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The Rhin–Viola method for $\log 2$

by

RAFFAELE MARCOVECCHIO (Wien)

1. Introduction. In the last two decades several different proofs have been published of Rukhadze's result [Ru] that the transcendental number $\log 2$ has the irrationality measure 3.89139978: see [H1], [HMV], [V], [Br] and the very recent paper [Sa]. Similar results are also given in [A] and [Rh]. Rukhadze's record essentially depends on a method of eliminating common prime factors from all the coefficients of certain polynomials. In his review of the paper [Ru], Bertrand [Be] suggests that it would be interesting to combine this method with the one introduced independently by [Rh] and [DV]. We say that an irrational number α has an *irrationality measure* μ if for all $\varepsilon > 0$ there exists a constant $v_0 = v_0(\varepsilon)$ for which

$$\left|\alpha - \frac{u}{v}\right| > v^{-\mu-\varepsilon}$$

for all integers u and v with $v \geq v_0$. We denote by $\mu(\alpha)$ the least of such μ.

One of the aims of this paper is to improve Rukhadze's result as follows:

THEOREM 1.1.

$$\mu(\log 2) < 3.57455391. \tag{1}$$

The best previously known non-quadraticity measure of $\log 2$ is 25.0463, and was proved by Hata [H3], after [C], [Re] and [So]. See also [AV] for a related approximation measure. We say that a non-quadratic number β has a *non-quadraticity measure* μ_2 if for all $\varepsilon > 0$ there exists a constant $H_0 = H_0(\varepsilon)$ for which

$$|\beta - U| > H(U)^{-\mu_2-\varepsilon}$$

for all quadratic numbers U with $H(U) \geq H_0$. Here, $H(U)$ denotes the *height* of U, i.e. the maximum of the absolute values of the coefficients of its minimal polynomial. We denote by $\mu_2(\beta)$ the least non-quadraticity measure of β. In the present paper we prove

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Theorem 1.2.

$$\mu_2(\log 2) < 15.65142025. \tag{2}$$

The powerful arithmetic method introduced by Rhin and Viola [RV1] in the diophantine study of the constant $\zeta(2)$, and extended by the same authors to $\zeta(3)$ [RV2] and to dilogarithms of some rational numbers [RV3], is also applied by Viola [V] to logarithms of some rational numbers, and by Amoroso and Viola [AV] to logarithms of some algebraic numbers. For example, Amoroso and Viola prove that $|\log 2 - U| > H(U)^{-6.2144}$ when $U \in \mathbb{Q}(\sqrt{2})$ and $H(U)$ is sufficiently large. Our method can be viewed as a two-dimensional variant of that of [V], and presents some analogies with [RV2]. It can be described in three steps.

The first step is to introduce a family of double integrals. Let h, j, k, l, m, q be six non-negative integers satisfying

$$h + j + q = k + l + m, \tag{3}$$

and such that

$$\begin{aligned} l + k - j &= q + h - m \geq 0,\\ h + j - k &= m + l - q \geq 0,\\ k + m - h &= j + q - l \geq 0. \end{aligned} \tag{4}$$

This idea of introducing six instead of five independent parameters is similar to what is done for the group structure of $\zeta(3)$ in [RV2]. Let x be a real number, and suppose $0 < x < 1$. We introduce the following family of double complex integrals:

$$\begin{aligned} I = I(h,j,k,l,m,q;x) &:= x^{\max\{0,q-l,m-h\}}(1-x)^{k+l+m+1} \\ &\times \int_{s=0}^{i\infty}\int_{t=0}^{-i\infty} \frac{s^h t^j\,dt\,ds}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}. \end{aligned} \tag{5}$$

In Section 2 we prove that the real and imaginary parts of the integral I take the form

$$\Re(I) = P(x)\,\frac{1}{2}\log^2(1/x) - Q(x)\log(1/x) + R(x),$$

$$\frac{\Im(I)}{\pi} = P(x)\log(1/x) - Q(x)$$

for some explicitly given polynomials with rational coefficients

$$\begin{aligned} P(x) &= P(h,j,k,l,m,q;x),\\ Q(x) &= Q(h,j,k,l,m,q;x),\\ R(x) &= R(h,j,k,l,m,q;x). \end{aligned}$$

By specializing $x = 1/2$, we see that $\Im(I)/\pi$ is a linear form with rational coefficients in 1 and $\log 2$ which is employed to get the bound (1). Moreover,

$$\frac{\Im(I)}{\pi}\log(1/x) - \Re(I) = P(x)\,\frac{1}{2}\log^2(1/x) - R(x),$$

thus giving simultaneous approximations to $\log(1/x)$ and $\frac{1}{2}\log^2(1/x)$. These are used to get the bound (2). We can also obtain non-quadraticity measures of logarithms of other rational numbers by taking different values of x.

In [H3] Hata introduced another double complex integral having real and imaginary parts of the same type as I. However, in his arithmetic analysis of the polynomials $P(x)$, $Q(x)$ and $R(x)$, the p-adic valuation of binomial coefficients is used, instead of the permutation group method due to Rhin and Viola.

An important feature of our treatment is that we give explicit expressions for the polynomials $P(x)$, $Q(x)$ and $R(x)$. We can do this by combining Sorokin's approach [So] with an idea introduced and developed by Rhin and Viola, which consists in finding a permutation group acting on the set of exponents appearing in the integral. Such a permutation group arises from suitable birational transformations which change an integral into another integral of the same type. Using the changes of variables

$$S = \frac{t}{s}, \quad T = t \quad \text{and} \quad S = s, \quad T = \frac{xs}{t}$$

we show the invariance of the integral $I(h,j,k,l,m,q;x)$ under the action on the set $\{h,j,k,l,m,q\}$ of a suitable permutation group $\mathbf{G}$ of order 6. One of the essential points of this step is to find good upper bounds for the degrees of $P(x)$, $Q(x)$ and $R(x)$. This is obtained by elementary computation of the derivatives of some rational functions. We shall also prove that the polynomial $P(x) = P(h,j,k,l,m,q;x)$ equals the double contour integral defined by

$$\text{(6)} \qquad J = J(h,j,k,l,m,q;x) := x^{\max\{0,q-l,m-h\}}(1-x)^{k+l+m+1} \times \frac{1}{(2\pi i)^2}\oint_{|s|=R}\oint_{|t|=r}\frac{s^h t^j\,dt\,ds}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}$$

for any r and R such that $x < r < R < 1$. This extends a formula of [So]. Again using the above changes of variables we see that $J(h,j,k,l,m,q;x)$ is also invariant under the action of the permutation group $\mathbf{G}$.

The second step is to apply another idea introduced by Rhin and Viola in order to get further arithmetic information on the coefficients of $P(x)$, $Q(x)$ and $R(x)$. We use the Euler integral representation of the Gauss hypergeometric function to show the invariance of

$$\frac{I(h,j,k,l,m,q;x)}{h!j!k!l!m!q!} \quad \text{and} \quad \frac{J(h,j,k,l,m,q;x)}{h!j!k!l!m!q!}$$

under the action of a group $\mathbf{\Phi}$ of 36 permutations on

$$h, j, k, l, m, q, l+k-j, h+j-k, k+m-h.$$

Of course, the group $\mathbf{G}$ is a subgroup of $\mathbf{\Phi}$, and has six left cosets in $\mathbf{\Phi}$. So in Section 3 we find $6-1=5$ non-trivial relations between integrals of the type $I(h_1, j_1, k_1, l_1, m_1, q_1; x), \ldots, I(h_6, j_6, k_6, l_6, m_6, q_6; x)$, where $h_i, \ldots, q_i$ are six suitably chosen integers among $h, j, k, l, m, q, l+k-j, h+j-k, k+m-h$. Such relations provide new information on the polynomials $P(x)$, $Q(x)$ and $R(x)$.

We replace, in each of these integrals, the six integers h, j, k, l, m, q with hn, jn, kn, ln, mn, qn respectively. Putting $I_n = I(hn, jn, kn, ln, mn, qn; x)$, we define P_n, Q_n and R_n accordingly. The third step consists in computing the limit

$$\lim_{n\to\infty} \frac{1}{n} \log P_n$$

and finding an upper bound of

$$\limsup_{n\to\infty} \frac{1}{n} \log |I_n|.$$

Then we can apply Hata's Lemma 2.1 of [H2] for our Theorem 1.1, and Lemma 2.3 of [H3] for our Theorem 1.2. At this point, it is natural to employ Hata's $\mathbb{C}^2$-saddle method [H3] in order to find the asymptotic behaviours of I_n and P_n, related to the three stationary points of the function appearing in the integrals I and J. However, in our arithmetic applications, only an upper bound of $|I_n|$ is needed, and this requires the $\mathbb{C}^2$-saddle method only in a weak version. As for P_n, its asymptotic behaviour can be obtained by the method introduced in the second proof of Lemma 3 of [BR]. Indeed, apart from controlled factors given by powers of x and $1-x$, we can express P_n by a power series with positive coefficients.

Our Theorems 1.1 and 1.2 are obtained by taking the value $x = 1/2$. In Theorem 1.1 the best choice for the parameters is $h = l = 5$, $j = m = 6$, $k = q = 4$, and gives the bound (1). The same choice also gives $\mu_2(\log 2) < 18.4166$.

The simplest choice $h = j = k = l = m = q = 1$ yields Cohen's [C] result $\mu_2(\log 2) < 287.8189$, and also gives the bound $\mu(\log 2) < 5.9382$, worse than Cohen's [C] estimate $\mu(\log 2) < 4.623$.

The choice $h = j = l = 11$, $k = m = 10$, $q = 9$ gives Hata's [H3] bound $\mu_2(\log 2) < 25.0463$.

The choice $h = l = 8$, $j = m = 9$, $k = q = 7$ gives $\mu_2(\log 2) < 15.6695$, and also $\mu(\log 2) < 3.76981$. Our Theorem 1.2 is proved with the choice $h = l = 65$, $j = m = 73$ and $k = q = 57$.

We now consider further examples, taking $x = a/(a+1)$, where a is a positive integer. We recover all the results in Table 1 on p. 4582, and in

Remark 4 on p. 4583 of [H3], by taking $h = j = l = \mu^{-1} + 1$, $k = m = \mu^{-1}$, $q = \mu^{-1} - 1$, where μ is Hata's parameter in [H3]. Improvements on the results of [H3, p. 4582], are given in the following table. All our new irrationality and non-quadraticity measures are obtained when the parameters satisfy

$$0 < k = q < h = l < j = m \quad \text{and} \quad 2h = j + k, \tag{7}$$

so that the non-quadraticity measure obtained for $\log(1 + 1/a)$ actually depends only on a rational parameter $0 < h/j < 1$. The value of this parameter yielding the best non-quadraticity measure seems to be an increasing function of a. Our method does not seem to give new irrationality measures of the logarithms of rational numbers different from 2.

a	h	j	h/j	$\mu_2(\log(1 + 1/a)) <$
1	65	73	0.89041...	15.651421
2	11	12	0.91666...	9.460812
3	29	31	0.93548...	7.902787
4	17	18	0.94444...	7.149533
5	98	103	0.95145...	6.695612
6	23	24	0.95833...	6.385084
7	25	26	0.96153...	6.156797
8	53	55	0.96363...	5.980276
9	29	30	0.96666...	5.838418
10	31	32	0.96875	5.721614
11	65	67	0.97014...	5.623186

2. Double complex integrals. Let h, j, k, l, m be any non-negative integers such that $q = k + l + m - h - j \geq 0$, and let $0 < x < 1$. We consider the double complex integral

$$\int_{s=0}^{\zeta\infty} \int_{t=0}^{\overline{\zeta}\infty} \frac{s^h t^j}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}\, dt\, ds, \tag{8}$$

where

$$\zeta = e^{2\pi i/3} = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2},$$

and the notation for the limits of integration means that the integration paths in s and t are the half-lines going from zero to infinity through the points ζ and $\overline{\zeta}$, respectively.

We claim that the integral (8) converges absolutely and uniformly for x in a neighbourhood of any fixed x_0 with $0 < x_0 < 1$. By the change of variables $s = \zeta X$, $t = \overline{\zeta} Y$, this is equivalent to proving that for any $0 < x < 1$,

$$\int_0^{+\infty}\int_0^{+\infty} \frac{X^h}{\sqrt{X^2+X+1}^{\,l+k-j+1}\sqrt{X^2+XY+Y^2}^{\,h+j-k+1}} \times \frac{Y^j}{\sqrt{Y^2+xY+x^2}^{\,k+m-h+1}}\,dX\,dY$$

is finite. This is seen by splitting this integral into the sum of the integrals over the regions:

(i) $0 \le X \le 1$, $0 \le Y \le 1$;
(ii) $X \ge 1$, $0 \le Y \le 1$;
(iii) $0 \le X \le 1$, $Y \ge 1$;
(iv) $X \ge 1$, $Y \ge 1$.

Over the square (i) the integral is finite since $h \ge 0$, $j \ge 0$ and $k \ge 0$, as is clear by changing to polar coordinates $X = \varrho\cos\vartheta$, $Y = \varrho\sin\vartheta$ and taking $0 \le \vartheta \le \pi/2$, $0 \le \varrho \le R$ for any fixed $R > 0$. Over the strip (ii) we write the integral as

$$\int_0^1 \frac{Y^j\,dY}{\sqrt{Y^2+xY+x^2}^{\,k+m-h+1}} \times \int_1^{+\infty} \frac{X^{-l-2}\,dX}{\sqrt{1+1/X+1/X^2}^{\,l+k-j+1}\sqrt{1+Y/X+Y^2/X^2}^{\,h+j-k+1}},$$

and we see that this is finite since $j \ge 0$ and $l \ge 0$. Similarly, over (iii) we use $h \ge 0$ and $m \ge 0$. For (iv) we put $X = 1/X_1$, $Y = 1/Y_1$ and again we change to polar coordinates $X_1 = \varrho\cos\vartheta$, $Y_1 = \varrho\sin\vartheta$, so that the integral is finite over (iv) since $l \ge 0$, $m \ge 0$ and $q = k+l+m-h-j \ge 0$.

The absolute convergence of (8) implies that we may interchange the integrations in s and t, and by the uniform convergence the derivative of (8) with respect to x equals

$$(9)\qquad (k+m-h+1) \times \int_{s=0}^{\zeta\infty}\int_{t=0}^{\overline{\zeta}\infty} \frac{s^h t^j}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}(t-x)^{k+(m+1)-h+1}}\,dt\,ds,$$

this being an integral of the same type as (8), with m and q changed to $m+1$ and $q+1$, respectively.

We remark that the value of (8) is unchanged if we rotate the integration path $(0, \zeta\infty)$ for s by moving it to the half-line $(0, \eta\infty)$ for any $\eta \in \mathbb{C}$ satisfying $|\eta| = 1$, $\varepsilon \le \arg\eta \le 4\pi/3 - \varepsilon$, with $\varepsilon > 0$ fixed. Indeed, for any fixed $t \in (0, \overline{\zeta}\infty)$ the function

$$\varphi(s) = \frac{s^h}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}}$$

has no poles for $\varepsilon \le \arg s \le 4\pi/3 - \varepsilon$. Thus, by the residue theorem, for any $\varrho > 0$ we get

$$\int_0^{\varrho\zeta} \varphi(s)\, ds = \int_0^{\varrho\eta} \varphi(s)\, ds + \int_{\gamma_\varrho} \varphi(s)\, ds,$$

where γ_ϱ is the arc $\{|s| = \varrho \mid \arg s \text{ from } \arg \eta \text{ to } \arg \zeta = 2\pi/3\}$. As $\varrho \to +\infty$ we have

$$(10) \qquad \Big| \int_{\gamma_\varrho} \varphi(s)\, ds \Big| \le \frac{2\pi \varrho^{h+1}}{(\varrho-1)^{l+k-j+1}(\varrho-|t|)^{h+j-k+1}} = O(\varrho^{-l-1}) \to 0,$$

whence

$$(11) \qquad \int_0^{\zeta\infty} \varphi(s)\, ds = \int_0^{\eta\infty} \varphi(s)\, ds.$$

Similarly, if the integration path $(0, \zeta\infty)$ for s in (8) is fixed, we may move the integration path $(0, \overline{\zeta}\infty)$ for t to the half-line $(0, \overline{\eta}\infty)$, again for any η satisfying $|\eta| = 1$, $\varepsilon \le \arg \eta \le 4\pi/3 - \varepsilon$. We conclude that the integral (8) equals

$$\int_{s=0}^{\eta_1\infty} \int_{t=0}^{\eta_2\infty} \frac{s^h t^j}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}\, dt\, ds$$

for any $\eta_1, \eta_2 \in \mathbb{C}$ satisfying $|\eta_1| = |\eta_2| = 1$, $0 < \arg \eta_1 < \arg \eta_2 < 2\pi$. In particular, (8) equals

$$\int_{s=0}^{i\infty} \int_{t=0}^{-i\infty} \frac{s^h t^j}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}\, dt\, ds.$$

Hence, by (5),

$$(12) \qquad I = I(h,j,k,l,m,q;x) = x^{\max\{0,q-l,m-h\}}(1-x)^{k+l+m+1} \times \int_{s=0}^{\zeta\infty} \int_{t=0}^{\overline{\zeta}\infty} \frac{s^h t^j\, dt\, ds}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}.$$

Similarly, in (6) we may take any r, R such that $x < r < R < 1$, in particular

$r = x^{2/3}$, $R = x^{1/3}$. Therefore,

$$J = J(h,j,k,l,m,q;x) = x^{\max\{0,q-l,m-h\}}(1-x)^{k+l+m+1} \times \frac{1}{(2\pi i)^2} \oint\limits_{|s|=x^{1/3}} \oint\limits_{|t|=x^{2/3}} \frac{s^h t^j \, dt\, ds}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}. \tag{13}$$

Using (12) and (13) we shall prove that the integrals I and J are invariant under the action of a permutation group of order 6 acting on the parameters h, j, k, l, m, q. For any fixed $t \in (0, \overline{\zeta}\infty)$, the involution $s \mapsto S$ defined by $S = t/s$ maps the half-line $(0, \zeta\infty)$ onto itself, and for any fixed t such that $|t| = x^{2/3}$ it maps the circle $|s| = x^{1/3}$ onto itself. Thus, if we make in (12) and (13) the substitution

$$s = T/S, \qquad t = T,$$

which preserves both the integration domains (up to the orientation) and the measure (up to the sign) in the integrals (12) and (13), i.e. satisfies

$$\frac{dt\, ds}{(1-s)(s-t)(t-x)} = -\frac{dT\, dS}{(1-S)(S-T)(T-x)}, \tag{14}$$

we get

$$I(h,j,k,l,m,q;x) = I(l,k,j,h,q,m;x),$$
$$J(h,j,k,l,m,q;x) = J(l,k,j,h,q,m;x).$$

Similarly, for any fixed $s \in (0, \zeta\infty)$ and $0 < x < 1$ the involution $t \mapsto T$ defined by $T = xs/t$ maps $(0, \overline{\zeta}\infty)$ onto itself, and for any fixed s such that $|s| = x^{1/3}$ it maps the circle $|t| = x^{2/3}$ onto itself. Thus with the substitution

$$s = S, \qquad t = xS/T,$$

which also satisfies (14), we get

$$I(h,j,k,l,m,q;x) = I(k,m,h,q,j,l;x),$$
$$J(h,j,k,l,m,q;x) = J(k,m,h,q,j,l;x).$$

This shows that the integrals $I(h,j,k,l,m,q;x)$ and $J(h,j,k,l,m,q;x)$ are invariant under all the permutations belonging to the group

$$\mathbf{G} = \langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle,$$

generated by $\boldsymbol{\sigma} = (h\ l)(j\ k)(m\ q)$ and $\boldsymbol{\tau} = (h\ k)(j\ m)(l\ q)$. The group $\mathbf{G}$ has six elements:

$$\mathbf{G} = \{\boldsymbol{\iota}, \boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\sigma\tau\sigma}, \boldsymbol{\tau\sigma}, \boldsymbol{\sigma\tau}\},$$

where $\boldsymbol{\iota}$ denotes the identity, and $\boldsymbol{\sigma\tau\sigma} = (h\ m)(j\ l)(k\ q)$, $\boldsymbol{\tau\sigma} = (h\ q\ j)(k\ m\ l)$, $\boldsymbol{\sigma\tau} = (h\ j\ q)(k\ l\ m)$ (according to Rhin and Viola's notation, for permutations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ we denote by $\boldsymbol{\beta\alpha}$ the product obtained by applying first $\boldsymbol{\alpha}$ and then $\boldsymbol{\beta}$). Since $\boldsymbol{\sigma\tau\sigma} = \boldsymbol{\tau\sigma\tau}$, we see that $\mathbf{G}$ is isomorphic to the symmetric group $\mathfrak{S}_3$. We remark that the relation (3) is preserved by the group $\mathbf{G}$. In other words, for any $\boldsymbol{\eta} \in \mathbf{G}$ we have $\boldsymbol{\eta}(h) + \boldsymbol{\eta}(j) + \boldsymbol{\eta}(q) = \boldsymbol{\eta}(k) + \boldsymbol{\eta}(l) + \boldsymbol{\eta}(m)$.

Let $a_1 \ge a_2 \ge \cdots \ge a_n$ be a finite sequence of integers, and let $b_1, \ldots, b_n$ be any reordering of $a_1, \ldots, a_n$. We then put $\max\{b_1, \ldots, b_n\} = a_1$ and $\max'\{b_1, \ldots, b_n\} = a_2$.

We define four integers H, K, γ and δ as follows:

$$
\begin{aligned}
H &= \max\{k+l-j, h+j-k, m+k-h\},\\
K &= \max\{[H/2], \max'\{k+l-j, h+j-k, m+k-h\}\},\\
\gamma &= \max\{\max'\{h+j, h+l, k+l\}, \max'\{k+m, k+q, h+q\},\\
&\qquad \max'\{j+m, j+q, l+m\}\},\\
\delta &= \max\{h+j, h+l, k+l, k+m, k+q, h+q, j+m, j+q, l+m\}.
\end{aligned}
\tag{15}
$$

We remark that H, K, γ and δ are invariant under the action of $\mathbf{G}$. Moreover, $0 \le \gamma \le \delta$. In what follows, we assume that (4) holds, so that H and K are also non-negative.

For any $n \in \mathbb{N}$, let $d_n = \operatorname{lcm}\{1, \ldots, n\}$ if $n > 0$, and $d_0 = 1$. We will prove the following

Proposition 2.1. *Let* $0 < x < 1$, *and let* h, j, k, l, m, q *be non-negative integers satisfying* (3). *Suppose that the integers* $k+l-j$, $h+j-k$ *and* $m+k-h$ *are also non-negative. Let* H, K, γ *and* δ *be defined by* (15). *Then the integral* $I(h, j, k, l, m, q; x)$ *defined by* (5) *satisfies*

$$
\begin{aligned}
I(h, j, k, l, m, q; x) = P(x)\,\frac{1}{2}\log^2(1/x) - Q(x)\log(1/x) + R(x)\\
+ \pi i(P(x)\log(1/x) - Q(x))
\end{aligned}
$$

for polynomials $P(x), Q(x), R(x)$ *such that*

$$
\deg P, \deg Q \le \gamma, \quad \deg R \le \delta \quad \textit{and} \quad P(x), d_H Q(x), d_H d_K R(x) \in \mathbb{Z}[x].
$$

Moreover, *the polynomial* $P(x)$ *equals the integral* $J(h, j, k, l, m, q; x)$ *defined by* (6).

We need some lemmas.

Lemma 2.1. *Up to applying a suitable permutation in the group* $\mathbf{G}$, *we may suppose*

$$
m \ge q \quad \textit{and} \quad j \ge l. \tag{16}
$$

Proof. We claim that at least one of the following conditions holds:

(i) $m \ge q$ and $j \ge l$;
(ii) $k \ge h$ and $q \ge m$;
(iii) $l \ge j$ and $h \ge k$.

Suppose, on the contrary, that (i), (ii) and (iii) are all false. Since (i) is false, we distinguish two cases:

First case. If $m < q$, then $k < h$, because (ii) is false. It follows that $m + k < q + h$. Using (3) we have $j < l$. Then (iii) is true.

Second case. If $j < l$, then $h < k$, because (iii) is false. It follows that $j + h < l + k$, that is, $m < q$. Then (ii) is true.

The lemma follows, because $\boldsymbol{\sigma}$ interchanges (i) and (ii), and $\boldsymbol{\tau}$ interchanges (ii) and (iii). ∎

Owing to (3) and (16),

$$h + j \geq h + l, \quad h + j \geq k + l, \quad k + m \geq k + q, \quad k + m \geq h + q,$$
$$j + m \geq j + q, \quad j + m \geq l + m.$$

So in this case we have

$$\begin{aligned} \gamma &= \max\{h + l, k + l, k + q, h + q, j + q, l + m\}, \\ \delta &= \max\{h + j, k + m, j + m\}. \end{aligned} \tag{17}$$

We define

$$E_1 = k + l - j,\ E_2 = h,\ E_3 = h + j - k,\ E_4 = j,\ E_5 = m + k - h. \tag{18}$$

With this notation we have $E_1, \ldots, E_5 \geq 0$, and

$$H = \max\{E_1, E_3, E_5\}, \qquad K = \max\{[H/2], \max{}'\{E_1, E_3, E_5\}\}.$$

The four non-negative integers k, l, m, q are equal to the integers

$$E_2 + E_4 - E_3, \quad E_1 + E_3 - E_2, \quad E_3 + E_5 - E_4, \quad E_1 + E_3 + E_5 - E_2 - E_4,$$

respectively, which therefore are all non-negative. Moreover, the inequalities $m \geq q$ and $j \geq l$ in (16) are equivalent to

$$E_1 \leq E_2 \quad \text{and} \quad E_1 + E_3 \leq E_2 + E_4, \tag{19}$$

respectively.

We shall use the notation

$$(f(x))^{[n]} := \frac{1}{n!} \frac{d^n}{dx^n}(f(x)).$$

We also denote by

$$\operatorname{ord} f(x)$$

the order of vanishing of $f(x)$ at $x = 0$.

In Lemmas 2.2–2.5 we extend Sorokin's method [So].

Lemma 2.2. *Let F be a non-negative integer; let $g(x) = A(x)/(1 - x)^{F+1}$ for a polynomial $A(x) \in \mathbb{Z}[x]$. Then for any $n \in \mathbb{N}$ we have $(g(x))^{[n]} = A_1(x)/(1 - x)^{F+n+1}$ with a suitable polynomial $A_1(x) \in \mathbb{Z}[x]$ satisfying* $\deg A_1 \leq \deg A$ *and* $\operatorname{ord} A_1 \geq \max\{0, \operatorname{ord} A - n\}$.

Proof. We consider a function $h(x) = x^m/(1-x)^{F+1}$, with $m_1 := \operatorname{ord} A \le m \le m_2 := \deg A$. Then, by Leibniz's formula,

$$(h(x))^{[n]} = \sum_{r=0}^{\min\{m,n\}} \binom{m}{r} x^{m-r} \binom{F+n-r}{F} \frac{1}{(1-x)^{F+n-r+1}} = \frac{B_m(x)}{(1-x)^{F+n+1}},$$

where

$$B_m(x) = \sum_{r=0}^{\min\{m,n\}} \binom{m}{r}\binom{F+n-r}{F} x^{m-r}(1-x)^r,$$

so that $B_m(x) \in \mathbb{Z}[x]$, $\deg B_m \le m \le \deg A$ and $\operatorname{ord} B_m \ge m - \min\{m,n\} = \max\{0, m-n\} \ge \operatorname{ord} A - n$.

If $A(x) = c_{m_1}x^{m_1} + c_{m_1+1}x^{m_1+1} + \cdots + c_{m_2}x^{m_2}$, with $c_{m_1}, c_{m_1+1}, \ldots, c_{m_2} \in \mathbb{Z}$, the lemma follows with $A_1(x) = c_{m_1}B_{m_1}(x) + c_{m_1+1}B_{m_1+1}(x) + \cdots + c_{m_2}B_{m_2}(x)$. ■

LEMMA 2.3. *Let h, j, k, l, m, q be non-negative integers satisfying* (3) *and* (4), *but not necessarily* (16). *Let $E_1, \ldots, E_5$ be defined by* (18). *Then*

$$P^*(x) := (1-x)^{E_1+E_3+E_5+1}\left(x^{E_4}\left(x^{E_2}\left(\frac{1}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5]} \in \mathbb{Z}[x],$$

and $\deg P^* \le \min\{E_1+E_3, E_1+E_4\}$.

Proof. Dividing the polynomial x^{E_2} by $(1-x)^{E_1+1}$, we find two polynomials $A_0, B_0 \in \mathbb{Z}[x]$ satisfying $\deg A_0 \le E_1$ and $\deg B_0 < E_2 - E_1$ (here and in what follows, we use the convention $\deg 0 = -\infty$) such that

$$x^{E_2}\left(\frac{1}{1-x}\right)^{[E_1]} = \frac{x^{E_2}}{(1-x)^{E_1+1}} = \frac{A_0(x)}{(1-x)^{E_1+1}} + B_0(x).$$

Since $E_1 + E_3 \ge E_2$, we have $(B_0(x))^{[E_3]} = 0$. Hence, by Lemma 2.2 with $A(x) = A_0(x)$, $F = E_1$ and $n = E_3$,

$$\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]} = \frac{A_1(x)}{(1-x)^{E_1+E_3+1}}, \tag{20}$$

where $A_1 \in \mathbb{Z}[x]$ and $\deg A_1 \le E_1$. Dividing $x^{E_4}A_1(x)$ by $(1-x)^{E_1+E_3+1}$ we get

$$x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]} = \frac{A_2(x)}{(1-x)^{E_1+E_3+1}} + B_1(x)$$

for some $A_2, B_1 \in \mathbb{Z}[x]$ with $\deg A_2 \le \min\{E_1+E_3, E_1+E_4\}$ and $\deg B_1 < (E_1+E_4) - (E_1+E_3) = E_4 - E_3$. As above, since $E_3 + E_5 \ge E_4$

we have $(B_1(x))^{[E_5]} = 0$. Then, by Lemma 2.2,

$$(21)\qquad \left(x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\right)^{[E_5]} = \frac{P^*(x)}{(1-x)^{E_1+E_3+E_5+1}}$$

with $P^* \in \mathbb{Z}[x]$ and $\deg P^* \le \min\{E_1+E_3, E_1+E_4\}$. ∎

Lemma 2.4. *Let h,j,k,l,m,q be non-negative integers satisfying* (3), (4) *and* (16). *Let $E_1,\dots,E_5$ be defined by* (18). *Then*

$$x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}}(1-x)^{E_1+E_3+E_5+1} \times \left(x^{E_4}\left(x^{E_2}\left(\frac{\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5]} = P(x)\log(1/x) - Q(x)$$

with

$$(22)\qquad P(x) = x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}}P^*(x) \in \mathbb{Z}[x],$$

where P^* is the polynomial in Lemma 2.3, *and $Q(x)$ satisfies*

$$d_H Q(x) \in \mathbb{Z}[x].$$

Moreover,

$$\deg P \le \min\{E_1+E_3, E_1+E_4\} + \max\{0, E_5-E_4, E_3+E_5-E_2-E_4\},$$
$$\deg Q \le \max\{E_1+E_3, E_1+E_4\} + \max\{0, E_5-E_4, E_3+E_5-E_2-E_4\},$$

whence $\deg P, \deg Q \le \gamma$.

Remark 2.1. Owing to (3), $E_1+E_3+E_5 = h+j+q = k+l+m$. Since $\max\{a_1,a_2\} + \max\{b_1,b_2,b_3\} = \max_{i=1,2,\,j=1,2,3} a_i + b_j$, by (17) we have

$$\max\{E_1+E_3, E_1+E_4\} + \max\{0, E_5-E_4, E_3+E_5-E_2-E_4\} = \max\{h+l, k+l, k+q, h+q, j+q, l+m\} = \gamma.$$

Proof of Lemma 2.4. By Leibniz's formula we obtain, for any $f(x)$ and for any integer $E \ge 0$,

$$(f(x)\log(1/x))^{(E)} = (f(x))^{(E)}\log(1/x) + \sum_{r=1}^{E}\binom{E}{r}(\log(1/x))^{(r)}(f(x))^{(E-r)},$$

whence, dividing by $E!$, we obtain

$$(23)\qquad (f(x)\log(1/x))^{[E]} = (f(x))^{[E]}\log(1/x) + \sum_{r=1}^{E}\frac{(-1)^r}{rx^r}(f(x))^{[E-r]}.$$

We apply the last formula with $f(x) = 1/(1-x)$ and $E = E_1$:

$$\left(\frac{\log(1/x)}{1-x}\right)^{[E_1]} = \frac{\log(1/x)}{(1-x)^{E_1+1}} + \sum_{r=1}^{E_1}\frac{(-1)^r}{r}\,\frac{x^{-r}}{(1-x)^{E_1-r+1}}.$$

We now multiply by x^{E_2}, and apply (23) again, with $f(x) = x^{E_2}/(1-x)^{E_1+1}$ and $E = E_3$:

$$\left(x^{E_2}\left(\frac{\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]} = \left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\log(1/x) + \sum_{r=1}^{E_1}\frac{(-1)^r}{r}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3]} + \sum_{r=1}^{E_3}\frac{(-1)^r}{r}\,x^{-r}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]}.$$

We multiply by x^{E_4}, and once again apply (23) with $E = E_5$ and

$$f(x) = x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}$$

to obtain

$$\begin{aligned}(24)\quad &\left(x^{E_4}\left(x^{E_2}\left(\frac{\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5]}\\ &= \left(x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\right)^{[E_5]}\log(1/x)\\ &\quad + \sum_{r=1}^{E_1}\frac{(-1)^r}{r}\left(x^{E_4}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3]}\right)^{[E_5]}\\ &\quad + \sum_{r=1}^{E_3}\frac{(-1)^r}{r}\left(x^{E_4-r}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]}\right)^{[E_5]}\\ &\quad + \sum_{r=1}^{E_5}\frac{(-1)^r}{rx^r}\left(x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\right)^{[E_5-r]}\\ &= \frac{P^*(x)\log(1/x)}{(1-x)^{E_1+E_3+E_5+1}} + S_1 + S_3 + S_5,\end{aligned}$$

say. Here P^* is exactly the polynomial in Lemma 2.3. Note that $S_i = 0$ if $E_i = 0$. The rest of the proof is similar to the proof of Lemma 2.3.

Since $E_2 \geq E_1$ by (19), in the sum S_1, for all $r = 1, \ldots, E_1$, the exponent $E_2 - r$ is non-negative. We repeat the argument given in the proof of Lemma 2.3, with E_1 and E_2 replaced by $E_1 - r$ and $E_2 - r$, respectively. Therefore, similarly to (21),

$$\left(x^{E_4}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3]}\right)^{[E_5]} = \frac{A_3(x)}{(1-x)^{E_1+E_3+E_5-r+1}}$$

with $A_3 \in \mathbb{Z}[x]$ and $\deg A_3 \leq \min\{E_1 + E_3, E_1 + E_4\} - r$. Hence

$d_H(1-x)^{E_1+E_3+E_5+1}S_1$ is a polynomial with integer coefficients, and degree $\le \min\{E_1+E_3, E_1+E_4\}$.

In the sum S_3, for each $r = 1,\dots,E_3$ we apply Lemma 2.2 with $A(x) = x^{E_2}$, $F = E_1$ and $n = E_3 - r$. Thus

$$\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]} = \frac{A_4(x)}{(1-x)^{E_1+E_3-r+1}} \tag{25}$$

with $A_4 \in \mathbb{Z}[x]$, $\deg A_4 \le E_2$ and $\operatorname{ord} A_4 \ge E_2 - E_3 + r$. Even if $E_4 - r$ may be negative, we see that $x^{E_4-r}A_4(x)$ is a polynomial, since $(E_4 - r) + (E_2 - E_3 + r) = E_2 + E_4 - E_3 = k \ge 0$. Then, dividing $x^{E_4-r}A_4(x)$ by $(1-x)^{E_1+E_3-r+1}$, we get

$$x^{E_4-r}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]} = \frac{A_5(x)}{(1-x)^{E_1+E_3-r+1}} + B_2(x),$$

with $\deg A_5 \le E_1 + E_3 - r$, and $\deg B_2 < (E_2 + E_4 - r) - (E_1 + E_3 - r) = E_2 + E_4 - E_1 - E_3$. Since $E_1 + E_3 + E_5 \ge E_2 + E_4$, again by Lemma 2.2 we have

$$\left(x^{E_4-r}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]}\right)^{[E_5]} = \frac{A_6(x)}{(1-x)^{E_1+E_3-r+E_5+1}}$$

with $A_6 \in \mathbb{Z}[x]$ and $\deg A_6 \le E_1 + E_3 - r$. Thus, $d_H(1-x)^{E_1+E_3+E_5+1}S_3$ is a polynomial with integer coefficients, and degree $\le E_1 + E_3$.

For S_5, if we apply Lemma 2.2 with $A(x) = x^{E_2}$, $F = E_1$ and $n = E_3$, we see that the polynomial $A_1(x)$ in (20) satisfies $\operatorname{ord} A_1 \ge \max\{0, E_2 - E_3\}$. But in the proof of Lemma 2.3 we found that $\deg A_1 \le E_1$. Hence, multiplying (20) by x^{E_4} and then applying Lemma 2.2 with $A(x) = x^{E_4}A_1(x)$, $F = E_1 + E_3$ and $n = E_5 - r$ for each $r = 1,\dots,E_5$, we obtain

$$\left(x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\right)^{[E_5-r]} = \frac{A_7(x)}{(1-x)^{E_1+E_3+E_5-r+1}}$$

with $\deg A_7 \le E_1 + E_4$, and $\operatorname{ord} A_7 \ge \max\{0, E_4 - E_5 + r, E_2 + E_4 - E_3 - E_5 + r\} \ge r - \min\{E_5 - E_4, E_3 + E_5 - E_2 - E_4\}$. It follows that $d_H x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}}(1-x)^{E_1+E_3+E_5+1}S_5$ is a polynomial with integer coefficients and degree $\le E_1+E_4+\max\{0, E_5-E_4, E_3+E_5-E_2-E_4\}$. This concludes the proof of the lemma, with

$$\begin{aligned} Q(x) = {} & -x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}} \\ & \times (1-x)^{E_1+E_3+E_5+1}(S_1+S_3+S_5). \ \blacksquare \end{aligned} \tag{26}$$

LEMMA 2.5. *Let h, j, k, l, m, q be non-negative integers satisfying* (3), (4) *and* (16). *Let $E_1,\dots,E_5$ be defined by* (18). *Then*

$$x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}}(1-x)^{E_1+E_3+E_5+1} \times \left(x^{E_4}\left(x^{E_2}\left(\frac{\frac12\log^2(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5]} = P(x)\,\frac12\log^2(1/x) - Q(x)\log(1/x) + R(x),$$

where $P(x)$ *and* $Q(x)$ *are the polynomials in Lemma* 2.4, *and the polynomial* $R(x)$ *satisfies* $d_H d_K R(x) \in \mathbb{Z}[x]$ *and*

$$\deg R \le E_2 + E_4 + \max\{0, E_5 - E_4, E_3 + E_5 - E_2 - E_4\} = \delta.$$

REMARK 2.2. By (17) we have

$$E_2 + E_4 + \max\{0, E_5 - E_4, E_3 + E_5 - E_2 - E_4\} = \max\{E_2 + E_4, E_2 + E_5, E_3 + E_5\} = \max\{h + j, k + m, j + m\} = \delta.$$

Proof of Lemma 2.5. As in the proof of Lemma 2.4, we successively apply formula (23) with

$$f(x) = \frac{\frac12\log(1/x)}{1-x},\quad x^{E_2}\left(\frac{\frac12\log(1/x)}{1-x}\right)^{[E_1]},\quad x^{E_4}\left(x^{E_2}\left(\frac{\frac12\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}$$

and $E = E_1$, E_3, E_5 respectively. We obtain

$$\begin{aligned}\left(x^{E_4}\left(x^{E_2}\left(\frac{\frac12\log^2(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5]} &= \left(x^{E_4}\left(x^{E_2}\left(\frac{\frac12\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5]}\log(1/x)\\ &\quad+\sum_{r=1}^{E_1}\frac{(-1)^r}{r}\left(x^{E_4}\left(x^{E_2-r}\left(\frac{\frac12\log(1/x)}{1-x}\right)^{[E_1-r]}\right)^{[E_3]}\right)^{[E_5]}\\ &\quad+\sum_{r=1}^{E_3}\frac{(-1)^r}{r}\left(x^{E_4-r}\left(x^{E_2}\left(\frac{\frac12\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3-r]}\right)^{[E_5]}\\ &\quad+\sum_{r=1}^{E_5}\frac{(-1)^r}{rx^r}\left(x^{E_4}\left(x^{E_2}\left(\frac{\frac12\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5-r]}.\end{aligned}$$

By (24), the first term is

$$\frac{P^*(x)\frac12\log^2(1/x)}{(1-x)^{E_1+E_3+E_5+1}} + \frac12(S_1 + S_3 + S_5)\log(1/x).$$

We apply the same process to each of the three remaining sums. For each $r = 1, \ldots, E_1$ we may apply (24) with E_1, E_2 replaced by $E_1 - r, E_2 - r$ respectively. Thus we get

$$\begin{aligned}
&\sum_{r=1}^{E_1}\frac{(-1)^r}{r}\left(x^{E_4}\left(x^{E_2-r}\left(\frac{\frac{1}{2}\log(1/x)}{1-x}\right)^{[E_1-r]}\right)^{[E_3]}\right)^{[E_5]}\\
&\qquad=\frac{1}{2}\sum_{r=1}^{E_1}\frac{(-1)^r}{r}\left(x^{E_4}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3]}\right)^{[E_5]}\log(1/x)\\
&\qquad\quad+\frac{1}{2}\sum_{r=1}^{E_1-1}\frac{(-1)^r}{r}\sum_{s=1}^{E_1-r}\frac{(-1)^s}{s}\left(x^{E_4}\left(\frac{x^{E_2-r-s}}{(1-x)^{E_1-r-s+1}}\right)^{[E_3]}\right)^{[E_5]}\\
&\qquad\quad+\frac{1}{2}\sum_{r=1}^{E_1}\frac{(-1)^r}{r}\sum_{s=1}^{E_3}\frac{(-1)^s}{s}\left(x^{E_4-s}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3-s]}\right)^{[E_5]}\\
&\qquad\quad+\frac{1}{2}\sum_{r=1}^{E_1}\frac{(-1)^r}{r}\sum_{s=1}^{E_5}\frac{(-1)^s}{sx^s}\left(x^{E_4}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3]}\right)^{[E_5-s]}\\
&\qquad=\frac{1}{2}S_1\log(1/x)+T_1+T_{13}+T_{15},
\end{aligned}$$

say. Similarly

$$\begin{aligned}
&\sum_{r=1}^{E_3}\frac{(-1)^r}{r}\left(x^{E_4-r}\left(x^{E_2}\left(\frac{\frac{1}{2}\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3-r]}\right)^{[E_5]}\\
&=\frac{1}{2}\sum_{r=1}^{E_3}\frac{(-1)^r}{r}\left(x^{E_4-r}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]}\right)^{[E_5]}\log(1/x)\\
&\quad+\frac{1}{2}\sum_{r=1}^{E_3}\frac{(-1)^r}{r}\sum_{s=1}^{E_1}\frac{(-1)^s}{s}\left(x^{E_4-r}\left(\frac{x^{E_2-s}}{(1-x)^{E_1-s+1}}\right)^{[E_3-r]}\right)^{[E_5]}\\
&\quad+\frac{1}{2}\sum_{r=1}^{E_3-1}\frac{(-1)^r}{r}\sum_{s=1}^{E_3-r}\frac{(-1)^s}{s}\left(x^{E_4-r-s}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r-s]}\right)^{[E_5]}\\
&\quad+\frac{1}{2}\sum_{r=1}^{E_3}\frac{(-1)^r}{r}\sum_{s=1}^{E_5}\frac{(-1)^s}{sx^s}\left(x^{E_4-r}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]}\right)^{[E_5-s]}\\
&=\frac{1}{2}S_3\log(1/x)+T_{13}+T_3+T_{35},
\end{aligned}$$

say, and

$$\begin{aligned}
&\sum_{r=1}^{E_5}\frac{(-1)^r}{rx^r}\left(x^{E_4}\left(x^{E_2}\left(\frac{\frac{1}{2}\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5-r]}\\
&\qquad=\frac{1}{2}\sum_{r=1}^{E_5}\frac{(-1)^r}{rx^r}\left(x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\right)^{[E_5-r]}\log(1/x)
\end{aligned}$$

$$+\frac{1}{2}\sum_{r=1}^{E_5}\frac{(-1)^r}{rx^r}\sum_{s=1}^{E_1}\frac{(-1)^s}{s}\left(x^{E_4}\left(\frac{x^{E_2-s}}{(1-x)^{E_1-s+1}}\right)^{[E_3]}\right)^{[E_5-r]}$$

$$+\frac{1}{2}\sum_{r=1}^{E_5}\frac{(-1)^r}{rx^r}\sum_{s=1}^{E_3}\frac{(-1)^s}{s}\left(x^{E_4-s}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-s]}\right)^{[E_5-r]}$$

$$+\frac{1}{2}\sum_{r=1}^{E_5-1}\frac{(-1)^r}{rx^r}\sum_{s=1}^{E_5-r}\frac{(-1)^s}{sx^s}\left(x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\right)^{[E_5-r-s]}$$

$$=\frac{1}{2}S_5\log(1/x)+T_{15}+T_{35}+T_5,$$

say. Note that $T_i=0$ if $E_i=0$ or $E_i=1$, and $T_{ij}=0$ if $E_i=0$ or $E_j=0$.

In the double sum T_1 we set $r+s=t$. We obtain

$$T_1=\frac{1}{2}\sum_{t=2}^{E_1}(-1)^t\left(x^{E_4}\left(\frac{x^{E_2-t}}{(1-x)^{E_1-t+1}}\right)^{[E_3]}\right)^{[E_5]}\sum_{s=1}^{t-1}\frac{1}{s(t-s)}.$$

Moreover, for all $t=2,\dots,E_1$, we see that

$$\frac{1}{2}\sum_{s=1}^{t-1}\frac{1}{s(t-s)}=\frac{1}{2t}\sum_{s=1}^{t-1}\left(\frac{1}{s}+\frac{1}{t-s}\right)=\frac{1}{2t}\sum_{s=1}^{t-1}\frac{1}{s}+\frac{1}{2t}\sum_{s=1}^{t-1}\frac{1}{t-s}=\frac{1}{t}\sum_{s=1}^{t-1}\frac{1}{s}.$$

A similar treatment can be made for the sums T_3 and T_5. In conclusion,

$$\left(x^{E_4}\left(x^{E_2}\left(\frac{\frac{1}{2}\log^2(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5]} \tag{27}$$

$$=\frac{P^*(x)\frac{1}{2}\log^2(1/x)}{(1-x)^{E_1+E_3+E_5+1}}+(S_1+S_3+S_5)\log(1/x)$$

$$+T_1+T_3+T_5+2(T_{13}+T_{15}+T_{35}),$$

where

$$T_1=\sum_{r=2}^{E_1}\sum_{s=1}^{r-1}\frac{(-1)^r}{rs}\left(x^{E_4}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3]}\right)^{[E_5]},$$

$$T_3=\sum_{r=2}^{E_3}\sum_{s=1}^{r-1}\frac{(-1)^r}{rs}\left(x^{E_4-r}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]}\right)^{[E_5]},$$

$$T_5=\sum_{r=2}^{E_5}\sum_{s=1}^{r-1}\frac{(-1)^r}{rsx^r}\left(x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\right)^{[E_5-r]},$$

$$2T_{13}=\sum_{r=1}^{E_1}\sum_{s=1}^{E_3}\frac{(-1)^{r+s}}{rs}\left(x^{E_4-s}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3-s]}\right)^{[E_5]},$$

$$2T_{15} = \sum_{r=1}^{E_1}\sum_{s=1}^{E_5} \frac{(-1)^{r+s}}{rsx^s}\left(x^{E_4}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3]}\right)^{[E_5-s]},$$

$$2T_{35} = \sum_{r=1}^{E_3}\sum_{s=1}^{E_5} \frac{(-1)^{r+s}}{rsx^s}\left(x^{E_4-r}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]}\right)^{[E_5-s]}.$$

The sums T_1, T_3 and T_5 can be treated as the sums S_1, S_3 and S_5 in the proof of Lemma 2.4. Note that for all $1 \le s < r \le E_i$ with $i = 1,3,5$, we have

$$\frac{1}{rs}\, d_{E_i} d_{[E_i/2]} \in \mathbb{N}.$$

Indeed, let $\lambda = \gcd(r,s)$. Since $\lambda \le s$ and $\lambda \le r-s$, we have $2\lambda \le s+(r-s) = r \le E_i$, whence $\lambda^{-1} d_{[E_i/2]} \in \mathbb{N}$. If $\mu,\nu \in \mathbb{Z}$ satisfy $\lambda = \mu r + \nu s$ then

$$\frac{1}{rs} = \frac{1}{\lambda}\left(\frac{\mu}{s} + \frac{\nu}{r}\right).$$

Therefore

$$\frac{1}{rs}\, d_{E_i} d_{[E_i/2]} = \frac{1}{\lambda}\, d_{[E_i/2]}\left(\frac{\mu}{s}\, d_{E_i} + \frac{\nu}{r}\, d_{E_i}\right) \in \mathbb{Z}.$$

Since $H = \max\{E_1, E_3, E_5\}$, by the same argument used for S_1, S_3, S_5 in Lemma 2.4 and by Remark 2.1 we see that

$$d_H d_{[H/2]} x^{\max\{0, E_5-E_4, E_3+E_5-E_2-E_4\}}(1-x)^{E_1+E_3+E_5+1}(T_1+T_3+T_5)$$

is a polynomial with integer coefficients and degree $\le \gamma \le \delta$.

As for $2T_{13}$, owing to (19), we may repeat the argument given for S_3 in the proof of Lemma 2.4, with E_1 and E_2 replaced by $E_1 - r$ and $E_2 - r$, respectively. We get

$$\left(x^{E_4-s}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3-s]}\right)^{[E_5]} = \frac{A_8(x)}{(1-x)^{E_1+E_3+E_5-r-s+1}}$$

with $\deg A_8 \le E_1 + E_3 - r - s$. Therefore $d_H d_K (1-x)^{E_1+E_3+E_5+1} 2T_{13}$ is a polynomial with integer coefficients and degree $\le E_1 + E_3 \le E_2 + E_4$, again by (19).

Concerning $2T_{15}$, we may apply the argument given for S_5, with E_1 and E_2 replaced by $E_1 - r$ and $E_2 - r$, respectively. We have

$$\left(x^{E_4}\left(\frac{x^{E_2-r}}{(1-x)^{E_1-r+1}}\right)^{[E_3]}\right)^{[E_5-s]} = \frac{A_9(x)}{(1-x)^{E_1+E_3+E_5-r-s+1}}$$

with $\deg A_9 \le E_1 + E_4 - r$ and $\operatorname{ord} A_9 \ge s - (E_5 - E_4)$. Hence, by (19) and by Remark 2.2, $d_H d_K x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}}(1-x)^{E_1+E_3+E_5+1} 2T_{15}$ is a polynomial with integer coefficients and degree $\le E_1 + E_4 + \max\{0, E_5 - E_4, E_3 + E_5 - E_2 - E_4\} \le \gamma \le \delta$.

On the other hand, for $2T_{35}$, we multiply (25) by x^{E_4-r} and then we apply Lemma 2.2. Thus, for all $r=1,\ldots,E_3$ and $s=1,\ldots,E_5$,

$$\left(x^{E_4-r}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3-r]}\right)^{[E_5-s]}=\frac{A_{10}(x)}{(1-x)^{E_1+E_3+E_5-r-s+1}}$$

with $\deg A_{10}\le E_2+E_4-r$ and $\operatorname{ord}A_{10}\ge(E_2+E_4-E_3)-(E_5-s)=s-(E_3+E_5-E_2-E_4)$. We conclude that $d_H d_K x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}}\times(1-x)^{E_1+E_3+E_5+1}2T_{35}$ is a polynomial with integer coefficients and with degree $\le\delta$.

The lemma follows from (26) and (27). ∎

In the following lemma we find the values of I and J in the simplest case.

Lemma 2.6. *For all* $0<x<1$ *we have*

$$I(0,0,0,0,0,0;x)=\frac{1}{2}\log^2(1/x)+i\pi\log(1/x),$$

and $J(0,0,0,0,0,0;x)=1$, *so that the conclusion of Proposition* 2.1 *holds for* $h=j=k=l=m=q=0$.

Proof. For brevity we write

$$K(x)=\frac{I(0,0,0,0,0,0;x)}{1-x}.$$

By (11), we have

$$K(x)=\int_{s=0}^{i\infty}\int_{t=0}^{-i\infty}\frac{dt\,ds}{(1-s)(s-t)(t-x)}=\int_{s=0}^{-\infty}\int_{t=0}^{-i\infty}\frac{dt\,ds}{(1-s)(s-t)(t-x)}.$$

Hence

$$\overline{K(x)}=\int_{s=0}^{-i\infty}\int_{t=0}^{i\infty}\frac{dt\,ds}{(1-s)(s-t)(t-x)}=\int_{s=0}^{-\infty}\int_{t=0}^{i\infty}\frac{dt\,ds}{(1-s)(s-t)(t-x)}.$$

Using the inequality similar to (10), for the integral over a large half-circle $\{|t|=\rho,\ -\pi/2\le\arg t\le\pi/2\}$, and applying the residue theorem, we see that for any fixed $s\in(0,-\infty)$ we may rotate the t-half-line $(0,-i\infty)$ to $(0,i\infty)$ in the positive direction. We get

$$\int_0^{-i\infty}\frac{dt}{(s-t)(t-x)}-\int_0^{i\infty}\frac{dt}{(s-t)(t-x)}=2\pi i\ \operatorname*{Res}_{t=x}\ \frac{1}{(s-t)(t-x)}=\frac{2\pi i}{s-x}.$$

Therefore

$$K(x) = \int_{s=0}^{-\infty} \int_{t=0}^{i\infty} \frac{dt\, ds}{(1-s)(s-t)(t-x)} + 2\pi i \int_{0}^{-\infty} \frac{ds}{(1-s)(s-x)}$$
$$= \overline{K(x)} + 2\pi i\, \frac{\log(1/x)}{1-x}.$$

Hence

$$\Im(K(x)) = \frac{K(x) - \overline{K(x)}}{2i} = \pi\, \frac{\log(1/x)}{1-x}.$$

On the other hand, writing $\frac{1}{(1-s)(s-t)} = \frac{1}{1-t}\big(\frac{1}{1-s} + \frac{1}{s-t}\big)$ and integrating with respect to s from 0 to $-\infty$, we see that

$$K(x) = \int_{0}^{-i\infty} \frac{\log(1/t)}{(1-t)(t-x)}\, dt,$$

where $\log(1/t) = \log(1/|t|) + i\pi/2$. Similarly, writing $\frac{1}{(s-t)(t-x)} = \frac{1}{s-x}\big(\frac{1}{s-t} + \frac{1}{t-x}\big)$ and integrating with respect to t from 0 to $i\infty$, we see that

$$\overline{K(x)} = \int_{0}^{-i\infty} \frac{\log(s/x)}{(1-s)(s-x)}\, ds,$$

where $\log(s/x) = \log(|s|/x) - i\pi/2$. It follows that

$$\Re(K(x)) = \frac{K(x) + \overline{K(x)}}{2} = \frac{1}{2} \int_{0}^{-i\infty} \frac{\log(1/s) + \log(s/x)}{(1-s)(s-x)}\, ds$$
$$= \frac{1}{2} \log(1/x) \int_{0}^{-i\infty} \frac{ds}{(1-s)(s-x)} = \frac{\frac{1}{2}\log^2(1/x)}{1-x}.$$

By (6) we have

$$J(0,0,0,0,0,0;x) = \frac{1-x}{(2\pi i)^2} \oint_{|s|=R} \oint_{|t|=r} \frac{dt\, ds}{(1-s)(s-t)(t-x)}$$

for $x < r < R < 1$. By the residue theorem applied twice we get

$$\frac{1}{2\pi i} \oint_{|s|=R} \left(\frac{1}{2\pi i} \oint_{|t|=r} \frac{dt}{(s-t)(t-x)} \right) \frac{ds}{1-s}$$
$$= \frac{1}{2\pi i} \oint_{|s|=R} \frac{ds}{(s-x)(1-s)} = \frac{1}{1-x}. \quad \blacksquare$$

Remark 2.3. For all integers $0 \le L \le M$,

$$\frac{1}{M!} \frac{d^M}{dx^M} \left(\frac{x^L}{t-x} \right) = \frac{t^L}{(t-x)^{M+1}}.$$

To see this, we first decompose $\frac{x^L}{t-x} = \frac{t^L}{t-x} - (x^{L-1} + tx^{L-2} + \cdots + t^{L-1})$, and then we differentiate M times. This remark is useful in the following proof.

Proof of Proposition 2.1. Lemma 2.1 allows us to suppose that (19) holds. By repeated application of (9) we have

$$x^{E_2}(K(x))^{[E_1]} = x^{E_2}\Bigg(\int_{s=0}^{\zeta\infty}\int_{t=0}^{\overline{\zeta}\infty} \frac{dt\,ds}{(1-s)(s-t)(t-x)}\Bigg)^{[E_1]}$$
$$= \int_{s=0}^{\zeta\infty}\int_{t=0}^{\overline{\zeta}\infty} \frac{x^{E_2}\,dt\,ds}{(1-s)(s-t)(t-x)^{E_1+1}}.$$

Using the change of variable $t = xs/T$ this integral becomes

$$\int_{s=0}^{\zeta\infty}\int_{t=0}^{\overline{\zeta}\infty} \frac{x^{E_2-E_1}t^{E_1}\,dt\,ds}{(1-s)(s-t)^{E_1+1}(t-x)}.$$

By Remark 2.3 and recalling that $E_3 \geq E_2 - E_1$, we get

$$(x^{E_2}(K(x))^{[E_1]})^{[E_3]} = \int_{s=0}^{\zeta\infty}\int_{t=0}^{\overline{\zeta}\infty} \frac{t^{E_2}\,dt\,ds}{(1-s)(s-t)^{E_1+1}(t-x)^{E_3+1}}.$$

After the change of variable $s = t/S$ we can rewrite the last integral in the following way:

$$\int_{s=0}^{\zeta\infty}\int_{t=0}^{\overline{\zeta}\infty} \frac{s^{E_1}t^{E_2-E_1}\,dt\,ds}{(1-s)^{E_1+1}(s-t)(t-x)^{E_3+1}}.$$

Now the change of variable $t = xs/T$ transforms the last integral into

$$\int_{s=0}^{\zeta\infty}\int_{t=0}^{\overline{\zeta}\infty} \frac{x^{E_2-E_1-E_3}s^{E_2}t^{E_1+E_3-E_2}\,dt\,ds}{(1-s)^{E_1+1}(s-t)^{E_3+1}(t-x)}.$$

Hence

$$x^{E_4}(x^{E_2}(K(x))^{[E_1]})^{[E_3]} = \int_{s=0}^{\zeta\infty}\int_{t=0}^{\overline{\zeta}\infty} \frac{x^{E_2+E_4-E_1-E_3}s^{E_2}t^{E_1+E_3-E_2}\,dt\,ds}{(1-s)^{E_1+1}(s-t)^{E_3+1}(t-x)}.$$

Since $E_5 \geq E_2 + E_4 - E_1 - E_3$, by Remark 2.3 we get

$$(x^{E_4}(x^{E_2}(K(x))^{[E_1]})^{[E_3]})^{[E_5]} = \int_{s=0}^{\zeta\infty}\int_{t=0}^{\overline{\zeta}\infty} \frac{s^{E_2}t^{E_4}\,dt\,ds}{(1-s)^{E_1+1}(s-t)^{E_3+1}(t-x)^{E_5+1}}.$$

Hence, by Lemma 2.6, the last integral equals

$$\left(x^{E_4}\left(x^{E_2}\left(\frac{\frac{1}{2}\log^2(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5]} + i\pi\left(x^{E_4}\left(x^{E_2}\left(\frac{\log(1/x)}{1-x}\right)^{[E_1]}\right)^{[E_3]}\right)^{[E_5]}.$$

By (5) and Lemmas 2.4 and 2.5 we obtain

$$\begin{aligned} I(h,j,k,l,m,q;x) &= x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}}(1-x)^{E_1+E_3+E_5+1} \\ &\quad\times \int_{s=0}^{\zeta\infty}\int_{t=0}^{\bar{\zeta}\infty} \frac{s^{E_2}t^{E_4}\,dt\,ds}{(1-s)^{E_1+1}(s-t)^{E_3+1}(t-x)^{E_5+1}} \\ &= P(x)\,\frac{1}{2}\log^2(1/x) - Q(x)\log(1/x) + R(x) + \pi i(P(x)\log(1/x) - Q(x)). \end{aligned}$$

By Cauchy's integral formula applied twice we get, for $x < r < R < 1$,

$$\begin{aligned} \left(x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\right)^{[E_5]} &= \frac{1}{2\pi i}\oint_{|t|=r}\left(\frac{t^{E_2}}{(1-t)^{E_1+1}}\right)^{[E_3]}\frac{t^{E_4}\,dt}{(t-x)^{E_5+1}} \\ &= \frac{1}{(2\pi i)^2}\oint_{|t|=r}\frac{t^{E_4}}{(t-x)^{E_5+1}}\oint_{|s|=R}\frac{s^{E_2}}{(1-s)^{E_1+1}(s-t)^{E_3+1}}\,ds\,dt, \end{aligned}$$

whence, by (6),

$$\begin{aligned} J(h,j,k,l,m,q;x) &= x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}}(1-x)^{E_1+E_3+E_5+1} \\ &\quad\times\left(x^{E_4}\left(\frac{x^{E_2}}{(1-x)^{E_1+1}}\right)^{[E_3]}\right)^{[E_5]}. \end{aligned}$$

By (22) and Lemma 2.3 we conclude that $J(h,j,k,l,m,q;x) = P(x)$, and Proposition 2.1 is proved. ∎

3. Hypergeometric identities. We now construct a larger permutation group, acting on the set of nine integers

$$\mathcal{S} = \{h,j,k,l,m,q,l+k-j,h+j-k,k+m-h\},$$

and we derive useful transformation formulae for the integrals $I(h,j,k,l,m,q;x)$ and $J(h,j,k,l,m,q;x)$. As in [RV1]–[RV3], we first extend the actions of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ to the set $\mathcal{S}$ by linearity. Taking account of (4), we have

$$\begin{aligned} \boldsymbol{\sigma} &= (h\ l)(j\ k)(m\ q)(h+j-k\ \ l+k-j), \\ \boldsymbol{\tau} &= (h\ k)(j\ m)(l\ q)(h+j-k\ \ k+m-h). \end{aligned}$$

Let $t \in (0, -i\infty)$. By the change of variable $s = u/(u-1)$ we have

$$\int_0^{-\infty} \frac{s^h}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}}\, ds = (-1)^{j+k} \int_0^1 \frac{u^h(1-u)^l}{(t+(1-t)u)^{h+j-k+1}}\, du.$$

Using the Euler integral representation of the classical hypergeometric function (see e.g. [RV1, formula (3.2)]), we get

$$\int_0^1 \frac{u^h(1-u)^l}{(t+(1-t)u)^{h+j-k+1}}\, du = \frac{h!l!}{(h+j-k)!(l+k-j)!} \int_0^1 \frac{u^{h+j-k}(1-u)^{l+k-j}t^{k-j}}{(t+(1-t)u)^{h+1}}\, du.$$

We now come back to the variable s, writing $u = s/(s-1)$. We have

$$\int_0^1 \frac{u^{h+j-k}(1-u)^{l+k-j}}{(t+(1-t)u)^{h+1}}\, du = (-1)^{j+k} \int_0^{-\infty} \frac{s^{h+j-k}}{(1-s)^{l+1}(s-t)^{h+1}}\, ds.$$

Therefore

$$\int_0^{-\infty} \frac{s^h}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}}\, ds = \frac{h!l!}{(h+j-k)!(l+k-j)!} \int_0^{-\infty} \frac{s^{h+j-k}t^{k-j}}{(1-s)^{l+1}(s-t)^{h+1}}\, ds.$$

Multiplying by $x^{\max\{0,q-l,m-h\}}(1-x)^{k+l+m+1}t^j/(t-x)^{k+m-h+1}$ and integrating over the half-line $(0, -i\infty)$ with respect to t we obtain, by (3),

$$I(h,j,k,l,m,q;x) = \frac{h!l!}{(h+j-k)!(l+k-j)!} I(h+j-k, k, j, l+k-j, m, q; x).$$

We infer that

$$\frac{I(h,j,k,l,m,q;x)}{h!j!k!l!m!q!} \tag{28}$$

is invariant under the action of the group

$$\boldsymbol{\Phi} = \langle \boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\chi} \rangle,$$

where

$$\boldsymbol{\chi} = (h\ h+j-k)(l\ l+k-j)(j\ k).$$

For $s \in (0, i\infty)$ we can also apply the change of variable $t = xv/(v-1)$ to the integral

$$\int_0^{-\infty} \frac{t^j}{(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}\, dt.$$

By repeating the previous argument, we see that (28) is also invariant under the action of the permutation $(h\ k)(j\ h{+}j{-}k)(m\ k{+}m{-}h)$, which however belongs to $\mathbf{\Phi}$, being equal to $\boldsymbol{\tau\sigma\tau\chi\tau\sigma}$.

The group $\mathbf{\Phi}$ has 36 elements. In order to prove this, we consider two partitions A and B of $\mathcal{S}$, precisely $A = \{U_1, U_2, U_3\}$ and $B = \{V_1, V_2, V_3\}$, where

$$U_1 = \{h, j, q\}, \quad U_2 = \{k, l, m\}, \quad U_3 = \{l + k - j, h + j - k, k + m - h\},$$
$$V_1 = \{k, q, h + j - k\}, \quad V_2 = \{j, m, l + k - j\}, \quad V_3 = \{h, l, k + m - h\}.$$

The permutations $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ carry the set U_3 onto itself and interchange U_1 and U_2, $\boldsymbol{\sigma}$ carries V_3 onto itself and interchanges V_1 and V_2, $\boldsymbol{\tau}$ carries V_2 onto itself and interchanges V_1 and V_3, and $\boldsymbol{\chi}$ interchanges U_1 and V_1, U_2 and V_2, U_3 and V_3. In other words, the permutations $\boldsymbol{\sigma}^*$, $\boldsymbol{\tau}^*$ and $\boldsymbol{\chi}^*$ of the set $A \cup B = \{U_1, U_2, U_3, V_1, V_2, V_3\}$ defined by

$$\begin{aligned} \boldsymbol{\sigma}^* &= (U_1\ U_2)(V_1\ V_2), \\ \boldsymbol{\tau}^* &= (U_1\ U_2)(V_1\ V_3), \\ \boldsymbol{\chi}^* &= (U_1\ V_1)(U_2\ V_2)(U_3\ V_3) \end{aligned} \tag{29}$$

are induced by $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ and $\boldsymbol{\chi}$, respectively, so that there exists a unique homomorphism $g : \mathbf{\Phi} \to \mathfrak{S}_6$ of the group $\mathbf{\Phi}$ into the symmetric group $\mathfrak{S}_6$ of the permutations of $A \cup B$ satisfying $g(\boldsymbol{\sigma}) = \boldsymbol{\sigma}^*$, $g(\boldsymbol{\tau}) = \boldsymbol{\tau}^*$ and $g(\boldsymbol{\chi}) = \boldsymbol{\chi}^*$. The table

	U_1	U_2	U_3
V_1	q	k	$h + j - k$
V_2	j	m	$l + k - j$
V_3	h	l	$k + m - h$

shows that each intersection $U_r \cap V_s$ $(r, s = 1, 2, 3)$ contains one and only one element of $\mathcal{S}$. Therefore, if $\boldsymbol{\varphi} \in \mathbf{\Phi}$ and $g(\boldsymbol{\varphi}) = \boldsymbol{\iota}^*$ is the identity of $\mathfrak{S}_6$, then, for all $r, s = 1, 2, 3$, $\boldsymbol{\varphi}$ must map $U_r \cap V_s$ onto itself, so that $\boldsymbol{\varphi}$ must be the identity $\boldsymbol{\iota} \in \mathbf{\Phi}$. This shows that g is injective. Thus the group

$$\mathbf{\Phi}^* := \langle \boldsymbol{\sigma}^*, \boldsymbol{\tau}^*, \boldsymbol{\chi}^* \rangle \subset \mathfrak{S}_6$$

is isomorphic to $\mathbf{\Phi}$, and in particular $|\mathbf{\Phi}| = |\mathbf{\Phi}^*|$. From (29) we get $\boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*\boldsymbol{\sigma}^* = (U_1\ U_2\ U_3)$ and $\boldsymbol{\tau}^*\boldsymbol{\sigma}^* = (V_1\ V_2\ V_3)$, whence $|\langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*\boldsymbol{\sigma}^*, \boldsymbol{\tau}^*\boldsymbol{\sigma}^* \rangle| = |\mathfrak{A}_3 \times \mathfrak{A}_3| = 3 \cdot 3 = 9$. In addition, $\boldsymbol{\sigma}^* \notin \langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*\boldsymbol{\sigma}^*, \boldsymbol{\tau}^*\boldsymbol{\sigma}^* \rangle$, since each element of this group is a product of 3-cycles, whereas $\boldsymbol{\sigma}^*$ is not. Thus $\langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*\boldsymbol{\sigma}^*, \boldsymbol{\tau}^*\boldsymbol{\sigma}^* \rangle$ is a proper subgroup of $\langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*\boldsymbol{\sigma}^*, \boldsymbol{\tau}^*\boldsymbol{\sigma}^*, \boldsymbol{\sigma}^* \rangle = \langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*, \boldsymbol{\tau}^*, \boldsymbol{\sigma}^* \rangle$. Similarly, $\boldsymbol{\chi}^* \notin \langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*, \boldsymbol{\tau}^*, \boldsymbol{\sigma}^* \rangle$, since $\boldsymbol{\chi}^*$ interchanges A and B, and is an odd permutation, whereas $\boldsymbol{\tau}^*$ and $\boldsymbol{\sigma}^*$, and hence also $\boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*$, map A onto itself and B onto itself, and are even permutations.

Therefore

$$|\boldsymbol{\Phi}^*| = |\langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*, \boldsymbol{\tau}^*, \boldsymbol{\sigma}^*, \boldsymbol{\chi}^* \rangle| \geq 2|\langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*, \boldsymbol{\tau}^*, \boldsymbol{\sigma}^* \rangle|$$
$$\geq 2 \cdot 2|\langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*\boldsymbol{\sigma}^*, \boldsymbol{\tau}^*\boldsymbol{\sigma}^* \rangle| = 36.$$

On the other hand, let $\widehat{\boldsymbol{\Phi}} \subset \mathfrak{S}_6$ be the subgroup of the permutations $\hat{\boldsymbol{\varphi}}$ of $A \cup B$ satisfying

$$\begin{cases} \hat{\varphi}(A) = A,\ \hat{\varphi}(B) = B & \text{if } \hat{\boldsymbol{\varphi}} \text{ is even,} \\ \hat{\varphi}(A) = B,\ \hat{\varphi}(B) = A & \text{if } \hat{\boldsymbol{\varphi}} \text{ is odd.} \end{cases}$$

We claim that $\boldsymbol{\Phi}^* = \widehat{\boldsymbol{\Phi}}$ and that $|\widehat{\boldsymbol{\Phi}}| = 36$. Since $\boldsymbol{\sigma}^*, \boldsymbol{\tau}^*, \boldsymbol{\chi}^* \in \widehat{\boldsymbol{\Phi}}$, we have $\boldsymbol{\Phi}^* \subset \widehat{\boldsymbol{\Phi}}$ and $|\widehat{\boldsymbol{\Phi}}| \geq 36$. Moreover, since the symmetric group $\mathfrak{S}_3$ of all permutations of A (or of B) contains three even permutations and three odd permutations, $\widehat{\boldsymbol{\Phi}}$ contains $3 \cdot 3 + 3 \cdot 3 = 18$ even permutations, hence $|\langle \boldsymbol{\chi}^*\boldsymbol{\tau}^*\boldsymbol{\chi}^*, \boldsymbol{\tau}^*, \boldsymbol{\sigma}^* \rangle| = \widehat{\boldsymbol{\Phi}} \cap \mathfrak{A}_6 = 18$. Note that $\hat{\boldsymbol{\varphi}} \in \widehat{\boldsymbol{\Phi}}$ is odd if and only if $\boldsymbol{\chi}^*\hat{\boldsymbol{\varphi}}$ is even. In conclusion, $|\widehat{\boldsymbol{\Phi}}| = 36$, whence $\boldsymbol{\Phi}^* = \widehat{\boldsymbol{\Phi}}$ and $|\boldsymbol{\Phi}| = |\boldsymbol{\Phi}^*| = 36$.

In the rest of this section we follow Rhin and Viola's notation and terminology ([RV2, Sections 4 and 5] and [RV3, Sections 3 and 4]). With any permutation $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ we associate the quotient of factorials

$$\frac{h!j!k!l!m!q!}{\boldsymbol{\varphi}(h)!\boldsymbol{\varphi}(j)!\boldsymbol{\varphi}(k)!\boldsymbol{\varphi}(l)!\boldsymbol{\varphi}(m)!\boldsymbol{\varphi}(q)!}. \tag{30}$$

Obviously, if the permutations $\boldsymbol{\varphi}, \boldsymbol{\varphi}' \in \boldsymbol{\Phi}$ lie in the same left coset of the subgroup $\mathbf{G}$ in $\boldsymbol{\Phi}$, the quotient (30) equals the similar quotient with $\boldsymbol{\varphi}'$ in place of $\boldsymbol{\varphi}$. Thus with each left coset of $\mathbf{G}$ in $\boldsymbol{\Phi}$ we may associate the corresponding quotient (30), where $\boldsymbol{\varphi}$ is any of the six permutations lying in the coset considered.

We say that a permutation $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$ has *level* v if the quotient (30) has v factorials in the numerator and v in the denominator, after removing the common factorials. For example, any element of $\mathbf{G}$ has level 0, and $\boldsymbol{\chi}$ has level 2. Since $|\mathbf{G}| = 6$ and $|\boldsymbol{\Phi}| = 36$, there are $36 : 6 = 6$ left cosets. If we choose one permutation in each of the five left cosets of $\mathbf{G}$ different from $\mathbf{G}$ itself, we get five transformation formulae for $I(h, j, k, l, m, q; x)$. The three permutations of level 2,

$$\boldsymbol{\chi} = (h\ h + j - k)(l\ l + k - j)(j\ k),$$
$$\boldsymbol{\tau\chi\tau} = (h\ m)(k\ k + m - h)(q\ q + h - m),$$
$$\boldsymbol{\sigma\tau\sigma\chi\sigma\tau\sigma} = (j\ j + q - l)(l\ q)(m\ m + l - q),$$

yield the identities

$$
\begin{aligned}
&I(h,j,k,l,m,q;x)\\
&\quad=\frac{h!l!}{(h+j-k)!(l+k-j)!}\,I(h+j-k,k,j,l+k-j,m,q;x)\\
&\quad=\frac{k!q!}{(k+m-h)!(q+h-m)!}\,I(m,j,k+m-h,l,h,q+h-m;x)\\
&\quad=\frac{j!m!}{(j+q-l)!(m+l-q)!}\,I(h,j+q-l,k,q,m+l-q,l;x),
\end{aligned}
$$

and the two permutations of level 3,

$$
\begin{aligned}
\boldsymbol{\chi\tau\chi}&=(h\ k+m-h)(j\ h+j-k)(q\ q+h-m)(k\ m),\\
\boldsymbol{\chi\sigma\tau\sigma\chi}&=(k\ l+k-j)(l\ j+q-l)(m\ m+l-q)(j\ q),
\end{aligned}
$$

yield

$$
\begin{aligned}
I(h,j,k,l,m,q;x)&=\frac{h!j!q!}{(k+m-h)!(h+j-k)!(q+h-m)!}\\
&\qquad\times I(k+m-h,h+j-k,m,l,k,q+h-m;x)\\
&=\frac{k!l!m!}{(l+k-j)!(j+q-l)!(m+l-q)!}\\
&\qquad\times I(h,q,l+k-j,j+q-l,m+l-q,j;x).
\end{aligned}
$$

We can separate the real and imaginary parts in all the previous identities, and to do this we apply Proposition 2.1. Moreover, if $x\in(0,1)$ is rational, then $P(h,j,k,l,m,q;x)$, $Q(h,j,k,l,m,q;x)$ and $R(h,j,k,l,m,q;x)$ are also rational, and $\log(1/x)$ is transcendental. Hence $P(h,j,k,l,m,q;x)$, $Q(h,j,k,l,m,q;x)$ and $R(h,j,k,l,m,q;x)$ are invariant under the action of $\mathbf{G}$. In addition,

$$
\begin{aligned}
&P(h,j,k,l,m,q;x)\\
&\quad=\frac{h!l!}{(h+j-k)!(l+k-j)!}\,P(h+j-k,k,j,l+k-j,m,q;x)\\
&\quad=\frac{k!q!}{(k+m-h)!(q+h-m)!}\,P(m,j,k+m-h,l,h,q+h-m;x)\\
&\quad=\frac{j!m!}{(j+q-l)!(m+l-q)!}\,P(h,j+q-l,k,q,m+l-q,l;x)\\
&\quad=\frac{h!j!q!}{(k+m-h)!(h+j-k)!(q+h-m)!}\\
&\qquad\times P(k+m-h,h+j-k,m,l,k,q+h-m;x)\\
&\quad=\frac{k!l!m!}{(l+k-j)!(j+q-l)!(m+l-q)!}\\
&\qquad\times P(h,q,l+k-j,j+q-l,m+l-q,j;x),
\end{aligned}
$$

and similarly for $Q(h,j,k,l,m,q;x)$ and $R(h,j,k,l,m,q;x)$. This means that the quotients similar to (28), with I replaced by P (i.e. by J), Q or R, are also invariant under the action of the permutation group $\mathbf{\Phi}$.

We remark that the integers γ and δ defined by (15) are invariant under the action of $\mathbf{\Phi}$, whereas H and K are not. We need to define two new integers M and N, not less than H and K, respectively, that are also invariant under the action of $\mathbf{\Phi}$. Let

$$\begin{aligned}M &= \max\{h,j,k,l,m,q,h+j-k,l+k-j,k+m-h\},\\ N &= \max\{[M/2], \max{}'\{h,j,k,l,m,q,h+j-k,l+k-j,k+m-h\}\}.\end{aligned}$$

We have $M \ge H$ and $N \ge K$. In practice we can disregard $[M/2]$ in the definition of N since in all our numerical examples we choose the parameters h,j,k,l,m,q satisfying (7), which implies $M = h+j-k$ and $N = j$. In fact, by (7) we have $l+k-j < k$, $k+m-h = k+j-h = h$ and $h+j-k > j$. Hence $M = h+j-k$, $h < 2h = j+k$, $M = h+j-k < 2j$ and $M/2 < j = N$.

For any natural number n Proposition 2.1 implies that $P(hn,jn,kn,ln,mn,qn;x)$ and $P((h+j-k)n,kn,jn,(l+k-j)n,mn,qn;x)$ are polynomials with integer coefficients, and we have just proved that

$$\begin{aligned}&((h+j-k)n)!((l+k-j)n)!P(hn,jn,kn,ln,mn,qn;x)\\ &\qquad = (hn)!(ln)!P((h+j-k)n,kn,jn,(l+k-j)n,mn,qn;x).\end{aligned}$$

Thus, following the arguments given in [RV1, pp. 44–47], we see that each prime $p > \sqrt{Mn}$ for which $[(l+k-j)\omega]+[(h+j-k)\omega] < [h\omega]+[l\omega]$, where $\omega = \{n/p\} = n/p - [n/p]$ denotes the fractional part of n/p, must divide all the coefficients of the polynomial $P(hn,jn,kn,ln,mn,qn;x)$. The same argument applies to all the five identities written above, and also to all the coefficients of $d_{Mn}Q(hn,jn,kn,ln,mn,qn;x)$ and $d_{Mn}d_{Nn}R(hn,jn,kn,ln,mn,qn;x)$. Therefore, each prime $p > \sqrt{Mn}$ satisfying at least one of

$$\begin{aligned}&[(h+j-k)\omega]+[(l+k-j)\omega] < [h\omega]+[l\omega],\\ &[(k+m-h)\omega]+[(q+h-m)\omega] < [k\omega]+[q\omega],\\ &[(j+q-l)\omega]+[(m+l-q)\omega] < [j\omega]+[m\omega],\\ &[(k+m-h)\omega]+[(h+j-k)\omega]+[(q+h-m)\omega] < [h\omega]+[j\omega]+[q\omega],\\ &[(l+k-j)\omega]+[(j+q-l)\omega]+[(m+l-q)\omega] < [k\omega]+[l\omega]+[m\omega]\end{aligned}\tag{31}$$

divides all the coefficients of the polynomials $P(hn,jn,kn,ln,mn,qn;x)$, $d_{Mn}Q(hn,jn,kn,ln,mn,qn;x)$ and $d_{Mn}d_{Nn}R(hn,jn,kn,ln,mn,qn;x)$.

Let Δ_n denote the product of all prime numbers $p > \sqrt{Mn}$ satisfying at least one of the inequalities (31), and let $D_n = d_{Mn}/\Delta_n$. We have proved

Proposition 3.1. *With the notation stated above*,

$$(\varDelta_n)^{-1}P(hn,jn,kn,ln,mn,qn;x),$$
$$D_nQ(hn,jn,kn,ln,mn,qn;x),$$
$$D_nd_{Nn}R(hn,jn,kn,ln,mn,qn;x)$$

are polynomials in x with integer coefficients.

Remark 3.1. The identities corresponding to permutations of level 3 actually allow one to eliminate divisors of the above polynomials of the types p and p^2. However, the best irrationality and non-quadraticity measures we can prove are all obtained when h,j,k,l,m,q satisfy (7). In this special case, the two quotients of three factorials corresponding to two permutations of level 3 lying in distinct left cosets of $\mathbf{G}$ in $\mathbf{\Phi}$, e.g. $\boldsymbol{\chi\tau\chi}$ and $\boldsymbol{\chi\sigma\tau\sigma\chi}$, coincide with one quotient of two factorials only. A substitution indeed shows that, under the assumption (7), the inequalities (31) become

$$\begin{aligned}[(h+k-j)\omega]+[(h+j-k)\omega]&<[j\omega]+[k\omega],\\ [(h+k-j)\omega]+[(h+j-k)\omega]&<2[h\omega],\\ [(h+k-j)\omega]+[h\omega]&<2[k\omega],\\ [(h+j-k)\omega]+[h\omega]&<2[j\omega].\end{aligned}$$

Again by the arguments in [RV1], these inequalities yield divisors of the above polynomials only of the type p.

4. Asymptotic behaviour of $P_n(x)$. Here and in the rest of this paper we assume that all the nine integers $h,j,k,l,m,q,l+k-j,h+j-k,k+m-h$ are strictly positive and satisfy (7). We shall keep h,j,k,l,m,q fixed, and make $n\to\infty$. Accordingly, we abbreviate $P_n(x)=P(hn,jn,kn,ln,mn,qn;x)$, $Q_n(x)=Q(hn,jn,kn,ln,mn,qn;x)$, $R_n(x)=R(hn,jn,kn,ln,mn,qn;x)$, and we define

$$S_n(x):=\left(x^{E_4n}\left(x^{E_2n}\left(\frac{1}{1-x}\right)^{[E_1n]}\right)^{[E_3n]}\right)^{[E_5n]},$$

where $E_1,\dots,E_5$ are given by (18). By (22) and Lemma 2.3 we have

$$P_n(x)=x^{\max\{0,E_5-E_4,E_3+E_5-E_2-E_4\}n}(1-x)^{(E_1+E_3+E_5)n+1}S_n(x). \tag{32}$$

We recall that $0<x<1$. We write the function $S_n(x)$ as a power series in x. Since the coefficients of this power series are positive, we may apply the method of [BR, pp. 201–202]. The condition (7) implies that $E_3-E_2=(h+j-k)-h=j-k>0$, $E_5-E_4=(m+k-h)-j=q-l<0$, and $E_3+E_5-E_2-E_4=j-k+q-l=m-h>0$. So $0<E_3+E_5-E_2-E_4$

$< E_3 - E_2$. Thus

$$((1-x)^{-1})^{[E_1 n]} = (1-x)^{-E_1 n-1} = \sum_{r\ge 0}\binom{r+E_1 n}{E_1 n}x^r,$$

so, using $E_3 - E_2 > 0$,

$$(x^{E_2 n}((1-x)^{-1})^{[E_1 n]})^{[E_3 n]} = \sum_{r\ge (E_3-E_2)n}\binom{r+E_1 n}{E_1 n}\binom{r+E_2 n}{E_3 n}x^{r+(E_2-E_3)n},$$

and finally, using $E_3 + E_5 - E_2 - E_4 < E_3 - E_2$,

$$\begin{aligned} S_n(x) &= \left(x^{E_4 n}\left(x^{E_2 n}\left(\frac{1}{1-x}\right)^{[E_1 n]}\right)^{[E_3 n]}\right)^{[E_5 n]} \\ &= \sum_{r\ge (E_3-E_2)n}\binom{r+E_1 n}{E_1 n}\binom{r+E_2 n}{E_3 n}\binom{r+(E_2+E_4-E_3)n}{E_5 n} \\ &\qquad\qquad \times x^{r+(E_2+E_4-E_3-E_5)n}. \end{aligned}$$

We want to prove that

$$\lim_{n\to\infty}\frac{1}{n}\log S_n(x) = \log\max_{y>E_3-E_2} F(y;x) = \log F(y_{\max};x), \tag{33}$$

where

$$F(y;x) = \frac{(y+E_1)^{y+E_1}(y+E_2)^{y+E_2}(y+E_2+E_4-E_3)^{y+E_2+E_4-E_3}}{y^y(y+E_2-E_3)^{y+E_2-E_3}(y+E_2+E_4-E_3-E_5)^{y+E_2+E_4-E_3-E_5}} \times \frac{x^{y+E_2+E_4-E_3-E_5}}{E_1^{E_1}E_3^{E_3}E_5^{E_5}}.$$

By computing $\frac{d}{dy}\log F(y;x)$, we see that $\frac{dF}{dy}$ has the sign of $x - H(y)$, where

$$H(y) := \left(1-\frac{E_1}{y+E_1}\right)\left(1-\frac{E_3}{y+E_2}\right)\left(1-\frac{E_5}{y+E_2+E_4-E_3}\right). \tag{34}$$

Note that $H(E_3-E_2)=0$, $\lim_{y\to+\infty}H(y)=1$ and that $H(y)$ is the product of three positive increasing functions for $y > E_3 - E_2$. Therefore $\frac{dF}{dy} = 0$ has one solution $y_{\max} = y_{\max}(x) > E_3 - E_2$ satisfying $H(y_{\max}) = x$, $F(y;x)$ is increasing for $y < y_{\max}$ and decreasing for $y > y_{\max}$, and $y_{\max}(x)$ is a continuous increasing function of x.

Let x_1 be such that $x < x_1 < \sqrt{x} < 1$. In the series $S_n(x_1)$ we consider the general term

$$a_r := \binom{r+E_1 n}{E_1 n}\binom{r+E_2 n}{E_3 n}\binom{r+(E_2+E_4-E_3)n}{E_5 n} \times x_1^{r+(E_2+E_4-E_3-E_5)n} \tag{35}$$

and we see that $a_{r-1} < a_r$ if and only if $r(r+(E_2-E_3)n)(r+(E_2+E_4-E_3-E_5)n) < (r+E_1n)(r+E_2n)(r+(E_2+E_4-E_3)n)x_1$, which is equivalent to $r < y_{\max}(x_1)n$. Similarly, $a_{r-1} > a_r$ if and only if $r > y_{\max}(x_1)n$. We define $r_{\max} = r_{\max}(x_1) := [y_{\max}(x_1)n]$ (we omit, for brevity, the dependence on n). We have $r_{\max} \geq (E_3-E_2)n$, and the previous argument shows that $a_{r_{\max}} = \max_{r\geq(E_3-E_2)n} a_r$.

Moreover, $r' := [y_{\max}(x)n] \leq r_{\max}(x_1) \leq r'' := [y_{\max}(\sqrt{x})n]$. Both r' and r'' are independent of x_1. In what follows we put

$$M_n(x_1) := \max_{r\geq(E_3-E_2)n} a_r = \max_{r'\leq r\leq r''} a_r.$$

Thus we have $\log M_n(x_1) = \log a_{r_{\max}}$. Taking the logarithm of (35) for $r = r_{\max}(x_1)$, and using Stirling's formula in the simple form $\log n! = n\log n - n + O(\log n)$, a straightforward computation yields

$$\lim_{n\to\infty} (M_n(x_1))^{1/n} = F(y_{\max}(x_1); x_1). \tag{36}$$

Let $x_2 := x/x_1$. Since $0 < x_2 < 1$, we have

$$\begin{aligned} S_n(x) &= \sum_{r\geq(E_3-E_2)n} a_r x_2^{r+(E_2+E_4-E_3-E_5)n} \\ &\leq M_n(x_1) \sum_{r\geq(E_3-E_2)n} x_2^{r+(E_2+E_4-E_3-E_5)n} = M_n(x_1)\,\frac{x_2^{(E_4-E_5)n}}{1-x_2}. \end{aligned}$$

Hence $\limsup_{n\to\infty}(S_n(x))^{1/n} \leq F(y_{\max}(x_1); x_1)x_2^{E_4-E_5}$. Since $x_2 \to 1$ for $x_1 \to x$, and $\lim_{x_1\to x} F(y_{\max}(x_1); x_1) = F(y_{\max}(x); x)$, it follows that we have $\limsup_{n\to\infty}(S_n(x))^{1/n} \leq F(y_{\max}(x); x)$.

On the other hand, $S_n(x) \geq M_n(x_1)x_2^{r''+(E_2+E_4-E_3-E_5)n}$. From (36) we deduce $\liminf_{n\to\infty}(S_n(x))^{1/n} \geq F(y_{\max}(x_1); x_1)x_2^{y_{\max}(\sqrt{x})+E_2+E_4-E_3-E_5}$, and then, for $x_1 \to x$ we have $\liminf_{n\to\infty}(S_n(x))^{1/n} \geq F(y_{\max}(x); x)$. Therefore $\lim_{n\to\infty}(S_n(x))^{1/n} = F(y_{\max}(x); x)$, as we claimed in (33).

We now prove that

$$F(y_{\max}(x); x) = \min_{x<t<s<1} f(s,t) = f(s_1,t_1), \tag{37}$$

where

$$f(s,t) = \frac{s^{E_2}t^{E_4}}{(1-s)^{E_1}(s-t)^{E_3}(t-x)^{E_5}}$$

is the function appearing in the integrals (5) and (6), and $x < t_1 < s_1 < 1$. We have $f(s,t) > 0$ inside the triangle $\{(s,t) \in \mathbb{R}^2 \mid x < t < s < 1\}$, and $f(s,t) = \infty$ on the boundary. Hence the minimum in (37) exists. By (32) and by Proposition 2.1, for all r and R such that $x < r < R < 1$, we have

$$S_n(x) = \frac{1}{(2\pi i)^2} \oint_{|s|=R} \oint_{|t|=r} (f(s,t))^n \frac{dt\, ds}{(1-s)(s-t)(t-x)}.$$

Thus, by (33),

$$\log F(y_{\max}; x) = \lim_{n\to\infty} \frac{1}{n} \log S_n(x) \le \log \max_{\substack{|s|=R\\ |t|=r}} |f(s,t)|. \tag{38}$$

For all $(s,t) \in \mathbb{C}^2$ with $|s| = R$ and $|t| = r$, we have $|1-s| \ge 1-R$, $|s-t| \ge R-r$ and $|t-x| \ge r-x$, whence

$$|f(s,t)| \le f(R,r). \tag{39}$$

On the other hand, the equation $H(y_{\max}) = x$, with $H(y)$ defined by (34), implies $F(y_{\max}; x) = f(s^*, t^*)$, where

$$s^* := \frac{y_{\max}}{y_{\max} + E_1}$$

and

$$t^* := \frac{(y_{\max} + E_2 + E_4 - E_3)x}{y_{\max} + E_2 + E_4 - E_3 - E_5} = \frac{y_{\max}(y_{\max} + E_2 - E_3)}{(y_{\max} + E_1)(y_{\max} + E_2)}.$$

Hence, by (38) and (39),

$$f(s^*, t^*) = F(y_{\max}; x) \le \max_{\substack{|s|=R\\ |t|=r}} |f(s,t)| = f(R,r)$$

for all r and R such that $x < r < R < 1$. Moreover, $x < t^* < s^* < 1$, whence $\min_{x<r<R<1} f(R,r) = f(s^*, t^*) = F(y_{\max}; x)$ is the minimum in (37). By (32), (33), (37) and (7),

$$\lim_{n\to\infty} \frac{1}{n} \log P_n(x) = \log f(s_1, t_1) + (j-h)\log x + 3h \log(1-x). \tag{40}$$

5. $\mathbb{C}^2$ saddle point method. In order to compute the irrationality and non-quadraticity measures of $\log(1/x)$ for suitable rational x, we require a good upper bound for $|I_n(x)|$, and this is obtained by a weak version of the $\mathbb{C}^2$ saddle point method given in [H3]. Such an upper bound depends on the values of the function

$$f(s,t) = \frac{s^h t^j}{(1-s)^{l+k-j}(s-t)^{h+j-k}(t-x)^{k+m-h}}$$

at its complex stationary points satisfying $st \ne 0$, i.e. at the complex solutions of $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial t} = 0$, $f(s,t) \ne 0$. Writing $\frac{\partial}{\partial s}\log f = \frac{\partial}{\partial t}\log f = 0$ and using (7), we are led to the system

$$\begin{cases} hs^2 - (j-k)s + (j-k)st - ht = 0, \\ jt^2 - (h-k)xt + (h-k)st - jxs = 0. \end{cases} \tag{41}$$

If for some solution (s,t) of (41) we had $s=h/(j-k)$, from the first equation in (41) we should get $s=h/(j-k)=(j-k)/h$, whence $h^2=(j-k)^2$, i.e. $(h+j-k)(h+k-j)=0$, which is impossible since $h+j-k>0$ and $h+k-j=l+k-j>0$. Hence the first equation of (41) yields

$$t=s\,\frac{hs-(j-k)}{h-(j-k)s}. \tag{42}$$

Substituting this in the second equation of (41) and dividing by $(h+j-k)s$, we obtain the cubic equation

$$hks^3-(j-h)((j-k)x+h+2k)s^2+(j-h)(j-k+(h+2k)x)s-hkx=0.$$

For all numerical values we choose in Section 6, this equation has only one real root $s_1>0$, and two complex conjugate roots s_2 and s_3 with negative real part, which we number so that $\Im(s_2)>0$. Let t_i be given by (42) for $s=s_i$, so that (s_i,t_i), for $i=1,2,3$, are the stationary points of $f(s,t)$ satisfying $f(s,t)\neq 0$, with $s_1,t_1\in\mathbb{R}^+$, and $s_2=\overline{s_3}$, $t_2=\overline{t_3}$. From (37) we know that $x<t_1<s_1<1$.

Let $I_n(x)=I(hn,jn,kn,ln,mn,qn;x)$. We claim that

$$\limsup_{n\to\infty}\frac{1}{n}\log|I_n(x)|\le\log|f(s_2,t_2)|+(j-h)\log x+3h\log(1-x). \tag{43}$$

For a given t, the equation (42) has two distinct solutions in s, unless the discriminant

$$(j-k)^2t^2+2(2h^2-(j-k)^2)t+(j-k)^2$$

vanishes. This occurs for two distinct negative values of t, say $\tau_1<\tau_2<0$, corresponding to the solutions σ_1 and σ_2 of $\frac{dt}{ds}=0$. Thus by (42) we have $\sigma_i\mapsto\tau_i$ $(i=1,2)$, where

$$\sigma_1=\frac{h+\sqrt{h^2-(j-k)^2}}{j-k},\qquad \sigma_2=\frac{h-\sqrt{h^2-(j-k)^2}}{j-k},$$

whence $\sigma_1>\sigma_2>0$. In other words, the inverse of (42) is a two-valued function with branch points at τ_1 and τ_2.

The function (42) maps the upper half-circumference having diameter $[\sigma_1,\sigma_2]$ onto the real interval $[\tau_1,\tau_2]$. Let

$$C=\left\{\Im(s)>0\text{ and }\left|s-\frac{h}{j-k}\right|>\frac{\sqrt{h^2-(j-k)^2}}{j-k}\right\},\qquad D=\{\Im(t)<0\},$$

whence $s_2\in C$ and $t_2\in D$. We denote by $t=T(s)$ the function (42), and by $s=S(t)$ the inverse function restricted to $t\in D$ with values in C. Clearly,

$$T:C\to D\quad\text{and}\quad S:D\to C$$

are one-to-one holomorphic functions. Let

$$\Gamma = \mathbb{R}^+ s_2 = \{\lambda s_2 \mid \lambda > 0\}, \qquad \Delta = T(\Gamma) = \left\{\lambda s_2 \, \frac{h\lambda s_2 - (j-k)}{h - (j-k)\lambda s_2} \,\middle|\, \lambda > 0\right\},$$

so that $\Gamma \subset C$ and $\Delta \subset D$. By (11), in the integral $I_n(x)$ we may rotate the integration path for s from $(0, i\infty)$ to Γ. Moreover, the curve $\Delta \subset D$ goes from 0 to infinity through t_2 with an oblique asymptote. Hence, by the same discussion yielding (10) and (11), for any fixed $s \in \Gamma$ we may move the integration path for t from $(0, -i\infty)$ to Δ. Therefore

$$I_n(x) = x^{(j-h)n}(1-x)^{3hn+1} \int\limits_{s\in\Gamma} \int\limits_{t\in\Delta} (f(s,t))^n \, \frac{dt\, ds}{(1-s)(s-t)(t-x)},$$

whence, by the absolute convergence of $\int_{s\in\Gamma}\int_{t\in\Delta} \frac{dt\,ds}{(1-s)(s-t)(t-x)}$, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log |I_n(x)| \le \log \max_{s\in\Gamma,\, t\in\Delta} |f(s,t)| + (j-h)\log x + 3h\log(1-x).$$

This implies (43), since for all $(s,t) \in \Gamma \times \Delta$ we have $|f(s,t)| \le |f(S(t),t)| \le |f(s_2,t_2)|$, as can be proved for all the numerical choices made in Section 6, as follows.

For any fixed $\mu > 0$, the real function

$$g(\lambda) := \log |f(\lambda\mu s_2, T(\mu s_2))| \qquad (0 < \lambda < +\infty)$$

satisfies

$$\lim_{\lambda\to 0} g(\lambda) = \lim_{\lambda\to+\infty} g(\lambda) = -\infty$$

and has only one stationary point $\lambda \in (0, +\infty)$, namely $\lambda = 1$. Indeed, for any s we have $\frac{\partial f}{\partial s} = 0$ at $(s, T(s))$, and in particular at the points $(\mu s_i, T(\mu s_i))$ $(i = 2, 3)$. Since

$$g(\lambda) = \frac{1}{2}\log f(\lambda\mu s_2, T(\mu s_2)) + \frac{1}{2}\log f(\lambda\mu s_3, T(\mu s_3))$$

we have $\frac{dg}{d\lambda} = 0$ at $\lambda = 1$. Moreover, for $i = 2, 3$,

$$\begin{aligned}\log f(\lambda\mu s_i, T(\mu s_i)) &= h\log\lambda - (h+k-j)\log(1-\lambda\mu s_i)\\ &\quad - (h+j-k)\log(h(\lambda-\mu s_i) + (j-k)(1-\lambda\mu s_i)) + L_i,\end{aligned}$$

where L_i is independent of λ. Thus the equation $\frac{dg}{d\lambda} = 0$ leads to a polynomial equation with real coefficients, having degree 4 in λ and the root $\lambda = 1$ independent of μ. Dividing by $\lambda - 1$, we are left with a polynomial of degree 3 in λ whose coefficients are polynomials in μ of degree not exceeding 4. The discriminant of this polynomial in λ is a polynomial in μ of degree 14 and vanishing of order 2 at $\mu = 0$, with negative leading coefficient and no real roots apart from $\mu = 0$. In particular this discriminant is negative for all real values of $\mu \ne 0$, so the polynomial of degree 3 in λ has only one real root, which must be negative for all positive μ since the leading coefficient and the constant term are both negative for $\mu > 0$. We conclude that $\max_{\lambda>0} g(\lambda) = g(1)$, i.e. for any $t \in \Delta$ we have

$$\max_{s\in\Gamma} |f(s,t)| = |f(S(t),t)|.$$

The real function

$$G(\lambda) := \log |f(\lambda s_2, T(\lambda s_2))| \qquad (0 < \lambda < +\infty)$$

satisfies

$$\lim_{\lambda\to 0} G(\lambda) = \lim_{\lambda\to+\infty} G(\lambda) = -\infty,$$

as is easily seen using the identity

$$\lambda s - T(\lambda s) = \frac{(h+j-k)(1-\lambda s)\lambda s}{h-(j-k)\lambda s}. \tag{44}$$

Moreover, $G(\lambda)$ has only one stationary point in $(0,+\infty)$, namely $\lambda = 1$. Indeed, we have $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial t} = 0$ at (s_i, t_i) $(i = 2, 3)$, and

$$G(\lambda) = \frac{1}{2}\log f(\lambda s_2, T(\lambda s_2)) + \frac{1}{2}\log f(\lambda s_3, T(\lambda s_3)),$$

whence $\frac{dG}{d\lambda} = 0$ at $\lambda = 1$. By (44) and (7) we have

$$\begin{aligned}\log f(\lambda s_i, T(\lambda s_i)) &= k\log\lambda + j\log(h\lambda s_i - (j-k)) + j\log(h-(j-k)\lambda s_i)\\ &\quad - 2h\log(1-\lambda s_i) - h\log(h(\lambda^2 s_i^2 - x) - (j-k)(1-x)\lambda s_i) + L_i',\end{aligned}$$

where L_i' is independent of λ. Thus the equation $\frac{dG}{d\lambda} = 0$ leads to a polynomial equation in λ with real coefficients, having degree 10 and only two real roots, i.e. $\lambda = 1$ and a negative root. Therefore $\max_{\lambda>0} G(\lambda) = G(1)$, whence $\max_{t\in\Delta} |f(S(t),t)| = |f(s_2,t_2)|$, and (43) follows.

6. The irrationality and non-quadraticity measures. Let $0 < x = a/b < 1$ be a rational number. By our Propositions 2.1 and 3.1 we have

$$b^{n\gamma} D_n P_n(a/b),\, b^{n\gamma} D_n Q_n(a/b) \in \mathbb{Z}.$$

Let Ω be the set of real numbers $\omega \in [0,1)$ satisfying at least one of (31). As a consequence of the Prime Number Theorem one can prove (see [RV1, p. 51]) that

$$\lim_{n\to\infty} \frac{1}{n}\log D_n = M - \int_{\Omega} d\psi(z), \tag{45}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of Euler's gamma-function. Let

$$\begin{aligned} c_0 &= -\log|f(s_2,t_2)| - (j-h)\log x - 3h\log(1-x),\\ c_1 &= \log f(s_1,t_1) + (j-h)\log x + 3h\log(1-x),\\ c_2 &= M + \gamma\log b - \int_\Omega d\psi(z),\\ c_3 &= M + N + \delta\log b - \int_\Omega d\psi(z). \end{aligned}$$

Note that the condition (7) implies $\gamma = h+j$, $\delta = 2j$, $M = h+j-k$, $N = j$. By Proposition 2.1 we have

$$\frac{1}{\pi}\Im(I_n(x)) = P_n(x)\log(1/x) - Q_n(x),$$

whence

$$|P_n(x)\log(1/x) - Q_n(x)| \le |I_n(x)|. \tag{46}$$

We set $x = a/b$, we multiply by $b^{n\gamma}D_n$ and we apply (40), (43) and (45). Since $\log(b/a)$ is transcendental, by Lemma 2.1 and Remark 2.1 of [H2, pp. 337–339], if $c_0 > c_2$ then

$$\mu(\log(b/a)) \le \frac{c_1+c_2}{c_0-c_2} + 1 = \frac{c_0+c_1}{c_0-c_2}.$$

With the choice $a = 1$, $b = 2$ (and then $x = a/b = 1/2$), $h = l = 5$, $j = m = 6$, $k = q = 4$, we have

$$\begin{aligned} &\gamma = 11, \qquad M = 7, \qquad \log 2 = 0.69314718\ldots,\\ &\Omega = [1/6,3/7)\cup[1/2,5/7)\cup[3/4,6/7), \qquad \int_\Omega d\psi(z) = 4.99510233\ldots,\\ &s_1 = 0.871065730\ldots, \qquad t_1 = 0.62975103\ldots, \qquad y_{\max} = 20.26766967\ldots,\\ &\log f(s_1,t_1) = \log F(y_{\max};x) = 22.84284685\ldots,\\ &s_2 = -0.08553286\ldots + i\cdot 0.75279055\ldots,\\ &t_2 = -0.35654218\ldots - i\cdot 0.51948046\ldots,\\ &-\log|f(s_2,t_2)| = 6.84429322\ldots,\\ &c_0 = 17.93464811\ldots, \qquad c_1 = 11.75249197\ldots, \qquad c_2 = 9.62951665\ldots, \end{aligned}$$

hence $\mu(\log 2) < 3.57455390\ldots$.

Moreover, again by our Propositions 2.1 and 3.1,

$$b^{n\delta}D_n d_{Nn}P_n(a/b),\, b^{n\delta}D_n d_{Nn}Q_n(a/b),\, b^{n\delta}D_n d_{Nn}R_n(a/b) \in \mathbb{Z}.$$

By Proposition 2.1 we get

$$\frac{2}{\pi}\Im(I_n(x))\log(1/x) - 2\Re(I_n(x)) = P_n(x)\log^2(1/x) - 2R_n(x),$$

whence

$$|P_n(x)\log^2(1/x) - 2R_n(x)| \le (\log(1/x)+2)|I_n(x)|. \tag{47}$$

If $c_0 > c_3$, we may apply Lemma 2.3 and Remark 1 of [H3, p. 4567]. Setting $x = a/b$ and multiplying (46) and (47) by $b^{n\delta} D_n d_{Nn}$, we get

$$\mu_2(\log(b/a)) \le \frac{c_1+c_3}{c_0-c_3} + 1 = \frac{c_0+c_1}{c_0-c_3}.$$

Taking again $a = 1$, $b = 2$ and $x = 1/2$, and for $h = l = 65$, $j = m = 73$, $k = q = 57$, we have

$$\delta = 146, \qquad M = 81, \qquad N = 73.$$

Now, Ω is the union of the following intervals:

$[1/73, 1/49), [2/73, 2/49), [3/73, 4/81), [1/19, 5/81), [5/73, 4/49),$
$[6/73, 7/81), [5/57, 8/81), [2/19, 1/9), [7/57, 10/81), [10/73, 1/7),$
$[11/73, 8/49), [12/73, 14/81), [10/57, 5/27), [14/73, 10/49), [15/73, 17/81),$
$[4/19, 2/9), [13/57, 19/81), [14/57, 20/81), [19/73, 13/49), [20/73, 2/7),$
$[21/73, 8/27), [17/57, 25/81), [23/73, 16/49), [24/73, 28/81), [20/57, 29/81),$
$[7/19, 10/27), [28/73, 19/49), [29/73, 20/49), [30/73, 34/81), [8/19, 35/81),$
$[32/73, 22/49), [33/73, 38/81), [9/19, 13/27), [28/57, 40/81), [37/73, 25/49),$
$[38/73, 26/49), [39/73, 44/81), [31/57, 5/9), [32/57, 4/7), [42/73, 16/27),$
$[34/57, 49/81), [35/57, 50/81), [46/73, 31/49), [47/73, 32/49), [48/73, 55/81),$
$[13/19, 34/49), [51/73, 58/81), [41/57, 59/81), [14/19, 20/27), [55/73, 37/49),$
$[56/73, 38/49), [57/73, 65/81), [46/57, 40/49), [60/73, 68/81), [16/19, 23/27),$
$[49/57, 70/81), [64/73, 43/49), [65/73, 44/49), [66/73, 25/27), [53/57, 46/49),$
$[69/73, 26/27), [55/57, 79/81), [56/57, 80/81),$

whence

$$\int_\Omega d\psi(z) = 52.18485975\ldots,$$

$$s_1 = 0.84050980\ldots, \qquad t_1 = 0.62988107\ldots, \qquad y_{\max} = 258.22891116\ldots,$$

$$\log f(s_1,t_1) = \log F(y_{\max}, x) = 303.76112912\ldots,$$

$$s_2 = -0.21836556\ldots + i\cdot 0.73972531\ldots,$$

$$t_2 = -0.33032145\ldots - i\cdot 0.53645881\ldots,$$

$$-\log|f(s_2,t_2)| = 87.29082912\ldots,$$

$$c_0 = 227.99970677\ldots, \qquad c_1 = 163.05225147\ldots, \qquad c_3 = 203.01462861\ldots,$$

hence $\mu_2(\log 2) < 15.65142024\ldots$.

Taking further values of a and b with $b = a+1$, and for h, j, k, l, m, q satisfying (7), we get the results in the table at the end of Section 1.

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Fakultät für Mathematik
Universität Wien
Nordbergstraße 15
1090 Vienna, Austria
E-mail: marcovr8@univie.ac.at

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