ACTA ARITHMETICA
146.1 (2011)

Parity results for broken k-diamond partitions and $(2k+1)$-cores

by

Silviu Radu (Linz) and James A. Sellers (University Park, PA)

1. Introduction. Broken k-diamond partitions were introduced recently by Andrews and Paule [1]. These are constructed in such a way that the generating functions of their counting sequences $(\Delta_k(n))_{n\geq 0}$ are closely related to modular forms. Namely,

$$\sum_{n=0}^{\infty} \Delta_k(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})}$$
$$= q^{(k+1)/12}\frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)}, \qquad k\geq 1,$$

where we recall the Dedekind eta function

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad (q=e^{2\pi i\tau}).$$

In [1], Andrews and Paule proved that, for all $n\geq 0$, $\Delta_1(2n+1)\equiv 0 \pmod 3$ and conjectured a few other congruences modulo 2 satisfied by certain families of k-broken diamond partitions.

Since then, a number of authors have provided proofs of additional congruences satisfied by broken k-diamond partitions. Hirschhorn and Sellers [9] provided a new proof of the modulo 3 result mentioned above as well as elementary proofs of the following parity results: For all $n\geq 1$,

$$\Delta_1(4n+2)\equiv 0 \pmod 2,$$
$$\Delta_1(4n+3)\equiv 0 \pmod 2,$$
$$\Delta_2(10n+2)\equiv 0 \pmod 2,$$
$$\Delta_2(10n+6)\equiv 0 \pmod 2.$$

The third result in the list above appeared in [1] as a conjecture while the

2010 *Mathematics Subject Classification*: Primary 11P83; Secondary 05A17.
Key words and phrases: broken k-diamonds, congruences, cores, modular forms, partitions.

other three did not. Soon after the publication of [9], Chan [3] provided a different proof of the parity results for Δ_2 mentioned above as well as a number of congruences modulo powers of 5.

In this paper, we significantly extend the list of known parity results for broken k-diamonds by proving a large number of congruences which are similar to those mentioned above. Indeed, we will do so by proving a similar set of parity results satisfied by certain t-core partitions.

A partition is called a *t-core* if none of its hook lengths is divisible by t. These partitions have been studied extensively by many, especially thanks to their strong connection to representation theory. Numerous congruence properties are known for t-cores, although few such results are known modulo 2. Such parity results can be found in [7], [6], [10], [8], [2], [4]. In all of these papers, the value of t which was considered was even; in this paper, we provide a new set of parity results for t-cores wherein t is odd.

The generating function for t-core partitions (for a fixed $t \geq 1$) is given by

$$\sum_{n=0}^{\infty} a_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}.$$

Given this fact, we can quickly see a connection between broken k-diamonds and $(2k+1)$-cores which we will utilize below.

LEMMA 1.1. *For all $k \geq 1$ we have*

$$\Big(\prod_{n=1}^{\infty}(1-q^{(4k+2)n})^{k+1}\Big)\Big(\sum_{n=0}^{\infty}\Delta_k(n)q^n\Big) \equiv \sum_{n=0}^{\infty} a_{2k+1}(n)q^n \pmod 2.$$

Proof. Using the relation $(1-q^n)^2 \equiv (1-q^{2n}) \pmod 2$ we find

$$\begin{aligned}
&\Big(\prod_{n=1}^{\infty}(1-q^{(4k+2)n})^{k+1}\Big)\Big(\sum_{n=0}^{\infty}\Delta_k(n)q^n\Big)\\
&\qquad = \prod_{n=1}^{\infty} \frac{(1-q^{(4k+2)n})^k(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3}\\
&\qquad \equiv \prod_{n=1}^{\infty} \frac{(1-q^{(2k+1)n})^{2k+1}}{(1-q^n)} \pmod 2 = \sum_{n=0}^{\infty} a_{2k+1}(n)q^n. \qquad \blacksquare
\end{aligned}$$

We assume throughout that $\Delta_k(v) = a_k(v) = 0$ if $v \leq 0$.

COROLLARY 1.2. *Let $r \in \mathbb{N}$. Then for all $k \geq 1$ we have*

$$\begin{aligned}
&\Delta_k((4k+2)n+r) \equiv 0 \pmod 2 \text{ for all } n \in \mathbb{Z}\\
&\qquad \Leftrightarrow a_{2k+1}((4k+2)n+r) \equiv 0 \pmod 2 \text{ for all } n \in \mathbb{Z}.
\end{aligned}$$

Proof. Let k and r be fixed and assume that $\Delta_k((4k+2)n+r) \equiv 0 \pmod 2$ for all $n \in \mathbb{Z}$. Let

$$\sum_{n\in\mathbb{Z}} b(n)q^{(4k+2)n} = \prod_{n=1}^{\infty}(1-q^{(4k+2)n})^{k+1}.$$

Then using Lemma 1.1 we find that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} a_{2k+1}((4k+2)n+r)q^{(4k+2)n+r} \\
&\qquad \equiv \sum_{\substack{n,m\in\mathbb{Z}\\ (4k+2)n+m\equiv r \ (\mathrm{mod}\ 4k+2)}} b(n)\Delta_k(m)q^{(4k+2)n+m} \\
&\qquad \equiv \sum_{\substack{n,m\in\mathbb{Z}\\ m\equiv r \ (\mathrm{mod}\ 4k+2)}} b(n)\Delta_k(m)q^{(4k+2)n+m} \\
&\qquad \equiv \sum_{n,v\in\mathbb{Z}} b(n)\Delta_k((4k+2)v+r)q^{(4k+2)n+(4k+2)v+r} \equiv 0 \pmod 2.
\end{aligned}$$

The reverse direction is analogous. ■

With this motivation, we now state the full list of parity results we will prove in this paper. With the goal of minimizing the notation, we will write

$$f(tn+r_1, r_2, \ldots, r_m) \equiv 0 \pmod 2$$

to mean that, for each $i \in \{1, \ldots, m\}$,

$$f(tn+r_i) \equiv 0 \pmod 2.$$

THEOREM 1.3. *For all $n \geq 0$,*

(1.1) $\Delta_2(10n+2, 6) \equiv 0 \pmod 2$,

(1.2) $\Delta_3(14n+7, 9, 13) \equiv 0 \pmod 2$,

(1.3) $\Delta_5(22n+2, 8, 12, 14, 16) \equiv 0 \pmod 2$,

(1.4) $\Delta_6(26n+2, 10, 16, 18, 20, 22) \equiv 0 \pmod 2$,

(1.5) $\Delta_8(34n+11, 15, 17, 19, 25, 27, 29, 33) \equiv 0 \pmod 2$,

(1.6) $\Delta_9(38n+2, 8, 10, 20, 24, 28, 30, 32, 34) \equiv 0 \pmod 2$,

(1.7) $\Delta_{11}(46n+11, 15, 21, 23, 29, 31, 35, 39, 41, 43, 45) \equiv 0 \pmod 2$.

(Note that (1.1) was proved in [9].) Thanks to Corollary 1.2, we see that Theorem 1.3 is proved once we prove the following corresponding theorem involving t-cores:

THEOREM 1.4. *For all $n \geq 0$,*

(1.8) $a_5(10n+2, 6) \equiv 0 \pmod 2$,

(1.9) $a_7(14n+7, 9, 13) \equiv 0 \pmod 8$,

$$a_{11}(22n+2,8,12,14,16) \equiv 0 \pmod{2}, \tag{1.10}$$

$$a_{13}(26n+2,10,16,18,20,22) \equiv 0 \pmod{2}, \tag{1.11}$$

$$a_{17}(34n+11,15,17,19,25,27,29,33) \equiv 0 \pmod{8}, \tag{1.12}$$

$$a_{19}(38n+2,8,10,20,24,28,30,32,34) \equiv 0 \pmod{2}, \tag{1.13}$$

$$a_{23}(46n+11,15,21,23,29,31,35,39,41,43,45) \equiv 0 \pmod{8}. \tag{1.14}$$

Note that every prime p, $5 \le p \le 23$, is represented in Theorem 1.4, which helps to explain why certain families of broken k-diamond partitions appear in Theorem 1.3 (and others do not). Our ultimate goal now is to provide a proof of Theorem 1.4. We close this section by developing the machinery necessary to prove this result.

For M a positive integer let $R(M)$ be the set of integer sequences indexed by the positive divisors δ of M. Let $1 = \delta_1, \dots, \delta_k = M$ be the positive divisors of M and $r \in R(M)$. Then we will write $r = (r_{\delta_1}, \dots, r_{\delta_k})$.

For s an integer and m a positive integer we denote by $[s]_m$ the set of all elements congruent to s modulo m, in other words $[s]_m \in \mathbb{Z}_m$. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $\mathbb{S}_m \subset \mathbb{Z}_m^*$ be the set of all squares in $\mathbb{Z}_m^*$.

Definition 1.5. For $m, M \in \mathbb{N}^*$, $(r_\delta) \in R(M)$ and $t \in \{0, \dots, m-1\}$ we define the map $\overline{\odot} : \mathbb{S}_{24m} \times \{0, \dots, m-1\} \to \{0, \dots, m-1\}$ by $([s]_{24m}, t) \mapsto [s]_{24m} \overline{\odot}\, t$ and the image is uniquely determined by the relation

$$[s]_{24m} \overline{\odot}\, t \equiv ts + \frac{s-1}{24} \sum_{\delta \mid M} \delta r_\delta \pmod{m}.$$

We define the set

$$P_{m,r}(t) := \{[s]_{24m} \overline{\odot}\, t \mid [s]_{24m} \in \mathbb{S}_{24m}\}.$$

Lemma 1.6. *Let $p \ge 5$ be a prime. Let $r^{(p)} := (r_1^{(p)}, r_p^{(p)}) = (-1, p) \in R(p)$. Then*

$$P_{2p,r^{(p)}}(t) = \left\{ t' \;\middle|\; \left(\frac{24t-1}{p}\right) = \left(\frac{24t'-1}{p}\right), \; t \equiv t' \pmod{2},\ 0 \le t' \le 2p-1 \right\}. \tag{1.15}$$

Proof. First note that

$$\frac{1}{24} \sum_{\delta \mid p} \delta r_\delta^{(p)} = \frac{p^2-1}{24} \in \mathbb{Z}.$$

Let $m = 2p$. If $s_1 \equiv s_2 \pmod{m}$ then $[s_1]_{24m} \overline{\odot}\, t = [s_2]_{24m} \overline{\odot}\, t$ because $(p^2-1)/24$ is an integer. This implies that

$$(1.16)\quad P_{2p,r^{(p)}}(t) = \left\{t' \,\middle|\, t' \equiv ts + (s-1)\frac{p^2-1}{24} \pmod p,\ s \in \mathbb{S}_m,\ 0 \le t \le 2p-1\right\}.$$

We see that

$$(1.17)\qquad P_{2p,r^{(p)}}(t) \pmod 2 = \{t \pmod 2\}.$$

Next we compute $P_{2p,r^{(p)}}(t) \pmod p$. By (1.16) we know

$$(1.18)\quad \begin{aligned} P_{2p,r^{(p)}}(t) &\pmod p \\ &= \left\{t' \pmod p \,\middle|\, t' \equiv ts + (s-1)\frac{p^2-1}{24} \pmod p,\ s \in \mathbb{S}_p\right\} \\ &= \{t' \pmod p \mid 24t'-1 \equiv s(24t-1) \pmod p,\ s \in \mathbb{S}_p\} \\ &= \left\{t' \pmod p \,\middle|\, \left(\frac{24t-1}{p}\right) = \left(\frac{24t'-1}{p}\right)\right\}. \end{aligned}$$

By (1.17) and (1.18) and the Chinese remainder theorem we obtain $P_{2p,r^{(p)}}(t) \pmod{2p}$ and we obtain the formula (1.15) by imposing that the elements of $P_{2p,r^{(p)}}(t)$ lie between 0 and $2p-1$. ∎

We now use Lemma 1.6 to compute $P_{2p,r^{(p)}}(t)$ for $p = 5, 7, 11, 13, 17, 19, 23$ and $t = 2, 7, 2, 2, 11, 2, 11$ below, respectively.

- $p = 5$, $t = 2$. We see that $\left(\frac{24t-1}{p}\right) = \left(\frac{2}{5}\right) = -1$. For $t' \in \{1, 2\}$ we have $\left(\frac{24t'-1}{5}\right) = -1$ and for $t' \in \{0, 3, 4\}$ we have $\left(\frac{24t'-1}{5}\right) \in \{0, 1\}$. This implies that $P_{10,r^{(5)}}(2) \equiv \{1, 2\} \pmod 5$. Since $t \equiv 0 \pmod 2$ we find that $P_{10,r^{(5)}}(2) \equiv 0 \pmod 2$. Hence by Lemma 1.6 we have

$$P_{10,r^{(5)}}(2) = \{2, 1+5\} = \{2, 6\}.$$

- $p = 7$, $t = 7$. We see that $\left(\frac{24t-1}{p}\right) = \left(\frac{-1}{7}\right) = -1$. Now, for $t' \in \{0, 2, 6\}$ we have $\left(\frac{24t'-1}{7}\right) = -1$ (and these are all t' with this property) so $P_{14,r^{(7)}} \equiv \{0, 2, 6\} \pmod 7$. Because $t \equiv 1 \pmod 2$, by Lemma 1.6 we obtain

$$P_{14,r^{(7)}}(7) = \{0+7, 2+7, 6+7\} = \{7, 9, 13\}.$$

- $p = 11$, $t = 2$. Here $\left(\frac{24t-1}{11}\right) = \left(\frac{5^2}{11}\right) = 1$. We see that for $t' \in \{1, 2, 3, 5, 8\}$ we have $\left(\frac{24t'-1}{11}\right) = 1$ so

$$P_{22,r^{(11)}}(2) = \{1+11, 2, 3+11, 5+11, 8\} = \{2, 8, 12, 14, 16\}.$$

- Similarly, by Lemma 1.6 we get

$$\begin{aligned} P_{26,r^{(13)}}(2) &= \{2, 10, 16, 18, 20, 22\}, \\ P_{34,r^{(17)}}(11) &= \{11, 15, 17, 19, 25, 27, 29, 33\}, \end{aligned}$$

$$P_{38,r^{(19)}}(2) = \{2, 8, 10, 20, 24, 28, 30, 32, 34\},$$
$$P_{46,r^{(23)}}(11) = \{11, 15, 21, 23, 29, 31, 35, 39, 41, 43, 45\}.$$

We see immediately from the above that Theorem 1.4 is equivalent to the following.

THEOREM 1.7. *Let $t : \{5, 7, 11, 13, 17, 19, 23\} \to \{2, 7, 11\}$ with $p \mapsto t_p$ be defined by*

$$(t_5, t_7, t_{11}, t_{13}, t_{17}, t_{19}, t_{23}) := (2, 7, 2, 2, 11, 2, 11).$$

Then for all $n \geq 0$, p prime with $5 \leq p \leq 23$, and $t' \in P_{2p,r^{(p)}}(t_p)$, we have

$$a_p(2pn + t') \equiv 0 \pmod{2^{i(p)}}, \tag{1.19}$$

where

$$i(p) = \begin{cases} 1 & \text{if } p = 5, 11, 13, 19, \\ 3 & \text{if } p = 7, 17, 23. \end{cases}$$

For each $r \in R(M)$ we assign a generating function

$$f_r(q) := \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta} = \sum_{n=0}^{\infty} c_r(n) q^n.$$

Given p a prime, $m \in \mathbb{N}$ and $t \in \{0, \ldots, m-1\}$ we are concerned with proving congruences of the type $c_r(mn + t) \equiv 0 \pmod p$, $n \in \mathbb{N}$. The congruences we are concerned with here have some additional structure; namely $a_r(mn + t') \equiv 0 \pmod p$, $n \geq 0$, $t' \in P_{m,r}(t)$. In other words a congruence is a tuple (r, M, m, t, p) with $r \in R(M)$, $m \geq 1$, $t \in \{0, \ldots, m-1\}$ and p a prime such that

$$a_r(mn + t') \equiv 0 \pmod p, \qquad n \geq 0,\ t' \in P(t).$$

Throughout, when we say that $a_r(mn + t) \equiv 0 \pmod p$ we mean that $a_r(mn + t') \equiv 0 \pmod p$ for all $n \geq 0$ and all $t' \in P(t)$. The purpose of this paper is to show the congruences

$$a_p(2pn + t_p) \equiv 0 \pmod 2$$

when $p = 5, 7, 11, 13, 17, 19, 23$ and $t_p = 2, 7, 2, 2, 11, 2, 11$.

In order to accomplish our goal we need a lemma ([11, Lemma 4.5]). We first state it and then explain the terminology.

LEMMA 1.8. *Let u be a positive integer, $(m, M, N, t, r = (r_\delta)) \in \Delta^*$, $a = (a_\delta) \in R(N)$, let n be the number of double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ and $\{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ a complete set of representatives of the double cosets. Assume that $p_{m,r}(\gamma_i) + p_a^*(\gamma_i) \geq 0$, $i \in \{1, \ldots, n\}$. Let $t_{\min} := \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24}\Big(\Big(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\Big)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta a_\delta\Big) - \frac{1}{24m} \sum_{\delta|M} \delta r_\delta - \frac{t_{\min}}{m}.$$

If

$$\sum_{n=0}^{\lfloor \nu \rfloor} c_r(mn+t')q^n \equiv 0 \pmod{u} \qquad \textit{for all } t' \in P_{m,r}(t)$$

then

$$\sum_{n=0}^{\infty} c_r(mn+t')q^n \equiv 0 \pmod{u} \qquad \textit{for all } t' \in P_{m,r}(t).$$

The lemma reduces the proof of a congruence modulo u to checking that finitely many values are divisible by u. We first define the set Δ^*. Let $\kappa = \kappa(m) = \gcd(m^2-1, 24)$ and $\pi(M,(r_\delta)) := (s,j)$ where s is a non-negative integer and j an odd integer uniquely determined by $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$. Then a tuple $(m, M, N, (r_\delta), t)$ belongs to Δ^* iff

- $m \geq 1$, $M \geq 1$, $N \geq 1$, $(r_\delta) \in R(M)$, $t \in \{0,\dots,m-1\}$;
- $p \mid m$ implies $p \mid N$ for every prime p;
- $\delta \mid M$ implies $\delta \mid mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$;
- $\kappa N \sum_{\delta|M} r_\delta mN/\delta \equiv 0 \pmod{24}$;
- $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$;
- $\displaystyle \frac{24m}{\gcd(\kappa(-24t - \sum_{\delta|M} \delta r_\delta), 24m)} \,\Big|\, N$;
- for $(s,j) = \pi(M,(r_\delta))$ we have either $(4 \mid \kappa N$ and $8 \mid Ns)$ or $(2 \mid s$ and $8 \mid N(1-j))$.

Next we need to define the groups $\Gamma, \Gamma_0(N)$ and Γ_∞:

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, N \mid c \right\} \qquad \text{for } N \text{ a positive integer},$$

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \,\middle|\, h \in \mathbb{Z} \right\}.$$

For the index we have $[\Gamma : \Gamma_0(N)] := N \prod_{p|N}(1+p^{-1})$ (see, e.g., [12]).

Finally for $m \geq 1, M \geq 1$, and $r \in R(M)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we define

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,\dots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta(a+\kappa\lambda c), mc)}{\delta m} \tag{1.20}$$

and

$$p_r^*(\gamma) := \frac{1}{24} \sum_{\delta|M} \frac{r_\delta \gcd^2(\delta, c)}{\delta}.$$

2. The congruences. Let $r^{(p)}=(-1,p)$ throughout this section where $p\geq 5$ is a prime. Before we prove the congruences we will show that $p_{2p,r^{(p)}}(\gamma)\geq 0$ for all $\gamma\in \mathrm{SL}_2(\mathbb{Z})$. For $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ we know by (1.20) that

$$\begin{aligned}
&p_{2p,r^{(p)}}(\gamma)\\
&\quad=\min_{\lambda\in\{0,\dots,2p-1\}}\frac{1}{24}\left(-\frac{\gcd^2(a+\kappa\lambda c,2pc)}{2p}+p\frac{\gcd^2(p(a+\kappa\lambda c),2pc)}{2p^2}\right)\\
&\quad=\min_{\lambda\in\{0,\dots,2p-1\}}\frac{1}{24}\left(-\frac{\gcd^2(a+\kappa\lambda c,2pc)}{2p}+p\frac{\gcd^2(a+\kappa\lambda c,2c)}{2}\right)\\
&\quad=\min_{\lambda\in\{0,\dots,2p-1\}}\frac{1}{24}\left(-\frac{\gcd^2(a+\kappa\lambda c,2p)}{2p}+p\frac{\gcd^2(a+\kappa\lambda c,2)}{2}\right).
\end{aligned}$$

The last rewriting follows from $\gcd(a,c)=1$ because $ad-bc=1$. Next we will show that $p_{2p,r^{(p)}}$ is non-negative by proving that

$$F(a,c,p,\lambda):=-\frac{\gcd^2(a+\kappa\lambda c,2p)}{2p}+p\frac{\gcd^2(a+\kappa\lambda c,2)}{2}\geq 0$$

for all integers a,c,p and λ. We split the proof in four cases:

$$\begin{aligned}
\gcd(a+\kappa\lambda c,2p)=1&\Rightarrow F(a,c,p,\lambda)=-\frac{1}{2p}+\frac{p}{2}\geq 0,\\
\gcd(a+\kappa\lambda c,2p)=2&\Rightarrow F(a,c,p,\lambda)=-\frac{2}{p}+2p\geq 0,\\
\gcd(a+\kappa\lambda c,2p)=p&\Rightarrow F(a,c,p,\lambda)=-\frac{p}{2}+\frac{p}{2}=0,\\
\gcd(a+\kappa\lambda c,2p)=2p&\Rightarrow F(a,c,p,\lambda)=-2p+2p=0.
\end{aligned}$$

Because $p_{2p,r^{(p)}}(\gamma)=\min_{\lambda\in\{0,\dots,2p-1\}}\frac{1}{24}F(a,c,p,\lambda)$ we know $p_{2p,r^{(p)}}(\gamma)\geq 0$.

We are now ready to prove the congruences in Theorem 1.4. We start with (1.8):

$$a_5(10n+2,6)\equiv 0 \pmod 2.$$

We apply Lemma 1.8. We see that $(10,5,10,2,r^{(5)}=(-1,5))\in\Delta^*$. We choose the sequence (a_δ) in Lemma 1.8 to be the zero sequence (this will be so for all the congruences in this paper). Because $(a_\delta)\equiv 0$ and because $p_{10,r^{(5)}}(\gamma)\geq 0$ we see that $p_{10,r^{(5)}}(\gamma)+p_a^*(\gamma)\geq 0$ for any $\gamma\in\mathrm{SL}_2(\mathbb{Z})$. Finally

$$\nu=\frac{1}{24}(5-1)(5+1)(2+1)-\frac{1}{10}-\frac{1}{5}=3-\frac{3}{10}.$$

We choose $u=2$ in the lemma and note that $c_{r^{(5)}}(n)=a_5(n)$ for all $n\geq 0$. Then (1.8) is true iff

$$a_5(2)\equiv a_5(12)\equiv a_5(22)\equiv a_5(6)\equiv a_5(16)\equiv a_5(26) \pmod 2.$$

These values of a_5 are all even as can be seen below, so (1.8) is proven.

A similar approach can be used to prove (1.9)–(1.14). In particular let t_p be as in Theorem 1.7 and $r^{(p)} = (-1, p)$. Then

$$(2p, p, 2^{(3-(-1)^{(p-1)/2})/2}p, t_p, r^{(p)}) \in \Delta^*.$$

We again set $(a_\delta) \equiv 0$ and see as before that

$$p_{2p,r^{(p)}}(\gamma) + p_a^*(\gamma) \geq 0$$

for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We further obtain

$$\begin{aligned}\nu = \nu_p &= \frac{1}{24}(p-1)2^{(3-(-1)^{(p-1)/2})/2}p\left(1+\frac{1}{p}\right)\left(1+\frac{1}{2}\right) - \frac{p^2-1}{48p} - \frac{t_p}{2p} \\ &= \frac{1}{8}(p^2-1)2^{(1-(-1)^{(p-1)/2})/2} - \frac{p^2-1}{48p} - \frac{t_p}{2p}.\end{aligned}$$

Putting these values in a table we obtain

p	ν_p	$\lfloor \nu_p \rfloor$
5	$3 - \frac{1}{10} - \frac{2}{10}$	2
7	$12 - \frac{1}{7} - \frac{1}{2}$	11
11	$30 - \frac{5}{22} - \frac{2}{22}$	29
13	$21 - \frac{7}{26} - \frac{2}{26}$	20
17	$36 - \frac{6}{17} - \frac{11}{34}$	35
19	$90 - \frac{15}{38} - \frac{2}{38}$	89
23	$132 - \frac{11}{23} - \frac{11}{46}$	131

We conclude by Lemma 1.8 that for all $n \geq 0$ we have

$$a_p(2pn + t') \equiv 0 \pmod{u}, \qquad t' \in P_{2p,r^{(p)}}(t_p),$$

if, for $0 \leq n \leq \lfloor \nu_p \rfloor$,

$$a_p(2pn + t') \equiv 0 \pmod{u}, \qquad t' \in P_{2p,r^{(p)}}(t_p).$$

In particular we choose $u = 2$ in the case $p = 5, 11, 13, 19$ and $u = 8$ for $p = 7, 17, 23$.

The values of $a_t(n)$ have been calculated in `MAPLE` for $5 \leq t \leq 23$ and we confirm that they satisfy the desired congruences. The authors would be happy to supply this data to anyone interested.

Given that all of these values are congruent to zero modulo 2 (or 8, respectively), Theorem 1.4 is proved.

Acknowledgements. The authors thank the referee for carefully reading the original manuscript and suggesting valuable changes to the paper.

Since the submission of this article, the authors have determined that (1.8), (1.11), and (1.12) of Theorem 1.4 appear in an alternative form in Garvan's work [5]. However, it should be noted that the proof technique of Garvan is different from ours, although both rely significantly on modular forms. Our belief is that our results provide a unified treatment of parity results for t-cores where t is a prime, $5 \leq t \leq 23$.

S. Radu was supported by DK grant W1214-DK6 of the Austrian Science Funds FWF.

References

[1] G. E. Andrews and P. Paule, *MacMahon's partition analysis XI: Broken diamonds and modular forms*, Acta Arith. 126 (2007), 281–294.

[2] M. Boylan, *Congruences for* 2^t*-core partition functions*, J. Number Theory 92 (2002), 131–138.

[3] S. H. Chan, *Some congruences for Andrews–Paule's broken 2-diamond partitions*, Discrete Math. 308 (2008), 5735–5741.

[4] S. Chen, *Arithmetical properties of the number of t-core partitions*, Ramanujan J. 18 (2009), 103–112.

[5] F. Garvan, *Some congruences for partitions that are p-cores*, Proc. London Math. Soc. 66 (1993), 449–478.

[6] M. D. Hirschhorn and J. A. Sellers, *Some amazing facts about 4-cores*, J. Number Theory 60 (1996), 51–69.

[7] —, —, *Two congruences involving 4-cores*, Electron. J. Combin. 3 (1996), no. 2, res. paper R10.

[8] —, —, *Some parity results for 16-cores*, Ramanujan J. 3 (1999), 281–296.

[9] —, —, *On recent congruence results of Andrews and Paule*, Bull. Austral. Math. Soc. 75 (2007), 121–126.

[10] L. W. Kolitsch and J. A. Sellers, *Elementary proofs of infinitely many congruences for 8-cores*, Ramanujan J. 3 (1999), 221–226.

[11] S. Radu, *An algorithmic approach to Ramanujan's congruences*, ibid. 20 (2009), 215–251.

[12] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton Univ. Press, 1971.

Silviu Radu
Research Institute for Symbolic Computation (RISC)
Johannes Kepler University
A-4040 Linz, Austria
E-mail: sradu@risc.uni-linz.ac.at

James A. Sellers
Department of Mathematics
Penn State University
University Park, PA 16802, U.S.A.
E-mail: sellersj@math.psu.edu

Received on 24.12.2009
and in revised form on 21.4.2010 (6247)