k_1)\sum_{\substack{m\le x/k\\ m\equiv b \bmod r}} d(k_2m).$$

Let $\mathcal{K}=\{\ell\in\mathbb{N}: p\,|\,\ell \Rightarrow p\,|\,k_2\}$. Then the above is

$$d(k_1)\sum_{\ell\in\mathcal{K}} d(k_2\ell)\sum_{\substack{n\le x/(k\ell)\\ n\equiv b[r,\ell] \bmod r\\ (n,k_2)=1}} d(n).$$

Let $\mu_2(n)$ denote the multiplicative function such that $\mu_2(p)=-2$, $\mu_2(p^2)=1$, $\mu_2(p^t)=0$ when $t\ge 3$. Now we have

$$\sum_{\substack{uv=n\\ u\in\mathcal{K}}} \mu_2(u)d(v) = \begin{cases} d(n) & \text{when } (n,k_2)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} d(n) = d(k_1)\sum_{\ell\in\mathcal{K}} d(k_2\ell)\sum_{u\in\mathcal{K}}\mu_2(u)\sum_{\substack{v\le x/(k\ell u)\\ v\equiv b[r,u][r,\ell] \bmod r}} d(v).$$

By the case $(q,a)=1$ of Theorem 1.1 this is

$$d(k_1)\sum_{\ell\in\mathcal{K}} d(k_2\ell)\sum_{\substack{u\in\mathcal{K}\\ u\le x/(k\ell)}}\mu_2(u)$$
$$\times\left(\frac{x}{k\ell u}\Big(f_1(r)\log\frac{x}{k\ell u}+g_1(r)\Big)+O\bigg(\Big(\frac{x}{k\ell u}\Big)^{1/3+\varepsilon}+r^{1/2}\Big(\frac{x}{k\ell u}\Big)^{\varepsilon}\bigg)\right),$$

where the functions $f_1(r)$ and $g_1(r)$ depend only on r. By use of the Rankin "trick" in the forms

$$\sum_{\substack{m\le x\\ m\in\mathcal{K}}} 1 \le \sum_{m\in\mathcal{K}}\Big(\frac{x}{m}\Big)^{\lambda}$$

with λ sufficiently small and

$$\sum_{\substack{m>x\\ m\in\mathcal{K}}} m^{-\theta} \le \sum_{m\in\mathcal{K}}\frac{m^{\lambda-\theta}}{x^{\lambda}}$$

with λ close to θ, we see that the various sums over $\mathcal{K}$ can either be completed to infinity with an error $\ll x^{\varepsilon-1}$ or bounded by x^{ε}. Thus the above is

$$x(f_2(q,a)\log x + g_2(q,a)) + O(x^{1/3+\varepsilon} + q^{1/2}x^{\varepsilon})$$

for some undetermined choice of $f_2(q,a)$ and $g_2(q,a)$.

Now, once more an appeal can be made to Lemma 2.5 to complete the proof. First of all dividing by $x\log x$ and letting $x \to \infty$ and comparing with Lemma 2.5 gives the leading coefficient. Now subtracting this here and in Lemma 2.5, dividing by x and letting $x \to \infty$ gives the second order term.

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