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Koecher–Maass series of a certain half-integral weight modular form related to the Duke–Imamoḡlu–Ikeda lift

by

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1. Introduction. Let l be an integer or a half-integer, and let F be a modular form of weight l for the congruence subgroup $\Gamma_0^{(m)}(N)$ of the symplectic group $Sp_m(\mathbb{Z})$. Then the *Koecher–Maass series* $L(s,F)$ of F is defined as

$$L(s,F)=\sum_A \frac{c_F(A)}{e(A)(\det A)^s},$$

where A runs over a complete set of representatives for the $SL_m(\mathbb{Z})$-equivalence classes of positive definite half-integral matrices of degree m, $c_F(A)$ is the Ath Fourier coefficient of F, and $e(A)$ denotes the order of the special orthogonal group of A. We note that $L(s,F)$ can also be obtained by the Mellin transform of F, and thus its analytic properties are relatively known. (For this, see Maass [19] and Arakawa [1–3].)

Now we are interested in an explicit form of the Koecher–Maass series for a specific choice of F. In particular, whenever F is a certain lift of an elliptic modular form h of either integral or half-integral weight, we may hope to express $L(s,F)$ in terms of certain Dirichlet series related to h. Indeed, this is realized in the case where F is a lift of h such that the weight l is an integer (cf. [8–10]). In this paper, we discuss a similar problem for lifts of elliptic modular forms to half-integral weight Siegel modular forms.

Let us explain our main result briefly. Let k and n be positive even integers. For a cuspidal Hecke eigenform h in the Kohnen plus space of weight $k-n/2+1/2$ for $\Gamma_0(4)$, let f be the primitive form of weight $2k-n$ for $SL_2(\mathbb{Z})$ corresponding to h under the Shimura correspondence, and let $I_n(h)$ be the Duke–Imamoḡlu–Ikeda lift of h (or of f) to the space of cusp forms of weight k for $Sp_n(\mathbb{Z})$. We note that $I_2(h)$ is nothing but the Saito–Kurokawa lift of h. Let $\phi_{I_n(h),1}$ be the first coefficient of the Fourier–Jacobi expansion of $I_n(h)$,

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and $\sigma_{n-1}(\phi_{I_n(h),1})$ the cusp form in the generalized Kohnen plus space of weight $k-1/2$ for $\Gamma_0^{(n-1)}(4)$ corresponding to $\phi_{I_n(h),1}$ under the Ibukiyama isomorphism σ_{n-1}. (For the precise definitions of the Duke–Imamoḡlu–Ikeda lift, the generalized Kohnen plus space and the Ibukiyama isomorphism, see Section 2.) Then our main result expresses $L(s,\sigma_{n-1}(\phi_{I_n(h),1}))$ in terms of $L(s,h)$ and $L(s,f)$ (cf. Theorem 2.1).

To prove Theorem 2.1, for a fundamental discriminant d_0 and a prime number p we define certain formal power series $P^{(1)}_{n-1,p}(d_0,\varepsilon^l,X,t) \in \mathbb{C}[X,X^{-1}][[t]]$ associated with some local Siegel series appearing in the p-factor of the Fourier coefficient of $\sigma_{n-1}(\phi_{I_n(h),1})$. Here ε is the Hasse invariant defined on the set of nondegenerate symmetric matrices with entries in $\mathbb{Q}_p$. We then rewrite $L(s,\sigma_{n-1}(\phi_{I_n(h),1}))$ in terms of the Euler products $\prod_p P^{(1)}_{n-1,p}(d_0,\varepsilon^l,\beta_p,p^{-s+k/2+n/4-1/4})$ with $l=0,1$, where β_p is the Satake p-parameter of f (cf. Theorem 3.2). By using a method similar to those in [9, 10], in Section 4 we get an explicit formula for the formal power series $P^{(1)}_{n-1,p}(d_0,\varepsilon^l,X,t)$ (cf. Theorem 4.4.1), which yields the desired formula for $L(s,\sigma_{n-1}(\phi_{I_n(h),1}))$ immediately. The above result is very simple, and the proof proceeds similarly to the one of [10], where we gave an explicit formula for the Koecher–Maass series of the Siegel–Eisenstein series of integral weight. However, it is more elaborate than the preceding one. For instance, we should be careful in dealing with the argument for $p=2$, which divides the level of $\sigma_{n-1}(\phi_{I_n(h),1})$. We also note that the method of this paper can be used to give an explicit formula for the Rankin–Selberg series of $\sigma_{n-1}(\phi_{I_n(h),1})$, and as a consequence, we can prove a conjecture of Ikeda [13] concerning the period of the Duke–Imamoḡlu–Ikeda lift; this will be done in [15].

Notation. Let R be a commutative ring. We denote by $R^\times$ and R^* the semigroup of nonzero elements of R and the unit group of R, respectively. We also put $S^\square = \{a^2 \mid a \in S\}$ for a subset S of R. We denote by $M_{ml}(R)$ the set of $m\times l$ matrices with entries in R. In particular we write $M_m(R) = M_{mm}(R)$. We put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, and $SL_m(R) = \{A \in GL_m(R) \mid \det A = 1\}$. For an $m\times l$ matrix X and an $m\times m$ matrix A, we write

$$A[X] = {}^tXAX,$$

where ${}^tX$ denotes the transpose of X. Let $S_m(R)$ denote the set of symmetric matrices of degree m with entries in R.

Furthermore, if R is an integral domain of characteristic different from 2, let $\mathcal{L}_m(R)$ denote the set of half-integral matrices of degree m over R, that is, $\mathcal{L}_m(R)$ is the subset of symmetric matrices of degree m with entries in the field of fractions of R, whose (i,j)th entry belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. In particular, we put $\mathcal{L}_m = \mathcal{L}_m(\mathbb{Z})$ and $\mathcal{L}_{m,p} = \mathcal{L}_m(\mathbb{Z}_p)$ for a prime number p.

For a subset S of $M_m(R)$ we denote by $S^\times$ the subset of S consisting of all nondegenerate matrices. If S is a subset of $S_m(\mathbb{R})$, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. The group $GL_m(R)$ acts on $S_m(R)$ as $GL_m(R) \times S_m(R) \ni (g, A) \mapsto A[g] \in S_m(R)$.

Let G be a subgroup of $GL_m(R)$. For a G-stable subset $\mathcal{B}$ of $S_m(R)$, we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ under the action of G. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives for $\mathcal{B}/G$. We abbreviate $\mathcal{B}/GL_m(R)$ as $\mathcal{B}/\sim$ if there is no risk of confusion.

For a given ring R', two symmetric matrices A and A' with entries in R are said to be *equivalent over* R', written $A \sim_{R'} A'$, if there is an element X of $GL_m(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no risk of confusion. For square matrices X and Y we write $X \perp Y = \left(\begin{smallmatrix} X & O \\ O & Y \end{smallmatrix}\right)$.

For an integer D with $D \equiv 0$ or $1 \bmod 4$, let $\mathfrak{d}_D$ be the discriminant of $\mathbb{Q}(\sqrt{D})$, and put $\mathfrak{f}_D = \sqrt{D/\mathfrak{d}_D}$. We call D a *fundamental discriminant* if it is either 1 or the discriminant of some quadratic field extension of $\mathbb{Q}$. For a fundamental discriminant D, let $\left(\frac{D}{*}\right)$ be the character corresponding to $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Here we make the convention that $\left(\frac{D}{*}\right) = 1$ if $D = 1$.

We put $\mathbf{e}(x) = \exp(2\pi i x)$ for $x \in \mathbb{C}$. For a prime number p we denote by $\nu_p(*)$ the additive valuation of $\mathbb{Q}_p$ normalized so that $\nu_p(p) = 1$, and by $\mathbf{e}_p(*)$ the continuous additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for $x \in \mathbb{Z}[p^{-1}]$.

For a nonnegative integer r we define a polynomial $\phi_r(x)$ in x by $\phi_r(x) = \prod_{i=1}^r (1 - x^i)$. Here we understand that $\phi_0(x) = 1$.

2. Main result. Put $J_m = \left(\begin{smallmatrix} O_m & -1_m \\ 1_m & O_m \end{smallmatrix}\right)$, where 1_m and O_m denote the unit matrix and the zero matrix of degree m, respectively. Furthermore, put

$$\Gamma^{(m)} = Sp_m(\mathbb{Z}) = \{M \in GL_{2m}(\mathbb{Z}) \mid J_m[M] = J_m\}.$$

Let l be an integer or a half-integer. For a congruence subgroup Γ of $\Gamma^{(m)}$, we denote by $\mathfrak{M}_l(\Gamma)$ the space of holomorphic modular forms of weight l for Γ. We denote by $\mathfrak{S}_l(\Gamma)$ the subspace of $\mathfrak{M}_l(\Gamma)$ consisting of all cusp forms. For a positive integer N, put $\Gamma_0^{(m)}(N) = \left\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(m)} \;\middle|\; C \equiv O_m \bmod N\right\}$.

Let $F(Z)$ be an element of $\mathfrak{M}_l(\Gamma_0^{(m)}(N))$. Then $F(Z)$ has Fourier expansion

$$F(Z) = \sum_{A \in (\mathcal{L}_m)_{\geq 0}} c_F(A)\mathbf{e}(\mathrm{tr}(AZ)),$$

where $\mathrm{tr}(X)$ denotes the trace of the matrix X. We then define the *Koecher–*

Maass series $L(s,F)$ of F as

$$L(s,F)=\sum_{A\in(\mathcal{L}_m)_{>0}/SL_m(\mathbb{Z})}\frac{c_F(A)}{e(A)(\det A)^s},$$

where $e(A)=\#\{X\in SL_m(\mathbb{Z})\mid A[X]=A\}$. We note that $L(s,F)$ is nothing but Hecke's L-function of F in the case where $m=1$ and l is an integer.

Now put

$$\mathcal{L}'_m=\{A\in\mathcal{L}_m\mid A\equiv -{}^t rr \bmod 4\mathcal{L}_m \text{ for some } r\in\mathbb{Z}^m\}.$$

For $A\in\mathcal{L}'_m$, the integral vector $r\in\mathbb{Z}^m$ in the above definition is uniquely determined by A modulo $2\mathbb{Z}^m$, and is denoted by r_A. Moreover it is easily shown that the matrix

$$\begin{pmatrix} 1 & r_A/2 \\ {}^t r_A/2 & ({}^t r_A r_A + A)/4 \end{pmatrix},$$

which will be denoted by $A^{(1)}$, belongs to $\mathcal{L}_{m+1}$, and that its $SL_{m+1}(\mathbb{Z})$-equivalence class is uniquely determined by A. Suppose that l is a positive even integer. We define the *generalized Kohnen plus space* of weight $l-1/2$ for $\Gamma_0^{(m)}(4)$ as

$$\mathfrak{M}^+_{l-1/2}(\Gamma_0^{(m)}(4))=\{F\in\mathfrak{M}_{l-1/2}(\Gamma_0^{(m)}(4))\mid c_F(A)=0 \text{ unless } A\in\mathcal{L}'_m\},$$

and put $\mathfrak{S}^+_{l-1/2}(\Gamma_0^{(m)}(4))=\mathfrak{M}^+_{l-1/2}(\Gamma_0^{(m)}(4))\cap\mathfrak{S}_{l-1/2}(\Gamma_0^{(m)}(4))$. Then there exists an isomorphism from the space of Jacobi forms of index 1 to the generalized Kohnen plus space, due to Ibukiyama. To explain this, let $\Gamma_J^{(m)}=\Gamma^{(m)}\ltimes \mathrm{H}_m(\mathbb{Z})$ be the Jacobi group, where $\Gamma^{(m)}$ is identified with its image inside $\Gamma^{(m+1)}$ via the natural embedding

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\mapsto\left(\begin{array}{cc|cc} 1 & & 0 & \\ & A & & B \\ \hline 0 & & 1 & \\ & C & & D \end{array}\right)$$

and

$\mathrm{H}_m(\mathbb{Z})=$

$$\left\{\left(\begin{array}{cc|cc} 1 & 0 & \kappa & \mu \\ 0 & 1_m & {}^t\mu & O_m \\ \hline & & 1 & 0 \\ & & 0 & 1_m \end{array}\right)\left(\begin{array}{cc|cc} 1 & \lambda & & \\ 0 & 1_m & & \\ \hline & & 1 & 0 \\ & & -{}^t\lambda & 1_m \end{array}\right)\ \middle|\ \begin{array}{l}(\lambda,\mu)\in\mathbb{Z}^m\oplus\mathbb{Z}^m, \\ \kappa\in\mathbb{Z}\end{array}\right\}.$$

Let $J_{l,1}(\Gamma_J^{(m)})$ denote the space of Jacobi forms of weight l and index 1 for $\Gamma_J^{(m)}$, and $J_{l,1}^{\mathrm{cusp}}(\Gamma_J^{(m)})$ the subspace of $J_{l,1}(\Gamma_J^{(m)})$ consisting of all cusp forms. Let $\phi(Z,z) \in J_{l,1}(\Gamma_J^{(m)})$. Then we have the following Fourier expansion:

$$\phi(Z,\, z) = \sum_{\substack{T \in \mathcal{L}_m,\, r \in \mathbb{Z}^m \\ 4T - {}^t rr \geq 0}} c_\phi(T, r)\mathbf{e}(\mathrm{tr}(TZ) + r^t z).$$

We then define $\sigma_m(\phi)$ as

$$\sigma_m(\phi) = \sum_{A \in (\mathcal{L}'_m)_{\geq 0}} c_\phi((A + {}^t r_A r_A)/4, r_A)\mathbf{e}(\mathrm{tr}(AZ)),$$

where $r = r_A$ denotes an element of $\mathbb{Z}^m$ such that $A + {}^t r_A r_A \in 4\mathcal{L}_m$. This r_A is uniquely determined modulo $2\mathbb{Z}^m$, and $c_\phi((A + {}^t r_A r_A)/4, r_A)$ does not depend on the choice of the representative of r_A mod $2\mathbb{Z}^m$. Ibukiyama [7] showed that σ_m gives a $\mathbb{C}$-linear isomorphism $J_{l,1}(\Gamma_J^{(m)}) \simeq \mathfrak{M}_{l-1/2}^+(\Gamma_0^{(m)}(4))$, and in particular, $\sigma_m(J_{l,1}^{\mathrm{cusp}}(\Gamma_J^{(m)})) = \mathfrak{S}_{l-1/2}^+(\Gamma_0^{(m)}(4))$. We call σ_m the *Ibukiyama isomorphism.*

Let p be a prime number. For a nonzero element $a \in \mathbb{Q}_p$ we put $\chi_p(a) = 1, -1$, or 0 according as $\mathbb{Q}_p(a^{1/2}) = \mathbb{Q}_p, \mathbb{Q}_p(a^{1/2})$ is an unramified quadratic extension of $\mathbb{Q}_p$, or $\mathbb{Q}_p(a^{1/2})$ is a ramified quadratic extension of $\mathbb{Q}_p$. We note that $\chi_p(D) = \left(\frac{D}{p}\right)$ if D is a fundamental discriminant.

For the rest of this section, let n be a positive even integer. For $T \in \mathcal{L}_{n,p}^\times$, put $\xi_p(T) = \chi_p((-1)^{n/2}\det T)$. Let $T \in \mathcal{L}_n^\times$. Then $(-1)^{n/2}\det(2T) \equiv 0$ or 1 mod 4, and we define $\mathfrak{d}_T = \mathfrak{d}_{(-1)^{n/2}\det(2T)}$ and $\mathfrak{f}_T = \mathfrak{f}_{(-1)^{n/2}\det(2T)}$. For $T \in \mathcal{L}_{n,p}^\times$, there exists $\widetilde{T} \in \mathcal{L}_n^\times$ such that $\widetilde{T} \sim_{\mathbb{Z}_p} T$. We then put $\mathfrak{e}_p(T) = \nu_p(\mathfrak{f}_{\widetilde{T}})$ and $[\mathfrak{d}_T] = \mathfrak{d}_{\widetilde{T}} \bmod (\mathbb{Z}_p^*)^\square$. These do not depend on the choice of $\widetilde{T}$. We note that $(-1)^{n/2}\det(2T)$ can be expressed as $(-1)^{n/2}\det(2T) = dp^{2\mathfrak{e}_p(T)} \bmod (\mathbb{Z}_p^*)^\square$ for any $d \in [\mathfrak{d}_T]$.

For each $T \in \mathcal{L}_{n,p}^\times$ we define the *local Siegel series* $b_p(T,s)$ by

$$b_p(T,s) = \sum_{R \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(TR))p^{-\nu_p(\mu_p(R))s},$$

where $\mu_p(R) = [\mathbb{Z}_p^n + \mathbb{Z}_p^n R : \mathbb{Z}_p^n]$. We remark that there exists a unique polynomial $F_p(T,X)$ in X such that

$$b_p(T,s) = F_p(T, p^{-s})\frac{(1-p^{-s})\prod_{i=1}^{n/2}(1-p^{2i-2s})}{1-\xi_p(T)p^{n/2-s}}$$

(cf. Kitaoka [16]). We then define a polynomial $\widetilde{F}_p(T,X)$ in X and X^{-1} as

$$\widetilde{F}_p(T,X) = X^{-\mathfrak{e}_p(T)}F_p(T, p^{-(n+1)/2}X).$$

We remark that $\widetilde{F}_p(T, X^{-1}) = \widetilde{F}_p(T, X)$ (cf. [14]). Now, for a positive even integer k, let

$$h(z) = \sum_{\substack{m \in \mathbb{Z}_{>0} \\ (-1)^{n/2} m \equiv 0,1 \bmod 4}} c_h(m)\mathbf{e}(mz)$$

be a Hecke eigenform in the Kohnen plus space $\mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$, and

$$f(z) = \sum_{m=1}^{\infty} c_f(m)\mathbf{e}(mz)$$

the primitive form in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$ corresponding to h under the Shimura correspondence (cf. Kohnen [18]). Let $\beta_p \in \mathbb{C}^\times$ be such that $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2} c_f(p)$, which we call the *Satake p-parameter* of f. We define a Fourier series $I_n(h)(Z)$ on $\mathbb{H}_n$ by

$$I_n(h)(Z) = \sum_{T \in (\mathcal{L}_n)_{>0}} c_{I_n(h)}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$c_{I_n(h)}(T) = c_h(|\mathfrak{d}_T|)\mathfrak{f}_T^{k-n/2-1/2} \prod_p \widetilde{F}_p(T, \beta_p).$$

Ikeda [12] showed that $I_n(h)(Z)$ is a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$ whose standard L-function coincides with $\zeta(s)\prod_{i=1}^n L(s+k-i, f)$, where $\zeta(s)$ is Riemann's zeta function. The existence of such a Hecke eigenform was conjectured by Duke and Imamoḡlu in an unpublished paper. We call $I_n(h)$ the *Duke–Imamoḡlu–Ikeda lift* of h (or of f), as in Section 1. Let $\phi_{I_n(h),1}$ be the first coefficient of the Fourier–Jacobi expansion of $I_n(h)$, that is,

$$I_n(h)\left(\begin{pmatrix} w & z \\ {}^t z & Z \end{pmatrix}\right) = \sum_{N=1}^{\infty} \phi_{I_n(h),N}(Z,\, z)\mathbf{e}(Nw),$$

where $Z \in \mathbb{H}_{n-1}$, $z \in \mathbb{C}^{n-1}$ and $w \in \mathbb{H}_1$. We easily see that $\phi_{I_n(h),1}$ belongs to $J^{\mathrm{cusp}}_{k-1/2,\,1}(\Gamma_J^{(n-1)})$ and

$$\phi_{I_n(h),1}(Z,\, z) = \sum_{\substack{T \in \mathcal{L}_{n-1},\, r \in \mathbb{Z}^{n-1} \\ 4T - {}^t rr > 0}} c_{I_n(h)}\left(\begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix}\right)\mathbf{e}(\mathrm{tr}(TZ) + r\,{}^t z).$$

Moreover we have

$$\sigma_{n-1}(\phi_{I_n(h),1})(Z) = \sum_{T \in (\mathcal{L}'_{n-1})_{>0}} c_{I_n(h)}(T^{(1)})\mathbf{e}(\mathrm{tr}(TZ)).$$

Put $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, and $\widetilde{\xi}(s) = \Gamma_{\mathbb{C}}(s)\zeta(s)$. Then our main result in this paper is stated as follows:

THEOREM 2.1. *Let h and f be as above. Then*

$$L(s,\sigma_{n-1}(\phi_{I_n(h),1})) = 2^{-\delta_{2,n}-s(n-2)-(n-2)/2}\prod_{i=1}^{(n-2)/2}\widetilde{\xi}(2i)$$
$$\times\Big\{L(s-n/2+1,h)\prod_{i=1}^{(n-2)/2}L(2s-n+2i+1,f)$$
$$+(-1)^{n(n-2)/8}L(s,h)\prod_{i=1}^{(n-2)/2}L(2s-n+2i,f)\Big\},$$

where $\delta_{2,n}$ denotes Kronecker's delta.

In the case of $n=2$, the modular form $\sigma_{n-1}(\phi_{I_n(h),1})$ is h itself, and then the above formula is trivial. We note that, unlike the cases of [8–10], there does not appear any convolution product of modular forms in the above theorem. However, the proof is not simple because the nature of Fourier coefficients of the modular form $\sigma_{n-1}(\phi_{I_n(h),1})$ is much more complicated than in the papers cited above.

3. Reduction to local computations. It turns out that the Fourier coefficient of $\sigma_{n-1}(\phi_{I_n(h),1})$ can be expressed in terms of a product of local Siegel series taken over all prime numbers p, and therefore we can reduce the problem to local computations. To explain this, we recall some terminology and notation. For given $a,b\in\mathbb{Q}_p^\times$ let $(a,b)_p$ denote the Hilbert symbol over $\mathbb{Q}_p$. Following Kitaoka [17], we define the *Hasse invariant* $\varepsilon(A)$ of $A\in S_m(\mathbb{Q}_p)^\times$ by

$$\varepsilon(A)=\prod_{1\le i\le j\le m}(a_i,a_j)_p$$

if A is equivalent to $a_1\perp\cdots\perp a_m$ over $\mathbb{Q}_p$ with some $a_1,\ldots,a_m\in\mathbb{Q}_p^\times$. We note that this definition does not depend on the choice of $a_1,\ldots,a_m$.

Now put

$$\mathcal{L}'_{m,p}=\{A\in\mathcal{L}_{m,p}\mid A\equiv -\,{}^t rr \bmod 4\mathcal{L}_{m,p}\text{ for some } r\in\mathbb{Z}_p^m\}.$$

Furthermore we set $S_m(\mathbb{Z}_p)_e=2\mathcal{L}_{m,p}$ and $S_m(\mathbb{Z}_p)_o=S_m(\mathbb{Z}_p)\setminus S_m(\mathbb{Z}_p)_e$. We note that $\mathcal{L}'_{m,p}=\mathcal{L}_{m,p}=S_m(\mathbb{Z}_p)$ if $p\ne 2$. Let $A\in\mathcal{L}'_{m-1,p}$. Then there exists an element $r\in\mathbb{Z}_p^{m-1}$ such that $\begin{pmatrix}1 & r/2\\ {}^t r/2 & (A+{}^t rr)/4\end{pmatrix}\in\mathcal{L}_{m,p}$. As is easily shown, r is uniquely determined by A modulo $2\mathbb{Z}_p^{m-1}$, and is denoted by r_A. Moreover, as will be shown in the next lemma, $\begin{pmatrix}1 & r_A/2\\ {}^t r_A/2 & (A+{}^t r_Ar_A)/4\end{pmatrix}$ is uniquely determined by A up to $GL_m(\mathbb{Z}_p)$-equivalence, and is denoted by $A^{(1)}$.

LEMMA 3.1. *Let m be a positive integer.*

(1) *Let A and B be elements of $\mathcal{L}'_{m-1,p}$. Then*

$$\begin{pmatrix} 1 & r_A/2 \\ {}^t r_A/2 & (A+{}^t r_A r_A)/4 \end{pmatrix} \sim \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B+{}^t r_B r_B)/4 \end{pmatrix} \quad \textit{if } A \sim B.$$

(2) *Let $A \in \mathcal{L}'_{m-1,p}$.*

(2.1) *Let $p \neq 2$. Then $A^{(1)} \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right)$.*

(2.2) *Let $p = 2$. If $r_A \equiv 0 \bmod 2$, then $A \sim 4B$ with $B \in \mathcal{L}_{m-1,2}$, and $A^{(1)} \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right)$. In particular, $\nu_2(\det B) \geq m$ or $m+1$ according as m is even or odd. If $r_A \not\equiv 0 \bmod 2$, then $A \sim a \perp 4B$ with $a \equiv -1 \bmod 4$ and $B \in \mathcal{L}_{m-2,2}$, and*

$$A^{(1)} \sim \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & (a+1)/4 & 0 \\ 0 & 0 & B \end{pmatrix}.$$

In particular, $\nu_2(\det B) \geq m$ or $m-1$ according as m is even or odd.

Proof. The assertion can be easily proved. ∎

Now suppose that m is even. For $T \in (\mathcal{L}'_{m-1})^\times$, put $\mathfrak{d}_T^{(1)} = \mathfrak{d}_{T^{(1)}}$ and $\mathfrak{f}_T^{(1)} = \mathfrak{f}_{T^{(1)}}$, and for $T \in (\mathcal{L}'_{m-1,p})^\times$, define $[\mathfrak{d}_T^{(1)}]$ and $\mathfrak{e}_T^{(1)}$ as $[\mathfrak{d}_{T^{(1)}}]$ and $\mathfrak{e}_{T^{(1)}}$, respectively. These do not depend on the choice of r_T. We note that $(-1)^{m/2}\det T = 2^{m-2}\mathfrak{f}_T^{(1)^2}\mathfrak{d}_T^{(1)}$ for $T \in (\mathcal{L}'_{m-1})^\times$. We define a polynomial $F_p^{(1)}(T,X)$ in X, and a polynomial $\widetilde{F}_p^{(1)}(T,X)$ in X and X^{-1}, by

$$F_p^{(1)}(T,X) = F_p(T^{(1)},X),$$
$$\widetilde{F}_p^{(1)}(T,X) = X^{-\mathfrak{e}_p^{(1)}(T)}F_p^{(1)}(T,p^{-(m+1)/2}X).$$

Let B be an element of $(\mathcal{L}'_{m-1,p})^\times$. Let $p \neq 2$. Then

$$\widetilde{F}_p^{(1)}(B,X) = \widetilde{F}_p(1 \perp B, X).$$

Let $p = 2$. Then

$$\widetilde{F}_2^{(1)}(B,X)$$
$$= \begin{cases} \widetilde{F}_2\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & (a+1)/4 \end{pmatrix} \perp B', X\right) & \text{if } B = a \perp 4B' \text{ with} \\ & \qquad a \equiv -1 \bmod 4,\ B' \in \mathcal{L}_{m-2,2}, \\ \widetilde{F}_2(1 \perp B', X) & \text{if } B = 4B' \text{ with } B' \in \mathcal{L}_{m-1,2}. \end{cases}$$

Now let m and l be positive integers such that $m \geq l$. Then for nondegenerate symmetric matrices A and B of degree m and l respectively with entries in $\mathbb{Z}_p$ we define the *local density* $\alpha_p(A,B)$ and the *primitive local density* $\beta_p(A,B)$ representing B by A as

$$\alpha_p(A,B) = 2^{-\delta_{m,l}} \lim_{a\to\infty} p^{a(-ml+l(l+1)/2)} \#\mathcal{A}_a(A,B),$$
$$\beta_p(A,B) = 2^{-\delta_{m,l}} \lim_{a\to\infty} p^{a(-ml+l(l+1)/2)} \#\mathcal{B}_a(A,B),$$

where

$$\mathcal{A}_a(A,B) = \{X \in M_{ml}(\mathbb{Z}_p)/p^a M_{ml}(\mathbb{Z}_p) \mid A[X] - B \in p^a S_l(\mathbb{Z}_p)_e\},$$
$$\mathcal{B}_a(A,B) = \{X \in \mathcal{A}_a(A,B) \mid \mathrm{rank}_{\mathbb{Z}_p/p\mathbb{Z}_p}(X \bmod p) = l\}.$$

In particular we write $\alpha_p(A) = \alpha_p(A,A)$. Put

$$\mathcal{F}_p = \{d_0 \in \mathbb{Z}_p \mid \nu_p(d_0) \le 1\}$$

if p is an odd prime, and

$$\mathcal{F}_2 = \{d_0 \in \mathbb{Z}_2 \mid d_0 \equiv 1 \bmod 4, \text{ or } d_0/4 \equiv -1 \bmod 4, \text{ or } \nu_2(d_0) = 3\}.$$

Let m be a positive integer. For $d \in \mathbb{Z}_p^\times$ put

$$S_m(\mathbb{Z}_p, d) = \{T \in S_m(\mathbb{Z}_p) \mid (-1)^{[(m+1)/2]} \det T = p^{2i} d \bmod \mathbb{Z}_p^{*\square} \text{ with some } i \in \mathbb{Z}\},$$

and $S_m(\mathbb{Z}_p,d)_x = S_m(\mathbb{Z}_p,d) \cap S_m(\mathbb{Z}_p)_x$ for $x = e$ or o. Set $\mathcal{L}_{m,p}^{(0)} = S_m(\mathbb{Z}_p)_e^\times$ and $\mathcal{L}_{m,p}^{(1)} = (\mathcal{L}'_{m,p})^\times$. We also define $\mathcal{L}_{m,p}^{(j)}(d) = S_m(\mathbb{Z}_p,d) \cap \mathcal{L}_{m,p}^{(j)}$ for $j = 0,1$. Let $\iota_{m,p}$ be the constant function on $\mathcal{L}_{m,p}^\times$ taking the value 1, and $\varepsilon_{m,p}$ the function on $\mathcal{L}_{m,p}^\times$ assigning the Hasse invariant of A to $A \in \mathcal{L}_{m,p}^\times$. We sometimes drop the suffix and write $\iota_{m,p}$ etc. as ι_p or ι if there is no risk of confusion.

From now on, we sometimes write $\omega = \varepsilon^l$ with $l = 0$ or 1 according as $\omega = \iota$ or ε.

Let n be a positive even integer. For $d_0 \in \mathcal{F}_p$ and $\omega = \varepsilon^l$ $(l = 0,1)$ we define a formal power series $P_{n-1,p}^{(1)}(d_0,\omega,X,t)$ in t by

$$P_{n-1,p}^{(1)}(d_0,\omega,X,t) = \kappa(d_0,n-1,l)^{-1} t^{\delta_{2,p}(2-n)} \sum_{B \in \mathcal{L}_{n-1,p}^{(1)}(d_0)} \frac{\widetilde{F}_p^{(1)}(B,X)}{\alpha_p(B)} \omega(B) t^{\nu(\det B)},$$

where

$$\kappa(d_0,r-1,l) = \kappa_p(d_0,r-1,l) = \{(-1)^{lr(r-2)/8} 2^{-(r-2)(r-1)/2}\}^{\delta_{2,p}} \cdot ((-1)^{r/2},(-1)^{r/2}d_0)_p^l \, p^{-(r/2-1)l\nu(d_0)}$$

for a positive even integer r. This type of formal power series appears in an explicit formula for the Koecher–Maass series associated with the Siegel–

Eisenstein series and the Duke–Imamoḡlu–Ikeda lift (cf. [9], [10]). Therefore we say that this formal power series is *of Koecher–Maass type*. An explicit formula for $P^{(1)}_{n-1,p}(d_0,\omega,X,t)$ will be given in the next section.

Let $\mathcal{F}$ denote the set of fundamental discriminants, and put

$$\mathcal{F}^{(\pm 1)} = \{d_0 \in \mathcal{F} \mid \pm d_0 > 0\}.$$

Now let h be a Hecke eigenform in $\mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$, and $f, I_n(h), \phi_{I_n(h),1}$ and $\sigma_{n-1}(\phi_{I_n(h),1})$ be as in Section 2.

Theorem 3.2. *Let the notation and assumptions be as above. Then for* $\mathrm{Re}(s) \gg 0$, *we have*

$$\begin{aligned}
L(s,\sigma_{n-1}(\phi_{I_n(h),1})) &= \kappa_{n-1} 2^{-(n-2)s-(n-2)/2-\delta_{2,n}} \\
&\times \Big\{ \sum_{d_0 \in \mathcal{F}^{((-1)^{n/2})}} c_h(|d_0|)|d_0|^{n/4-k/2+1/4} \\
&\qquad\qquad \times \prod_p P^{(1)}_{n-1,p}(d_0,\iota_p,\beta_p,p^{-s+k/2+n/4-1/4}) \\
&\quad + (-1)^{n(n-2)/8} \sum_{d_0 \in \mathcal{F}^{((-1)^{n/2})}} c_h(|d_0|)|d_0|^{-n/4-k/2+5/4} \\
&\qquad\qquad \times \prod_p P^{(1)}_{n-1,p}(d_0,\varepsilon_p,\beta_p,p^{-s+k/2+n/4-1/4}) \Big\},
\end{aligned}$$

where $\kappa_{n-1} = \prod_{i=1}^{(n-2)/2} \Gamma_{\mathbb{C}}(2i)$.

Proof. Let $T \in (\mathcal{L}'_{n-1})_{>0}$. It follows from Lemma 3.1 that the Tth Fourier coefficient $c_{\sigma_{n-1}(\phi_{I_n(h),1})}(T)$ of $\sigma_{n-1}(\phi_{I_n(h),1})$ is uniquely determined by the genus to which T belongs, and by definition, it can be expressed as

$$c_{\sigma_{n-1}(\phi_{I_n(h),1})}(T) = c_{I_n(h)}(T^{(1)}) = c_h(|\mathfrak{d}^{(1)}_T|)(\mathfrak{f}^{(1)}_T)^{k-n/2-1/2} \prod_p \widetilde{F}^{(1)}(T,\beta_p).$$

We note that

$$(\mathfrak{f}^{(1)}_T)^{k-n/2-1/2} = |\mathfrak{d}^{(1)}_T|^{-(k/2-n/4-1/4)} (\det T)^{(k/2-n/4-1/4)} 2^{-(n-2)(k/2-n/4-1/4)}$$

for $T \in (\mathcal{L}_{n-1})_{>0}$. We also note that

$$\sum_{T' \in \mathcal{G}(T)} \frac{1}{e(T')} = \kappa_{n-1} 2^{3-n-\delta_{2,n}} \det T^{n/2} \prod_p \alpha_p(T)^{-1}$$

for $T \in S_{n-1}(\mathbb{Z})_{>0}$, where $\mathcal{G}(T)$ denotes the set of $SL_{n-1}(\mathbb{Z})$-equivalence classes belonging to the genus of T (cf. [17, Theorem 6.8.1]). Hence

$$\sum_{T'\in\mathcal{G}(T)} \frac{c_{\sigma_{n-1}(\phi_{I_n(h),1})}(T)}{e(T')} = \kappa_{n-1} 2^{3-n-\delta_{2,n}-(n-2)(k/2-n/4-1/4)} \times \det(T^{k/2+n/4-1/4}) |\mathfrak{d}_T^{(1)}|^{-k/2+n/4+1/4} \prod_p \frac{\widetilde{F}_p^{(1)}(T,\beta_p)}{\alpha_p(T)}.$$

Thus, by using the same method as in [11, Proposition 2.2], similarly to [8, Theorem 3.3(1)], and [9, Theorem 3.2], we obtain

$$\begin{aligned}
&L(s,\sigma_{n-1}(\phi_{I_n(h),1}))\\
&= \kappa_{n-1} 2^{-(k/2-n/4-1/4)(n-2)+2-n-\delta_{2,n}} \sum_{d_0\in\mathcal{F}^{((-1)^{n/2})}} c_h(|d_0|)|d_0|^{n/4-k/2+1/4}\\
&\times\Big\{2^{(-s+k/2+n/4-1/4)(n-2)} \prod_p \kappa_p(d_0,n-1,0) P_{n-1,p}^{(1)}(d_0,\iota_p,\beta_p,p^{-s+k/2+n/4-1/4})\\
&+2^{(-s+k/2+n/4-1/4)(n-2)} \prod_p \kappa_p(d_0,n-1,1) P_{n-1,p}^{(1)}(d_0,\varepsilon_p,\beta_p,p^{-s+k/2+n/4-1/4})\Big\}.
\end{aligned}$$

We note that $\prod_p((-1)^{n/2},(-1)^{n/2}d_0)_p = 1$. Hence $\prod_p \kappa_p(d_0,n-1,0) = 2^{-(n-2)(n-1)/2}$, and $\prod_p \kappa_p(d_0,n-1,1) = (-1)^{n(n-2)/8}|d_0|^{-n/2+1}2^{-(n-2)(n-1)/2}$. This proves the assertion. ■

4. Formal power series associated to local Siegel series. Throughout this section we fix a positive even integer n. We also simply write ν_p etc. as ν if the prime number p is clear from the context. In this section, we give an explicit formula for $P_{n-1}^{(1)}(d_0,\omega,X,t)$ with $\omega=\varepsilon^l$ ($l=0$, 1) to prove Theorem 3.2 (cf. Theorem 4.4.1). The idea of the proof is to express the power series in question as a sum of certain subseries (cf. Proposition 4.4.3). Henceforth, for a $GL_m(\mathbb{Z}_p)$-stable subset $\mathcal{B}$ of $S_m(\mathbb{Q}_p)$, we simply write $\sum_{T\in\mathcal{B}}$ instead of $\sum_{T\in\mathcal{B}/\sim}$ if there is no risk of confusion.

4.1. Local densities. Put

$$\mathcal{D}_{m,i} = GL_m(\mathbb{Z}_p)\begin{pmatrix} 1_{m-i} & 0 \\ 0 & p1_i \end{pmatrix} GL_m(\mathbb{Z}_p).$$

For $S,T\in S_m(\mathbb{Z}_p)^\times$ and a nonnegative integer $i\le m$, we define

$$\alpha_p(S,T,i) = 2^{-1}\lim_{e\to\infty} p^{-e(m-1)m/2}\mathcal{A}_e(S,T,i),$$

where

$$\mathcal{A}_e(S,T,i) = \{\overline{X}\in\mathcal{A}_e(S,T) \mid X\in\mathcal{D}_{m,i}\}.$$

LEMMA 4.1.1. *Let $S, T \in S_m(\mathbb{Z}_p)^\times$.*

(1) *Let $\Omega(S,T) = \{W \in M_m(\mathbb{Z}_p)^\times \mid S[W] \sim T\}$ and $\Omega(S,T,i) = \Omega(S,T) \cap \mathcal{D}_{m,i}$. Then*
$$\frac{\alpha_p(S,T)}{\alpha_p(T)} = \#(\Omega(S,T)/GL_m(\mathbb{Z}_p))p^{-m(\nu(\det T)-\nu(\det S))/2},$$
$$\frac{\alpha_p(S,T,i)}{\alpha_p(T)} = \#(\Omega(S,T,i)/GL_m(\mathbb{Z}_p))p^{-m(\nu(\det T)-\nu(\det S))/2}.$$

(2) *Let $\widetilde{\Omega}(S,T) = \{W \in M_m(\mathbb{Z}_p)^\times \mid S \sim T[W^{-1}]\}$ and $\widetilde{\Omega}(S,T,i) = \widetilde{\Omega}(S,T) \cap \mathcal{D}_{m,i}$. Then*
$$\frac{\alpha_p(S,T)}{\alpha_p(S)} = \#(GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}(S,T))p^{(\nu(\det T)-\nu(\det S))/2},$$
$$\frac{\alpha_p(S,T,i)}{\alpha_p(S)} = \#(GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}(S,T,i))p^{(\nu(\det T)-\nu(\det S))/2}.$$

Proof. The assertion (1) follows from [4, Lemma 2.2]. Now by [14, Proposition 2.2] we have
$$\alpha_p(S,T) = \sum_{W \in GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}(S,T)} \beta_p(S,T[W^{-1}])p^{\nu(\det W)}.$$
Then $\beta_p(S,T[W^{-1}]) = \alpha_p(S)$ or 0 according as $S \sim T[W^{-1}]$ or not. Thus (2) holds. ∎

A nondegenerate $m \times m$ matrix $D = (d_{ij})$ with entries in $\mathbb{Z}_p$ is said to be *reduced* if it satisfies the following two conditions:

(a) For $i = j$, $d_{ii} = p^{e_i}$ with a nonnegative integer e_i.
(b) For $i \neq j$, d_{ij} is a nonnegative integer satisfying $d_{ij} \leq p^{e_j} - 1$ if $i < j$ and $d_{ij} = 0$ if $i > j$.

It is well known that we can take the set of all reduced matrices as a complete set of representatives for $GL_m(\mathbb{Z}_p)\backslash M_m(\mathbb{Z}_p)^\times$. Let $j = 0$ or 1 according as m is even or odd. For $B \in \mathcal{L}^{(j)}_{m,p}$ put
$$\widetilde{\Omega}^{(j)}(B) = \{W \in M_m(\mathbb{Z}_p)^\times \mid B[W^{-1}] \in \mathcal{L}^{(j)}_{m,p}\}.$$
Furthermore set $\widetilde{\Omega}^{(j)}(B,i) = \widetilde{\Omega}^{(j)}(B) \cap \mathcal{D}_{m,i}$. Let $n_0 \leq m$, and $\psi_{n_0,m}$ be the mapping from $GL_{n_0}(\mathbb{Q}_p)$ into $GL_m(\mathbb{Q}_p)$ defined by $\psi_{n_0,m}(D) = 1_{m-n_0} \perp D$.

LEMMA 4.1.2.

(1) *Suppose $p \neq 2$. Let $\Theta \in GL_{n_0}(\mathbb{Z}_p) \cap S_{n_0}(\mathbb{Z}_p)$ and $B_1 \in S_{m-n_0}(\mathbb{Z}_p)^\times$.*

(1.1) *Let n_0 be even. Then $\psi_{m-n_0,m}$ induces a bijection*
$$GL_{m-n_0}(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j)}(B_1) \simeq GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j)}(\Theta \perp B_1),$$

where $j=0$ *or* 1 *according as* m *is even or odd. In particular,*

$$GL_{m-n_0}(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j)}(pB_1)\simeq GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j)}(\Theta\perp pB_1).$$

(1.2) *Let* n_0 *be odd. Then* $\psi_{m-n_0,m}$ *induces a bijection*

$$GL_{m-n_0}(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j')}(B_1)\simeq GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j)}(\Theta\perp B_1),$$

where $j=0$ *or* 1 *according as* m *is even or odd, and* $j'=1$ *or* 0 *according as* m *is even or odd. In particular,*

$$GL_{m-n_0}(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j')}(pB_1)\simeq GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j)}(\Theta\perp pB_1).$$

(2) *Suppose that* $p=2$. *Let* m *be a positive integer,* n_0 *an even integer not greater than* m, *and* $\Theta\in GL_{n_0}(\mathbb{Z}_2)\cap S_{n_0}(\mathbb{Z}_2)_e$.

(2.1) *Let* $B_1\in S_{m-n_0}(\mathbb{Z}_2)^{\times}$. *Then* $\psi_{m-n_0,m}$ *induces a bijection*

$$GL_{m-n_0}(\mathbb{Z}_2)\backslash\widetilde{\Omega}^{(j)}(2^{j+1}B_1)\simeq GL_m(\mathbb{Z}_2)\backslash\widetilde{\Omega}^{(j)}(2^j\Theta\perp 2^{j+1}B_1),$$

where $j=0$ *or* 1 *according as* m *is even or odd.*

(2.2) *Suppose that* m *is even. Let* $a\in\mathbb{Z}_2$ *be such that* $a\equiv -1 \bmod 4$, *and* $B_1\in S_{m-n_0-2}(\mathbb{Z}_2)^{\times}$. *Then* $\psi_{m-n_0-1,m}$ *induces a bijection*

$$\begin{aligned}&GL_{m-n_0-1}(\mathbb{Z}_2)\backslash\widetilde{\Omega}^{(1)}(a\perp 4B_1)\\&\qquad\simeq GL_m(\mathbb{Z}_2)\backslash\widetilde{\Omega}^{(0)}\left(\Theta\perp\begin{pmatrix}2 & 1\\ 1 & (1+a)/2\end{pmatrix}\perp 2B_1\right).\end{aligned}$$

(2.3) *Suppose that* m *is even, and let* $B_1\in S_{m-n_0-1}(\mathbb{Z}_2)^{\times}$. *Then there exists a bijection* $\widetilde{\psi}_{m-n_0-1,m}$

$$GL_{m-n_0-1}(\mathbb{Z}_2)\backslash\widetilde{\Omega}^{(1)}(4B_1)\simeq GL_m(\mathbb{Z}_2)\backslash\widetilde{\Omega}^{(0)}(\Theta\perp 2\perp 2B_1).$$

(*As will be seen,* $\widetilde{\psi}_{m-n_0-1,m}$ *is not induced from* $\psi_{m-n_0-1,m}$.)

(3) *Assertions* (1), (2) *remain valid if one replaces* $\widetilde{\Omega}^{(j)}(B)$ *by* $\widetilde{\Omega}^{(j)}(B,i)$.

Proof. (1) Clearly the mapping $\psi_{m-n_0,m}$ induces an injection from the set $GL_{m-n_0}(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j)}(B_1)$ to $GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j)}(\Theta\perp B_1)$, denoted by the same symbol. To prove the surjectivity of $\psi_{m-n_0,m}$, take a representative D of an element of $GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(j)}(\Theta\perp B_1)$. Without loss of generality we may suppose that D is a reduced matrix. Since $(\Theta\perp B_1)[D^{-1}]\in S_m(\mathbb{Z}_p)$, we have $D=\left(\begin{smallmatrix}1_{n_0} & 0\\ 0 & D_1\end{smallmatrix}\right)$ with $D_1\in\widetilde{\Omega}^{(j)}(B_1)$. This proves (1.1); and (1.2) can be proved in the same way.

(2) First we prove (2.1). As in (1), the mapping $\psi_{m-n_0,m}$ induces an injection from $GL_{m-n_0}(\mathbb{Z}_2)\backslash\widetilde{\Omega}^{(j)}(2^{j+1}B_1)$ to $GL_m(\mathbb{Z}_2)\backslash\widetilde{\Omega}^{(j)}(2^j\Theta\perp 2^{j+1}B_1)$, denoted by the same symbol. Then the surjectivity of $\psi_{m-n_0,m}$ in case $j=0$ can be proved in the same manner as (1). To prove the surjectivity of $\psi_{m-n_0,m}$ in case $j=1$, take a reduced matrix $D=\left(\begin{smallmatrix}D_1 & D_{12}\\ 0 & D_2\end{smallmatrix}\right)$ with

$D_1 \in M_{n_0}(\mathbb{Z}_2)^\times$, $D_2 \in M_{m-n_0}(\mathbb{Z}_2)^\times$, $D_{12} \in M_{n_0,m-n_0}(\mathbb{Z}_2)$. If $(2\Theta \perp 4B_1)[D^{-1}] \in \mathcal{L}^{(1)}_{m,2}$, then there exists $(r_1, r_2) \in \mathbb{Z}_2^{n_0} \times \mathbb{Z}_2^{m-n_0}$ such that

$$2\Theta[D_1^{-1}] \equiv -{}^t r_1 r_1 \bmod 4\mathcal{L}_{n_0,2},$$
$$-2\Theta[D_1^{-1}]D_{12}D_2^{-1} \equiv -{}^t r_2 r_1 \bmod 2M_{n_0,m-n_0}(\mathbb{Z}_2),$$
$$2\Theta[D_1^{-1}D_{12}D_2^{-1}] + 4B_1[D_2^{-1}] \equiv -{}^t r_2 r_2 \bmod 4\mathcal{L}_{m-n_0,2}.$$

We have $\nu(\det(2\Theta[D_1^{-1}])) \geq n_0$ and $\nu(2\Theta) = n_0$. Hence $D_1 = 1_{n_0}$ and $r_1 \equiv 0 \bmod 2$. Therefore $4B_1[D_2^{-1}] \in \mathcal{L}^{(1)}_{m-n_0}$ and $D_{12}D_2^{-1} \in M_{n_0,m-n_0}(\mathbb{Z}_2)$. Consequently, $D = U\left(\begin{smallmatrix} 1_{n_0} & 0 \\ 0 & D_2 \end{smallmatrix}\right)$ with $U \in GL_m(\mathbb{Z}_p)$. Thus the surjectivity of $\psi_{m-n_0,m}$ can be proved as above. The assertion (2.2) can be proved in the same way.

To prove (2.3), we may suppose $n_0 = 0$ by (2.1). Let $D \in \widetilde{\Omega}^{(1)}(4B_1)$. Then

$$4B_1[D^{-1}] = -{}^t r_0 r_0 + 4B'$$

with $r_0 \in \mathbb{Z}_2^{m-1}$ and $B' \in \mathcal{L}_{m-1,2}$. Then we can take $r \in \mathbb{Z}_2^{m-1}$ such that

$$4\,{}^tD^{-1}\,{}^t r r D^{-1} \equiv {}^t r_0 r_0 \bmod 4\mathcal{L}_{m-1,2}.$$

Furthermore, $2rD^{-1}$ is uniquely determined modulo $2\mathbb{Z}_2^{m-1}$ by r_0. Put $\widetilde{D} = \left(\begin{smallmatrix} 1 & r \\ 0 & D \end{smallmatrix}\right)$. Then $\widetilde{D}$ belongs to $\widetilde{\Omega}^{(0)}(2 \perp 2B_1)$, and the mapping $D \mapsto \widetilde{D}$ induces the bijection in question. ■

Corollary. *Suppose that m is even. Let $B \in \mathcal{L}^{(1)}_{m-1,p}$, and*

$$B^{(1)} = \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B + {}^t r_B r_B)/4 \end{pmatrix}$$

with $r_B \in \mathbb{Z}_p^{m-1}$ as defined in Section 3. Then there exists a bijection

$$\psi : GL_{m-1}(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(1)}(B) \simeq GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(0)}(2^{\delta_{2,p}}B^{(1)})$$

such that $\nu(\det\psi(W)) = \nu(\det W)$ for any $W \in GL_{m-1}(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(1)}(B)$. Moreover, ψ induces a bijection

$$\psi_i : GL_{m-1}(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(1)}(B, i) \to GL_m(\mathbb{Z}_p)\backslash\widetilde{\Omega}^{(0)}(2^{\delta_{2,p}}B^{(1)}, i)$$

for $i = 0, \dots, m-1$.

Proof. Let $p \neq 2$. Then we may suppose $r_B = 0$, and the assertion follows from (1.2). Let $p = 2$. If $r_B \equiv 0 \bmod 2$ we may suppose that $r_B = 0$, and the assertion follows from (2.3). If $r_B \not\equiv 0 \bmod 2$, we may suppose that $B = a \perp 4B_1$ with $B_1 \in \mathcal{L}^\times_{m-2,2}$ and $r_B = (1, 0, \dots, 0)$. Then the assertion follows from (2.2). ■

LEMMA 4.1.3. *Suppose that $p \neq 2$.*

(1) *Let $B \in S_m(\mathbb{Z}_p)^\times$. Then*
$$\alpha_p(p^r dB) = p^{rm(m+1)/2}\alpha_p(B)$$
for any nonnegative integer r and $d \in \mathbb{Z}_p^*$.

(2) *Let $U_1 \in GL_{n_0}(\mathbb{Z}_p) \cap S_{n_0}(\mathbb{Z}_p)$ and $B_1 \in S_{m-n_0}(\mathbb{Z}_p)^\times$. Then*
$$\alpha_p(pB_1 \perp U_1) = 2^{r(n_0)}\,\alpha_p(pB_1) \times \begin{cases} \displaystyle\prod_{i=1}^{n_0/2}(1-p^{-2i})(1+\chi((-1)^{n_0/2}\det U_1)p^{-n_0/2})^{-1} & \text{if } n_0 \text{ even},\\ \displaystyle\prod_{i=1}^{(n_0-1)/2}(1-p^{-2i}) & \text{if } n_0 \text{ odd}, \end{cases}$$
for $n_0 \geq 1$, where $r(n_0) = 0$ or 1 according as $n_0 = m$ or not.

Proof. The assertion (1) follows from [17, Theorem 5.6.4(a)], while (2) follows from [17, p. 110, line 4 from the bottom]. ∎

LEMMA 4.1.4.

(1) *Let $B \in S_m(\mathbb{Z}_2)^\times$. Then*
$$\alpha_2(2^r dB) = 2^{rm(m+1)/2}\alpha_2(B)$$
for any nonnegative integer r and $d \in \mathbb{Z}_2^*$.

(2) *Let n_0 be even and let $U_1 \in GL_{n_0}(\mathbb{Z}_2) \cap S_{n_0}(\mathbb{Z}_2)_e$. Then for $B_1 \in S_{m-n_0}(\mathbb{Z}_2)^\times$ we have*
$$\alpha_2(U_1 \perp 2B_1) = 2^{r(n_0)}\alpha_2(2B_1) \times \begin{cases} \displaystyle\prod_{i=1}^{n_0/2}(1-2^{-2i})(1+\chi((-1)^{n_0/2}\det U_1)p^{-n_0/2})^{-1} & \text{if } B_1 \in S_{m-n_0}(\mathbb{Z}_2)_e,\\ \displaystyle\prod_{i=1}^{(n_0-1)/2}(1-2^{-2i}) & \text{if } B_1 \in S_{m-n_0}(\mathbb{Z}_2)_o, \end{cases}$$
and for $u_0 \in \mathbb{Z}_2^*$ and $B_2 \in S_{m-n_0-1}(\mathbb{Z}_2)^\times$ we have
$$\alpha_2(u_0 \perp 2U_1 \perp 4B_2) = \alpha_2(2B_2)2^{m(m-1)/2+1}\prod_{i=1}^{n_0/2}(1-2^{-2i}).$$

(3) *Let $u_0 \in \mathbb{Z}_2^*$ and $B_1 \in S_{m-1}(\mathbb{Z}_2)^\times$. Then*
$$\alpha_2(u_0 \perp 5B_1) = \alpha_2(u_0 \perp B_1).$$

Proof. The assertion (1) follows from [17, Theorem 5.6.4(a)], and (2) follows from [17, (4), p. 111]. For a nondegenerate half-integral matrix A, let W_A be the quadratic space over $\mathbb{Z}_p$ associated with A, and $n_{A,j}, q_{A,j}$ and $E_{A,j}$ be the quantities n_j, q_j and E_j, respectively, from [17, p. 109] defined for W_A. Then the transformation $u_0 \perp B_1 \mapsto u_0 \perp 5B_1$ does not change these quantities. This proves (3). ∎

Now let R be a commutative ring. Then the group $GL_m(R) \times R^*$ acts on $S_m(R)$ in a natural way. We write $B_1 \approx_R B_2$ if $B_2 \sim_R \xi B_1$ with some $\xi \in R^*$. Let m be a positive integer. Then for $B \in S_m(\mathbb{Z}_p)$ let $\widetilde{\mathcal{S}}_{m,p}(B)$ denote the set of elements $B' \in S_m(\mathbb{Z}_p)$ such that $B' \approx_{\mathbb{Z}_p} B$, and let $\mathcal{S}_{m-1,p}(B)$ denote the set of elements $B' \in S_{m-1}(\mathbb{Z}_p)$ such that $1 \perp B' \approx_{\mathbb{Z}_p} B$.

LEMMA 4.1.5. *Let m be a positive even integer. Let $B \in S_m(\mathbb{Z}_2)_o^{\times}$. Then*

$$\sum_{B' \in \mathcal{S}_{m-1,2}(B)/\sim} \frac{1}{\alpha_2(B')} = \frac{\#(\widetilde{\mathcal{S}}_{m,2}(B)/\sim)}{2\alpha_2(B)}.$$

Proof. For a positive integer l let $l = l_1 + \cdots + l_r$ be the partition of l by positive integers, and $\{s_i\}_{i=1}^r$ the set of nonnegative integers such that $0 \le s_1 < \cdots < s_r$. Then for a positive integer e let $S_l^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{l_i\}, \{s_i\})$ be the subset of $S_l(\mathbb{Z}_2/2^e\mathbb{Z}_2)$ consisting of all symmetric matrices of the form $2^{s_1}U_1 \perp \cdots \perp 2^{s_r}U_r$ with $U_i \in S_{l_i}(\mathbb{Z}_2/2^e\mathbb{Z}_2)$ unimodular. Let $B \in S_m(\mathbb{Z}_2)_o$ and $\det B = (-1)^{m/2}d$. Then B is equivalent, over $\mathbb{Z}_2$, to a matrix of the form

$$2^{t_1}W_1 \perp \cdots \perp 2^{t_r}W_r,$$

where $0 = t_1 < \cdots < t_r$ and $W_1, \ldots, W_r$ are unimodular matrices of degree $n_1, \ldots, n_r$, respectively, and in particular, W_1 is odd unimodular. Then by [11, Lemma 3.2], similarly to [11, (3.5)], for a sufficiently large integer e, we have

$$\begin{aligned}\frac{\#(\widetilde{\mathcal{S}}_{m,2}(B)/\sim)}{\alpha_2(B)} &= \sum_{\widetilde{B} \in \widetilde{\mathcal{S}}_{m,2}(B)/\sim} \frac{1}{\alpha_2(\widetilde{B})} \\ &= 2^{m-1} 2^{-\nu(d) + \sum_{i=1}^r n_i(n_i-1)e/2 - (r-1)(e-1) - \sum_{1 \le j < i \le r} n_i n_j t_j} \\ &\quad \times \prod_{i=1}^r \#(SL_{n_i}(\mathbb{Z}_2/2^e\mathbb{Z}_2))^{-1} \# \widetilde{S}_m^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i\}, \{t_i\}, B),\end{aligned}$$

where $\widetilde{S}_m^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i\}, \{t_i\}, B)$ is the subset of $S_m^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i\}, \{t_i\})$ consisting of all matrices A such that $A \approx_{\mathbb{Z}_2/2^e\mathbb{Z}_2} B$. We note that our local density $\alpha_2(\widetilde{B})$ is 2^{-m} times that in [11] for $\widetilde{B} \in S_m(\mathbb{Z}_2)$. If $n_1 \ge 2$, put $r' = r$, $n_1' = n_1 - 1$, $n_2' = n_2, \ldots, n_r' = n_r$, and $t_i' = t_i$ for $i = 1, \ldots, r'$, and if $n_1 = 1$, put $r' = r - 1$, $n_i' = n_{i+1}$ and $t_i' = t_{i+1}$ for $i = 1, \ldots, r'$. Let $S_{m-1}^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i'\}, \{t_i'\}, B)$ be the subset of $S_{m-1}^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i'\}, \{t_i'\})$ consisting of all matrices $B' \in S_{m-1}(\mathbb{Z}_2/2^e\mathbb{Z}_2)$ such that $1 \perp B' \approx_{\mathbb{Z}_2/2^e\mathbb{Z}_2} B$.

Then, similarly, we obtain

$$\sum_{B' \in \mathcal{S}_{m-1,2}(B)/\sim} \frac{1}{\alpha_2(B')}$$
$$= 2^{m-2} 2^{-\nu(d)+\sum_{i=1}^{r'} n_i'(n_i'-1)e/2-(r'-1)(e-1)-\sum_{1\le j<i\le r'} n_i'n_j't_j'}$$
$$\times \prod_{i=1}^{r'} \#(SL_{n_i'}(\mathbb{Z}_2/2^e\mathbb{Z}_2))^{-1} \# S_{m-1}^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i'\}, \{t_i'\}, B).$$

Take $A \in \widetilde{S}_m^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i\}, \{t_i\}, B)$. Then

$$A = 2^{t_1}U_1 \perp \cdots \perp 2^{t_r}U_r$$

with $U_i \in S_{n_i}(\mathbb{Z}_2/2^e\mathbb{Z}_2)$ unimodular. Put $U_1 = (u_{\lambda\mu})$. Then by the assumption there exists an integer $1 \le \lambda \le n_1$ such that $u_{\lambda\lambda} \in \mathbb{Z}_2^*$. Let λ_0 be the least such integer, and V_1 be the matrix obtained from U_1 by interchanging the first and λ_0th rows and the first and λ_0th columns. Write

$$V_1 = \begin{pmatrix} v_1 & \mathbf{v}_1 \\ {}^t\mathbf{v}_1 & V' \end{pmatrix}$$

with $v_1 \in \mathbb{Z}_2^*$, $\mathbf{v}_1 \in \mathbb{Z}_2^{n_1-1}$ and $V' \in S_{n_1-1}(\mathbb{Z}_2)$. Here we understand that $V' - {}^t\mathbf{v}_1\mathbf{v}_1$ is the empty matrix if $n_1 = 1$. Then

$$V_1 \sim \begin{pmatrix} v_1 & 0 \\ 0 & V' - v_1^{-1}[\mathbf{v}_1] \end{pmatrix}.$$

Then the map $A \mapsto v_1^{-1}(2^{t_1}(V' - v_1^{-1}[\mathbf{v}_1]) \perp 2^{t_2}U_2 \perp \cdots \perp 2^{t_r}U_r)$ induces a map Υ from $\widetilde{S}_m^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i\}, \{t_i\}, B)$ to $S_{m-1}^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i'\}, \{t_i'\}, B)$. By a simple calculation, we obtain

$$\#\Upsilon^{-1}(B') = 2^{(e-1)n_1}(2^{n_1}-1)$$

for any $B' \in S_{m-1}^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2, \{n_i'\}, \{t_i'\}, B)$. We also note that

$$\#SL_{n_1}(\mathbb{Z}_2/2^e\mathbb{Z}_2) = 2^{(e-1)(2n_1-1)}2^{n_1-1}(2^{n_1}-1)\#(SL_{n_1-1}(\mathbb{Z}_2/2^e\mathbb{Z}_2)) \text{ or } 1$$

according as $n_1 \ge 2$ or $n_1 = 1$, and

$$\sum_{i=1}^{r} n_i(n_i-1)e/2 - (r-1)(e-1) - \sum_{1\le j<i\le r} n_in_jt_j$$
$$= e_{n_1} + \sum_{i=1}^{r'} n_i'(n_i'-1)e/2 - (r'-1)(e-1) + \sum_{1\le j<i\le r'} n_i'n_j't_j',$$

where $e_{n_1} = (n_1-1)e$ or $e_{n_1} = 1-e$ according as $n_1 \ge 2$ or $n_1 = 1$. Hence

$$2^{m-1}2^{-\nu(d)+\sum_{i=1}^{r} n_i(n_i-1)e/2-(r-1)(e-1)-\sum_{1\le j<i\le r} n_in_jt_j}$$
$$\times \prod_{i=1}^{r} \#(SL_{n_i}(\mathbb{Z}_2/2^e\mathbb{Z}_2))^{-1}\#\widetilde{S}_m^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2,\{n_i\},\{t_i\},B)$$
$$= 2\cdot 2^{m-2}2^{-\nu(d)+\sum_{i=1}^{r'} n_i'(n_i'-1)e/2-(r'-1)(e-1)-\sum_{1\le j\le i\le r'} n_i'n_j't_j'}$$
$$\times \prod_{i=1}^{r'} \#(SL_{n_i'}(\mathbb{Z}_2/2^e\mathbb{Z}_2))^{-1}\#S_{m-1}^{(0)}(\mathbb{Z}_2/2^e\mathbb{Z}_2,\{n_i'\},\{t_i'\},B).$$

This proves the assertion. ∎

4.2. Siegel series. For a half-integral matrix B of degree m over $\mathbb{Z}_p$, let $(\overline{W},\overline{q})$ denote the quadratic space over $\mathbb{Z}_p/p\mathbb{Z}_p$ defined by the quadratic form $\overline{q}(\mathbf{x}) = B[\mathbf{x}] \bmod p$, and define the radical $R(\overline{W})$ of $\overline{W}$ by

$$R(\overline{W}) = \{\mathbf{x}\in\overline{W} \mid \overline{B}(\mathbf{x},\mathbf{y}) = 0 \text{ for any } \mathbf{y}\in\overline{W}\},$$

where $\overline{B}$ denotes the symmetric bilinear form associated to $\overline{q}$. We then put $l_p(B) = \mathrm{rank}_{\mathbb{Z}_p/p\mathbb{Z}_p} R(\overline{W})^{\perp}$, where $R(\overline{W})^{\perp}$ is the orthogonal complement of $R(\overline{W})^{\perp}$ in $\overline{W}$. Furthermore, in case $l_p(B)$ is even, set $\overline{\xi}_p(B) = 1$ or -1 according as $R(\overline{W})^{\perp}$ is hyperbolic or not. In case $l_p(B)$ is odd, we put $\overline{\xi}_p(B) = 0$. Here we make the convention that $\xi_p(B) = 1$ if $l_p(B) = 0$. We note that $\overline{\xi}_p(B)$ is different from $\xi_p(B)$ in general, but they coincide if $B\in\mathcal{L}_{m,p}\cap\frac{1}{2}GL_m(\mathbb{Z}_p)$.

Let n be a positive even integer. For $B\in\mathcal{L}_{n-1,p}^{(1)}$ put

$$B^{(1)} = \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & (B+{}^t rr)/4 \end{pmatrix},$$

where $r\in\mathbb{Z}_p^{n-1}$ is such that $B+{}^t rr\in 4\mathcal{L}_{n-1,p}$. Then we set $\xi^{(1)}(B) = \xi(B^{(1)})$ and $\overline{\xi}^{(1)}(B) = \overline{\xi}(B^{(1)})$. These do not depend on the choice of r, and we have $\xi^{(1)}(B) = \chi((-1)^{n/2}\det B)$.

Let $p\neq 2$. Let $j = 0$ or 1. Then an element B of $\mathcal{L}_{n-j,p}^{(j)}$ is equivalent, over $\mathbb{Z}_p$, to $\Theta\perp pB_1$ with $\Theta\in GL_{n-n_1-j}(\mathbb{Z}_p)\cap S_{n-n_1-j}(\mathbb{Z}_p)$ and $B_1\in S_{n_1}(\mathbb{Z}_p)^{\times}$. Thus $\overline{\xi}^{(j)}(B) = 0$ if n_1 is odd, and $\overline{\xi}^{(1)}(B) = \chi((-1)^{(n-n_1)/2}\det\Theta)$ if n_1 is even.

Let $p = 2$. Then an element $B\in\mathcal{L}_{n-1,2}^{(1)}$ is equivalent, over $\mathbb{Z}_2$, to a matrix of the form $2\Theta\perp B_1$, where $\Theta\in GL_{n-n_1-2}(\mathbb{Z}_2)\cap S_{n-n_1-2}(\mathbb{Z}_2)_e$ and B_1 is one of the following types:

(I) $B_1 = a\perp 4B_2$ with $a\equiv -1 \bmod 4$, and $B_2\in S_{n_1}(\mathbb{Z}_2)_e^{\times}$;

(II) $B_1\in 4S_{n_1+1}(\mathbb{Z}_2)^{\times}$;

(III) $B_1 = a\perp 4B_2$ with $a\equiv -1 \bmod 4$, and $B_2\in S_{n_1}(\mathbb{Z}_2)_o$.

Thus $\overline{\xi}^{(1)}(B) = 0$ if B_1 is of type (II) or (III). If B_1 is of type (I), then $(-1)^{(n-n_1)/2} a \det \Theta \bmod (\mathbb{Z}_2^*)^\square$ is uniquely determined by B, and we have $\overline{\xi}^{(1)}(B) = \chi((-1)^{(n-n_1)/2} a \det \Theta)$. Moreover, an element $B \in \mathcal{L}_{n,2}^{(0)}$ is equivalent, over $\mathbb{Z}_2$, to a matrix of the form $\Theta \perp 2B_1$, where $\Theta \in GL_{n-n_1-2}(\mathbb{Z}_2) \cap S_{n-n_1-2}(\mathbb{Z}_2)_e$ and $B_1 \in S_{n_1}(\mathbb{Z}_2)^\times$.

Suppose that $p \neq 2$, and let $\mathcal{U} = \mathcal{U}_p$ be a complete set of representatives for $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^\square$. Then, for each positive integer l and $d \in \mathcal{U}_p$, there exists a unique, up to $\mathbb{Z}_p$-equivalence, element of $S_l(\mathbb{Z}_p) \cap GL_l(\mathbb{Z}_p)$ whose determinant is $(-1)^{[(l+1)/2]} d$, which will be denoted by $\Theta_{l,d}$.

Suppose that $p = 2$, and put $\mathcal{U} = \mathcal{U}_2 = \{1, 5\}$. Then for each positive even integer l and $d \in \mathcal{U}_2$ there exists a unique, up to $\mathbb{Z}_2$-equivalence, element of $S_l(\mathbb{Z}_2)_e \cap GL_l(\mathbb{Z}_2)$ whose determinant is $(-1)^{l/2} d$, which will also be denoted by $\Theta_{l,d}$.

In particular, if p is any prime number and l is even, we put $\Theta_l = \Theta_{l,1}$. We make the convention that $\Theta_{l,d}$ is the empty matrix if $l = 0$. For $d \in \mathcal{U}$ we use the same symbol d to denote the coset $d \bmod (\mathbb{Z}_p^*)^\square$.

For $B \in \mathcal{L}_{n-1,p}^{(1)}$, let $\widetilde{F}_p^{(1)}(B, X)$ be the polynomial in X and X^{-1} defined in Section 3. We also define a polynomial $G_p^{(1)}(B, X)$ in X by

$$G_p^{(1)}(B, X) = \sum_{i=0}^{n-1} (-1)^i p^{i(i-1)/2} (X^2 p^n)^i \sum_{D \in GL_{n-1}(\mathbb{Z}_p) \backslash \mathcal{D}_{n-1,i}} F_p^{(1)}(B[D^{-1}], X).$$

LEMMA 4.2.1. *Let n be a positive even integer. Let $B \in \mathcal{L}_{n-1,p}^{(1)}$, and put $\xi_0 = \chi((-1)^{n/2} \det B)$.*

(1) *Let $p \neq 2$, and suppose that $B = \Theta_{n-n_1-1,d} \perp pB_1$ with $d \in \mathcal{U}$ and $B_1 \in S_{n_1}(\mathbb{Z}_p)^\times$. Then*

$$G_p^{(1)}(B, X) = \begin{cases} \dfrac{1 - \xi_0 p^{n/2} X}{1 - p^{n_1/2+n/2} \overline{\xi}^{(1)}(B) X} \displaystyle\prod_{i=1}^{n_1/2} (1 - p^{2i+n} X^2) & \text{if } n_1 \text{ is even}, \\ (1 - \xi_0 p^{n/2} X) \displaystyle\prod_{i=1}^{(n_1-1)/2} (1 - p^{2i+n} X^2) & \text{if } n_1 \text{ is odd}. \end{cases}$$

(2) *Let $p = 2$. Suppose that n_1 is even and that $B = 2\Theta \perp B_1$ with $\Theta \in S_{n-n_1-2}(\mathbb{Z}_2)_e \cap GL_{n-n_1-2}(\mathbb{Z}_2)$ and $B_1 \in S_{n_1+1}(\mathbb{Z}_2)^\times$. Then*

$$G_2^{(1)}(B,X) = \begin{cases} \dfrac{1-\xi_0 2^{n/2}X}{1-2^{n_1/2+n/2}\overline{\xi}^{(1)}(B)X} \displaystyle\prod_{i=1}^{n_1/2}(1-2^{2i+n}X^2) & \textit{if } B_1 \textit{ is of type } (\mathrm{I}), \\ (1-\xi_0 2^{n/2}X)\displaystyle\prod_{i=1}^{n_1/2}(1-2^{2i+n}X^2) & \textit{if } B_1 \textit{ is of type } (\mathrm{II}) \textit{ or } (\mathrm{III}). \end{cases}$$

Proof. By the Corollary to Lemma 4.1.2 and by definition we have $G_p^{(1)}(B,X) = G_p(B^{(1)},X)$. Thus the assertion follows from [16, Lemma 9]. ∎

Remark. In the above lemma, we have $\overline{\xi}^{(1)}(B) = \xi_0(B)$ if $n_1 = 0$. Hence $G_p^{(1)}(B,X) = 1$ in this case.

Lemma 4.2.2. *Let $B \in \mathcal{L}_{n-1,p}^{(1)}$. Then*

$$\widetilde{F}_p^{(1)}(B,X) = \sum_{B' \in \mathcal{L}_{n-1,p}^{(1)}/GL_{n-1}(\mathbb{Z}_p)} X^{-\mathfrak{e}^{(1)}(B')} \frac{\alpha_p(B',B)}{\alpha_p(B')} \times G_p^{(1)}(B', p^{(-n-1)/2}X)(p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2}.$$

Proof. We have

$$\begin{aligned} \widetilde{F}_p^{(1)}(B,X) &= \sum_{W \in GL_{n-1}(\mathbb{Z}_p)\backslash \widetilde{\Omega}^{(1)}(B)} X^{-\mathfrak{e}^{(1)}(B)} G_p^{(1)}(B[W^{-1}], p^{(-n-1)/2}X) X^{2\nu(\det W)} \\ &= \sum_{B' \in \mathcal{L}_{n-1,p}^{(1)}/GL_{n-1}(\mathbb{Z}_p)} \sum_{W \in GL_{n-1}(\mathbb{Z}_p)\backslash \widetilde{\Omega}(B',B)} X^{-\mathfrak{e}^{(1)}(B)} \\ &\qquad \times G_p^{(1)}(B', p^{(-n-1)/2}X) X^{2\nu(\det W)} \\ &= \sum_{B' \in \mathcal{L}_{n-1,p}^{(1)}/GL_{n-1}(\mathbb{Z}_p)} X^{-\mathfrak{e}^{(1)}(B')} \#(GL_{n-1}(\mathbb{Z}_p)\backslash \widetilde{\Omega}(B',B)) p^{(\nu(\det B)-\nu(\det B'))/2} \\ &\qquad \times G_p^{(1)}(B', p^{(-n-1)/2}X)(p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2}. \end{aligned}$$

Thus the assertion follows from Lemma 4.1.1(2). ∎

4.3. Certain reduction formulas. To give an explicit formula for the power series $P_{n-1}^{(1)}(d_0,\omega,X,t)$, we give certain reduction formulas, by means of which we can express $P_{n-1}^{(1)}(d_0,\omega,X,t)$ in terms of the power series defined in [11]. First we review the notion of canonical forms of quadratic forms over $\mathbb{Z}_2$ in the sense of Watson [20].

Let $B \in \mathcal{L}_{m,2}^{\times}$. Then B is equivalent, over $\mathbb{Z}_2$, to a matrix of the form

$$\mathop{\perp}_{i=0}^{r} 2^i(V_i \perp U_i),$$

where $V_i = \perp_{j=1}^{k_i} c_{ij}$ with $0 \le k_i \le 2$, $c_{ij} \in \mathbb{Z}_2^*$ and $U_i = \frac{1}{2}\Theta_{m_i,d}$ with $0 \le m_i$, $d \in \mathcal{U}$. The degrees k_i and m_i of the matrices are uniquely determined by B. Furthermore we can choose the matrix $\perp_{i=0}^{r} 2^i(V_i \perp U_i)$ uniquely so that it satisfies the following conditions:

- (c.1) $c_{i1} = \pm 1$ or ± 3 if $k_i = 1$, and $(c_{i1}, c_{i2}) = (1, \pm 1), (1, \pm 3), (-1, -1)$, or $(-1, 3)$ if $k_i = 2$;
- (c.2) $k_{i+2} = k_i = 0$ if $U_{i+2} = \frac{1}{2}\Theta_{m_{i+2},5}$ with $m_{i+2} > 0$;
- (c.3) $-\det V_i \equiv 1 \bmod 4$ if $k_i = 2$ and $U_{i+1} = \frac{1}{2}\Theta_{m_{i+1},5}$ with $m_{i+1} > 0$;
- (c.4) $(-1)^{k_i - 1} \det V_i \equiv 1 \bmod 4$ if $k_i, k_{i+1} > 0$;
- (c.5) $V_i \ne \left(\begin{smallmatrix} -1 & 0 \\ 0 & c_{i2} \end{smallmatrix}\right)$ if $k_{i-1} > 0$;
- (c.6) $\deg V_i = 0$, or $V_i = (\pm 1)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ if $k_{i+2} > 0$.

The matrix satisfying the conditions (c.1)–(c.6) is called the *canonical form* of B, and denoted by $C(B)$. Now for $V = \perp_{j=1}^{k} c_j$ with $1 \le k \le 2$, put $\widetilde{V} = 5c_1$ or $\widetilde{V} = 5c_1 \perp c_2$ according as $V = c_1$ or $V = c_1 \perp c_2$.

Lemma 4.3.1. *For $B \in S_m(\mathbb{Z}_2)_o^{\times}$, let $C(B) = V_0 \perp \perp_{i=1}^{r}(U_i \perp V_i)$ be the canonical form of B stated as above. Let $l = l_B$ be the smallest integer such that $k_{2l+2} = 0$. Then*

$$C\Big(\widetilde{V_0} \perp \mathop{\perp}_{i=1}^{r}(U_i \perp V_i)\Big) = V_0 \perp \mathop{\perp}_{i=1}^{2l-1}(U_i \perp V_i) \perp U_{2l} \perp C(\widetilde{V_{2l}}) \perp \mathop{\perp}_{i=2l+1}^{r}(U_i \perp V_i).$$

Proof. We note that $5a_1 \perp 4a_2 \sim a_1 \perp 4 \cdot 5a_2$ for $a_1, a_2 \in \mathbb{Z}_2^*$. Hence

$$\widetilde{V_0} \perp \mathop{\perp}_{i=1}^{l} V_{2i} \sim V_0 \perp \mathop{\perp}_{i=1}^{l-1} V_{2i} \perp \widetilde{V_{2l}}.$$

This proves the assertion. ∎

Corollary. *For $B, B' \in S_{2m+1}(d_0)_o$, let $C(B) = V_0 \perp \perp_{i=1}^{r}(U_i \perp V_i)$ and $C(B') = V_0' \perp \perp_{i=1}^{r'}(U_i' \perp V_i')$ with $V_0 = \perp_{j=1}^{k_0} c_{0j}$ and $V_0' = \perp_{j=1}^{k_0} c_{0j}'$. Put*

$$B_1 = \mathop{\perp}_{j=2}^{k_0} c_{0j} \perp \mathop{\perp}_{i=1}^{r}(U_i \perp V_i) \quad \text{and} \quad B_1' = \mathop{\perp}_{j=2}^{k_0'} c_{0j}' \perp \mathop{\perp}_{i=1}^{r'}(U_i' \perp V_i').$$

Then $B \sim B'$ if and only if $c_{01} \perp 5B_1 \sim c_{01}' \perp 5B_1'$.

Proof. We note that $c_{01} \perp 5B_1 \sim c_{01}' \perp 5B_1'$ if and only if $5c_{01} \perp B_1 \sim 5c_{01}' \perp B_1'$. Hence the assertion follows from the lemma. ∎

The following lemma follows from [17, Theorem 3.4.2].

Lemma 4.3.2. *Let m and r be integers such that $0\le r\le m$, and $d_0\in\mathbb{Z}_p^\times$.*

(1) *Let $p\neq 2$ and $T\in S_r(\mathbb{Z}_p,d_0)$. Then for any $d\in\mathcal{U}$ we have*

$$\varepsilon(\Theta_{m-r,d}\perp T)=((-1)^{[(m-r+1)/2]}d,d_0)_p\varepsilon(T).$$

Furthermore

$$\varepsilon(pT)=\begin{cases}(p,d_0)_p\varepsilon(T) & \text{if } r \text{ even},\\ (p,(-1)^{(r+1)/2})_p\varepsilon(T) & \text{if } r \text{ odd},\end{cases}$$

and $\varepsilon(aT)=(a,d_0)_p^{r+1}\varepsilon(T)$ for any $a\in\mathbb{Z}_p^*$.

(2) *Let $p=2$ and $T\in S_r(\mathbb{Z}_2,d_0)$. Suppose that $m-r$ is even, and let $d\in\mathcal{U}$. Then for $\Theta=2\Theta_{m-r,d}$ or $2\Theta_{m-r-2}\perp(-d)$, we have*

$$\varepsilon(\Theta\perp T)=(-1)^{(m-r)(m-r+2)/8}((-1)^{(m-r)/2}d,(-1)^{[(r+1)/2]}d_0)_2\varepsilon(T)$$

and

$$\begin{aligned}&\varepsilon(\Theta_{m-r,d}\perp T)\\ &\quad=(-1)^{(m-r)(m-r+2)/8}(2,d)_2((-1)^{(m-r)/2}d,(-1)^{[(r+1)/2]}d_0)_2\epsilon(T).\end{aligned}$$

Furthermore, $\varepsilon(2T)=(2,d_0)_2^{r+1}\varepsilon(T)$,

$$\varepsilon(a\perp T)=(a,(-1)^{[(r+1)/2]+1}d_0)_2\varepsilon(T)$$

for any $a\in\mathbb{Z}_2^*$, and

$$\varepsilon(aT)=\begin{cases}(a,d_0)_2\varepsilon(T) & \text{if } r \text{ even},\\ (a,(-1)^{(r+1)/2})_2\varepsilon(T) & \text{if } r \text{ odd}\end{cases}$$

for any $a\in\mathbb{Z}_2^*$.

Henceforth, we sometimes abbreviate $S_r(\mathbb{Z}_p)$ and $S_r(\mathbb{Z}_p,d)$ as $S_{r,p}$ and $S_{r,p}(d)$, respectively. Furthermore we abbreviate $S_r(\mathbb{Z}_2)_x$ and $S_r(\mathbb{Z}_2,d)_x$ as $S_{r,2;x}$ and $S_{r,2}(d)_x$, respectively, for $x=e,o$.

Let R be a commutative ring. A function H defined on a subset $\mathcal{S}$ of $S_m(\mathbb{Q}_p)$ with values in R is said to be $GL_m(\mathbb{Z}_p)$*-invariant* if $H(A[U])=H(A)$ for any $U\in GL_m(\mathbb{Z}_p)$ and $A\in\mathcal{S}$.

Let $p\neq 2$. Let $\{H^{(j)}_{2r+j,\xi}\mid j\in\{0,1\},\ 1-j\le r\le n/2-j,\ \xi=\pm1\}$ be a set of $GL_{2r+j}(\mathbb{Z}_p)$-invariant functions on $S_{2r+j}(\mathbb{Z}_p)^\times$ with values in R satisfying the following conditions for any positive even integer $m\le n$:

(H-p-0) $H^{(0)}_{m,\xi}(\Theta_{m,d})=1$ and $H^{(1)}_{m-1,\xi}(\Theta_{m-1,d})=1$ for $d\in\mathcal{U}$;

(H-p-1) $H^{(0)}_{m,\xi}(\Theta_{m-2r,d}\perp pB)=H^{(0)}_{2r,\xi\chi(d)}(pB)$ for any $r\le m/2-1$, $\xi=\pm1$, $d\in\mathcal{U}$ and $B\in S_{2r}(\mathbb{Z}_p)^\times$;

(H-p-2) $H^{(1)}_{m-1,\xi}(\Theta_{m-2r-2,d}\perp pB)=H^{(1)}_{2r+1,\xi}(pdB)$ for any $r\le m/2-2$, $\xi=\pm1$, $d\in\mathcal{U}$ and $B\in S_{2r+1}(\mathbb{Z}_p)^\times$;

(H-p-3) $H^{(0)}_{m,\xi}(\Theta_{m-2r-1,d} \perp pB) = H^{(1)}_{2r+1,\xi}(-pdB)$ for any $r \le m/2-2$, $\xi = \pm 1$ and $B \in S_{2r+1}(\mathbb{Z}_p)^\times$;

(H-p-4) $H^{(1)}_{m-1,\xi}(\Theta_{m-2r-1,d} \perp pB) = H^{(0)}_{2r,\xi\chi(d)}(pB)$ for any $r \le m/2-2$, $\xi = \pm 1$, $d \in \mathcal{U}$ and $B \in S_{2r}(\mathbb{Z}_p)^\times$;

(H-p-5) $H^{(0)}_{2r,\xi}(dB) = H^{(0)}_{2r,\xi}(B)$ for any $r \le m/2$, $\xi = \pm 1$, $d \in \mathbb{Z}_p^*$ and $B \in S_{2r}(\mathbb{Z}_p)^\times$.

Let $d_0 \in \mathcal{F}_p$, and m be a positive even integer such that $m \le n$. Then for each $0 \le r \le m/2-1$ we put

$$Q^{(1)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \epsilon^l, t) = \kappa(d_0, m-1, l)^{-1} \times \sum_{d \in \mathcal{U}} \sum_{B \in p^{-1}S_{2r+1,p}(d_0 d) \cap S_{2r+1,p}} \frac{H^{(1)}_{m-1,\xi}(\Theta_{m-2r-2,d} \perp pB)\epsilon(\Theta_{m-2r-2,d} \perp pB)^l}{\alpha_p(\Theta_{m-2r-2,d} \perp pB)} \times t^{\nu(\det(pB))}.$$

Let $d \in \mathcal{U}$. Then we put

$$Q^{(1)}(d_0, d, H^{(1)}_{m-1,\xi}, 2r, \epsilon^l, t) = \kappa(d_0, m-1, l)^{-1} \times \sum_{B \in S_{2r,p}(d_0 d)} \frac{H^{(1)}_{m-1,\xi}(\Theta_{m-2r-1,d} \perp pB)\epsilon(\Theta_{m-2r-1,d} \perp pB)^l}{\alpha_p(\Theta_{m-2r-1,d} \perp pB)} t^{\nu(\det(pB))}$$

for each $1 \le r \le m/2-1$, and

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \epsilon^l, t) = \sum_{B \in S_{2r,p}(d_0 d)} \frac{H^{(0)}_{m,\xi}(\Theta_{m-2r,d} \perp pB)\epsilon(\Theta_{m-2r,d} \perp pB)^l}{\alpha_p(\Theta_{m-2r,d} \perp pB)} t^{\nu(\det(pB))}$$

for each $1 \le r \le m/2$. Here we make the convention that

$$Q^{(0)}(d_0, 1, H^{(0)}_{m,\xi}, m, \epsilon^l, t) = \sum_{B \in S_{m,p}(d_0)} \frac{H^{(0)}_{m,\xi}(pB)\epsilon(pB)^l}{\alpha_p(pB)} t^{\nu(\det(pB))}.$$

We also define

$$Q^{(1)}(d_0, d, H^{(1)}_{m-1,\xi}, 0, \epsilon^l, t) = Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 0, \epsilon^l, t) = \delta(d, d_0),$$

where $\delta(d, d_0) = 1$ or 0 according as $d = d_0$ or not. Furthermore put

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \epsilon^l, t)$$
$$= \sum_{d \in \mathcal{U}} \sum_{B \in p^{-1}S_{2r+1,p}(d_0 d) \cap S_{2r+1,p}} \frac{H^{(0)}_{m,\xi}(-\Theta_{m-2r-1,d} \perp pB)\epsilon(-\Theta_{m-2r-1,d} \perp pB)^l}{\alpha_p(-\Theta_{m-2r-1,d} \perp pB)} \times t^{\nu(\det(pB))}$$

for each $0 \le r \le m/2-1$.

Let $\{H^{(j)}_{2r+j,\xi} \mid j \in \{0,1\}, 1-j \le r \le n/2-j, \xi = \pm 1\}$ be a set of $GL_{2r+j}(\mathbb{Z}_2)$-invariant functions on $S_{2r+j}(\mathbb{Z}_2)^{\times}$ with values in R satisfying the following conditions for any positive even integer $m \le n$:

(H-2-0) $H^{(0)}_{m,\xi}(\Theta_{m,d}) = H^{(1)}_{m-1,\xi}(-d \perp 2\Theta_{m-2}) = 1$ for $d \in \mathcal{U}$;

(H-2-1) $H^{(0)}_{m,\xi}(\Theta_{m-2r,d} \perp 2B) = H^{(0)}_{2r,\xi\chi(d)}(2B)$ for any $r \le m/2-1$, $\xi = \pm 1$, $d \in \mathcal{U}$ and $B \in S_{2r}(\mathbb{Z}_2)^{\times}$;

(H-2-2) $H^{(1)}_{m-1,\xi}(2\Theta_{m-2r-2,d} \perp 4B) = H^{(1)}_{2r+1,\xi}(4dB)$ for any $r \le m/2-2$, $\xi = \pm 1$, $d \in \mathcal{U}$ and $B \in S_{2r+1}(\mathbb{Z}_2)^{\times}$;

(H-2-3) $H^{(0)}_{m,\xi}(2 \perp \Theta_{m-2r-2} \perp 2B) = H^{(1)}_{2r+1,\xi}(4B)$ for any $r \le m/2-2$, $\xi = \pm 1$ and $B \in S_{2r+1}(\mathbb{Z}_2)^{\times}$;

(H-2-4) $H^{(1)}_{m-1,\xi}(-a \perp 2\Theta_{m-2r-2} \perp 4B) = H^{(0)}_{2r,\xi\chi(a)}(2B)$ for any $r \le m/2-2$, $\xi = \pm 1$, $a \in \mathcal{U}$ and $B \in S_{2r}(\mathbb{Z}_2)^{\times}$;

(H-2-5) $H^{(0)}_{2r,\xi}(dB) = H^{(0)}_{2r,\xi}(B)$ for any $r \le m/2$, $\xi = \pm 1$, $d \in \mathbb{Z}_2^*$ and $B \in S_{2r}(\mathbb{Z}_2)^{\times}$;

(H-2-6) $H^{(1)}_{2r+1,\xi}(4(u_0 \perp B)) = H^{(1)}_{2r+1,\xi}(4(u_0 \perp 5B))$ for any $r \le m/2-1$, $\xi = \pm 1$ and $u_0 \in \mathbb{Z}_2^*$, $B \in S_{2r}(\mathbb{Z}_2)^{\times}$.

Let $d_0 \in \mathcal{F}_2$, and m be a positive even integer such that $m \le n$. Then for each $0 \le r \le m/2-1$, we put

$$Q^{(11)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = \kappa(d_0, m-1, l)^{-1} t^{2-m} \times$$
$$\sum_{d \in \mathcal{U}} \sum_{B \in S_{2r+1,2}(d_0 d)_e} \frac{H^{(1)}_{m-1,\xi}(2\Theta_{m-2r-2,d} \perp 4B)\varepsilon^l(2\Theta_{m-2r-2,d} \perp 4B)}{\alpha_2(2\Theta_{m-2r-2,d} \perp 4B)} \times t^{m-2r-2+\nu(\det(4B))},$$

$$Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = \kappa(d_0, m-1, l)^{-1} t^{2-m} \times$$
$$\sum_{B \in S_{2r+1,2}(d_0)_o} \frac{H^{(1)}_{m-1,\xi}(2\Theta_{m-2r-2} \perp 4B)\varepsilon^l(2\Theta_{m-2r-2} \perp 4B)}{\alpha_2(2\Theta_{m-2r-2} \perp 4B)} t^{m-2r-2+\nu(\det(4B))},$$

and

$$Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t)$$
$$= \kappa(d_0, m-1, l)^{-1} t^{2-m} \sum_{B \in S_{2r+2,2}(d_0)_o} H^{(1)}_{m-1,\xi}(-1 \perp 2\Theta_{m-2r-4} \perp 4B)$$
$$\times \frac{\varepsilon^l(-1 \perp 2\Theta_{m-2r-4} \perp 4B)}{\alpha_2(-1 \perp 2\Theta_{m-2r-4} \perp 4B)} t^{m-2r-4+\nu(\det(4B))}.$$

Moreover put

$$Q^{(1)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = Q^{(11)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t)$$
$$+ Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) + Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t).$$

We note that

$$Q^{(1)}(d_0, H^{(1)}_{m-1,\xi}, m-1, \epsilon^l, t)$$
$$= \kappa(d_0, m-1, l)^{-1} t^{2-m} \sum_{B \in S_{m-1,2}(d_0)} H^{(1)}_{m-1,\xi}(4B) \frac{\epsilon(4B)^l}{\alpha_2(4B)} t^{\nu(\det(4B))}.$$

Let $d \in \mathcal{U}$. Then we put

$$Q^{(1)}(d_0, d, H^{(1)}_{m-1,\xi}, 2r, \epsilon^l, t)$$
$$= \kappa(d_0, m-1, l)^{-1} t^{2-m} \sum_{B \in S_{2r,2}(d_0 d)_e} H^{(1)}_{m-1,\xi}(-d \perp 2\Theta_{m-2r-2} \perp 4B)$$
$$\times \frac{\epsilon(-d \perp 2\Theta_{m-2r-2} \perp 4B)^l}{\alpha_2(-d \perp 2\Theta_{m-2r-2} \perp 4B)} t^{m-2r-2+\nu(\det(4B))}$$

for each $1 \le r \le m/2 - 1$, and

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \epsilon^l, t)$$
$$= \kappa(d_0, m, l)^{-1} \sum_{B \in S_{2r,2}(d_0 d)_e} \frac{H^{(0)}_{m,\xi}(\Theta_{m-2r,d} \perp 2B) \epsilon(\Theta_{m-2r,d} \perp 2B)^l}{\alpha_2(\Theta_{m-2r,d} \perp 2B)} t^{\nu(\det(2B))}$$

for each $1 \le r \le m/2$, where $\kappa(d_0, m, l) = \{(-1)^{m(m+2)/8}((-1)^{m/2}2, d_0)_2\}^l$. Here we make the convention that

$$Q^{(0)}(d_0, 1, H^{(0)}_{m,\xi}, m, \epsilon^l) = \kappa(d_0, m, l)^{-1} \sum_{B \in S_{m,2}(d_0)_e} \frac{H^{(0)}_{m,\xi}(2B) \epsilon(2B)^l}{\alpha_2(2B)} t^{\nu(\det(2B))}.$$

We also define

$$Q^{(1)}(d_0, d, H^{(1)}_{m-1,\xi}, 0, \epsilon^l, t) = Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 0, \epsilon^l, t) = \delta(d, d_0).$$

Furthermore put

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \epsilon^l, t)$$
$$= \kappa(d_0, m, l)^{-1} \sum_{B \in S_{2r+2,2}(d_0)_o} \frac{H^{(0)}_{m,\xi}(\Theta_{m-2r-2} \perp 2B)\epsilon(\Theta_{m-2r-2} \perp 2B)^l}{\alpha_2(\Theta_{m-2r-2} \perp 2B)} t^{\nu(\det(2B))}$$

for $0 \le r \le m/2 - 1$. Henceforth, for $d_0 \in \mathcal{F}_p$ and nonnegative integers m, r such that $r \le m$, set $\mathcal{U}(m, r, d_0) = \{1\}$, $\mathcal{U} \cap \{d_0\}$, or $\mathcal{U}$ according as $r = 0$, $r = m \ge 1$, or $1 \le r \le m-1$.

Proposition 4.3.3. *Let the notation be as above.*

(1) *For $0 \le r \le (m-2)/2$, we have*

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \varepsilon^l, t) = \frac{Q^{(1)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-2)/2}(p^{-2})}$$

if $l\nu(d_0) = 0$, and

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \varepsilon, t) = 0$$

if $\nu(d_0) > 0$.

(2) *For $1 \le r \le m/2$ and $d \in \mathcal{U}(m, m-2r, d_0)$, we have*

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \varepsilon^l, t)$$
$$= \frac{(1 + p^{-(m-2r)/2}\chi(d))Q^{(0)}(d_0 d, 1, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t)}{2\phi_{(m-2r)/2}(p^{-2})}$$

if $l\nu(d_0) = 0$, and

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \varepsilon, t) = 0$$

if $\nu(d_0) > 0$.

Proof. First suppose that $p \neq 2$. We note that

$$(-\Theta_{m-2r-1,d}) \perp pB \sim d(-\Theta_{m-2r-1}) \perp pB \approx (-\Theta_{m-2r-1}) \perp dpB$$

for $d \in \mathcal{U}$ and $B \in p^{-1}S_{2r+1,p}(d_0 d)$ and the mapping

$$p^{-1}S_{2r+1,p}(d_0 d) \cap S_{2r+1,p} \ni B \mapsto dB \in p^{-1}S_{2r+1,p}(d_0) \cap S_{2r+1,p}$$

is a bijection. By Lemma 4.3.2, $\varepsilon((-\Theta_{m-2r-1,d}) \perp pB) = (d, d_0)_p \varepsilon(pB)$ and $\varepsilon(dpB) = \varepsilon(pB)$ for $B \in p^{-1}S_{2r+1,p}(d_0 d)$. Thus (1) follows from (H-p-3), (H-p-5) and Lemma 4.1.3.

By (H-p-2) and Lemmas 4.1.3 and 4.3.2, we have

$$Q^{(0)}(d_0, d, H_{m,\xi}^{(0)}, 2r, \varepsilon^l, t) = \frac{(1+p^{-(m-2r)/2}\chi(d))((-1)^{(m-2r)/2}d, d_0)_p^l}{2\phi_{(m-2r)/2}(p^{-2})} \times Q^{(0)}(d_0 d, 1, H_{2r,\xi\chi(d)}^{(0)}, 2r, \varepsilon^l, t).$$

Thus (2) follows immediately in case $l\nu(d_0) = 0$.

Now suppose that $l = 1$ and $\nu(d_0) = 1$. Take $a \in \mathbb{Z}_p^*$ such that $(a, p)_p = -1$. Then the mapping $S_{2r}(\mathbb{Z}_p) \ni B \mapsto aB \in S_{2r}(\mathbb{Z}_p)$ induces a bijection from $S_{2r,p}(dd_0)$ to itself, and $\varepsilon(apB) = -\varepsilon(pB)$ and $\alpha_p(apB) = \alpha_p(pB)$ for $B \in S_{2r,p}(dd_0)$. Furthermore by (H-p-5) we have

$$\begin{aligned} Q^{(0)}(d_0 d, 1, H_{2r,\xi\chi(d)}^{(0)}, 2r, \varepsilon^l, t) &= \sum_{B \in S_{2r}(dd_0)} \frac{H_{2r,\xi\chi(d)}^{(0)}(apB)\varepsilon(apB)}{\alpha_p(apB)} \\ &= -Q^{(0)}(d_0 d, 1, H_{2r,\xi\chi(d)}^{(0)}, 2r, \varepsilon^l, t). \end{aligned}$$

Hence $Q^{(0)}(d_0 d, 1, H_{2r,\xi\chi(d)}^{(0)}, 2r, \varepsilon^l, t) = 0$. This proves (2) in this case.

Next suppose that $p = 2$. First suppose that $l = 0$, or $l = 1$ and $d_0 \equiv 1 \bmod 4$. Fix a complete set $\mathcal{B}$ of representatives for $(S_{2r+2,2}(d_0)_o)/\approx$. For $B \in \mathcal{B}$, let $\mathcal{S}_{2r+1,2}(B)$ and $\widetilde{\mathcal{S}}_{2r+2,2}(B)$ be those defined in Subsection 4.1. Then, by (H-2-1) and (H-2-5) we have

$$\begin{aligned} &Q^{(0)}(d_0, H_{m,\xi}^{(0)}, 2r+1, \iota, t) \\ &\quad = \sum_{B \in \mathcal{B}} \frac{H_{2r+2,\xi}^{(0)}(2B)}{\phi_{(m-2r-2)/2}(2^{-2})2^{(r+1)(2r+3)}\alpha_2(B)} \#(\widetilde{\mathcal{S}}_{2r+2,2}(B)/\sim) t^{\nu(\det(2B))}. \end{aligned}$$

We have $S_{2r+1,2}(d_0) = \bigcup_{B \in \mathcal{B}} \mathcal{S}_{2r+1,2}(B)$, and $1 \perp B' \approx B$ for any $B' \in \mathcal{S}_{2r+1,2}(B)$. Hence $\nu(\det(2B)) = \nu(\det(4B')) - 2r$ and $H_{2r+2,\xi}^{(0)}(2B) = H_{2r+2,\xi}^{(0)}(2 \perp 2B') = H_{2r+1,\xi}^{(1)}(4B')$. Thus by Lemma 4.1.5 we have

$$\begin{aligned} &Q^{(0)}(d_0, H_{m,\xi}^{(0)}, 2r+1, \varepsilon^l, t) \\ &\quad = 2 \sum_{B' \in S_{2r+1,2}(d_0)} \frac{H_{2r+1,\xi}^{(1)}(4B')}{2^{(r+1)(2r+3)}\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(B')} t^{\nu(\det(4B'))-2r} \\ &\quad = 2^{(2r+1)r} t^{-2r} \sum_{B' \in 2^{-1}S_{2r+1,2}(d_0) \cap S_{2r+1,2}} \frac{H_{2r+1,\xi}^{(1)}(4B')}{\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B')} t^{\nu(\det(4B'))}. \end{aligned}$$

This proves (1) for $l = 0$. Now let $d_0 \equiv 1 \bmod 4$, and put $\xi_0 = (2, d_0)_2$. Then by Lemma 4.3.2 we have

$$\varepsilon(\Theta_{m-2r-2} \perp 2B) = (-1)^{m(m+2)/8+r(r+1)/2+(r+1)^2}\xi_0\varepsilon(B).$$

Furthermore for any $a \in \mathbb{Z}_2^*$ we have $\varepsilon(aB)^l = \varepsilon(B)^l$ and $\alpha_2(aB) = \alpha_2(B)$. Thus, by using the same argument as above we obtain

$$Q^{(0)}(d_0, H_{m,\xi}^{(0)}, 2r+1, \varepsilon, t)$$
$$= (-1)^{m(m+2)/8}\xi_0 \sum_{B \in \mathcal{B}} \frac{H_{2r+2,\xi}^{(0)}(2B)(-1)^{m(m+2)/8+r(r+1)/2+(r+1)^2}\xi_0\varepsilon(B)}{\phi_{(m-2r-2)/2}(2^{-2})2^{(r+1)(2r+3)}\alpha_2(B)}$$
$$\times \#(\widetilde{\mathcal{S}}_{2r+2,2}(B)/\sim)t^{\nu(\det(2B))}.$$

We note that $\varepsilon(1 \perp B') = \varepsilon(4B')$ for $B' \in S_{2r+1,2}$. Therefore, again by Lemma 4.1.5, we have

$$Q^{(0)}(d_0, H_{m,\xi}^{(0)}, 2r+1, \varepsilon^l, t) = (-1)^{r(r+1)/2}((-1)^{r+1}, (-1)^{r+1})_2 2^{(2r+1)r}t^{-2r}$$
$$\times \sum_{B' \in S_{2r+1,2}(d_0)} \frac{H_{2r+1,\xi}^{(1)}(4B')\varepsilon(B)}{\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B')}t^{\nu(\det(4B'))}.$$

This proves (1) for $l = 1$ and $d_0 \equiv 1 \bmod 4$.

Next suppose that $l = 1$ and $4^{-1}d_0 \equiv -1 \bmod 4$, or $l = 1$ and $8^{-1}d_0 \in \mathbb{Z}_2^*$. Then there exists $a \in \mathbb{Z}_2^*$ such that $(a, d_0)_2 = -1$. Then the map $2B \mapsto 2aB$ induces a bijection of $2S_{2r+2,2}(d_0)_o$ to itself. Furthermore $H_{2r+2,\xi}^{(0)}(2aB) = H_{2r+2,\xi}^{(0)}(2B)$, $\varepsilon(2aB) = -\varepsilon(2B)$, and $\alpha_2(2aB) = \alpha_2(2B)$. Thus (1) can be proved by using the same argument as in the proof of (2) for $p \neq 2$. The assertion (2) for $p = 2$ can be proved by using (H-2-1), Lemmas 4.1.4 and 4.3.2 similarly to (2) for $p \neq 2$. ■

Proposition 4.3.4. *Let the notation be as above.*

(1) *For* $0 \le r \le (m-2)/2$ *we have*

$$Q^{(1)}(d_0, H_{m-1,\xi}^{(1)}, 2r+1, \varepsilon^l, t) = \frac{Q^{(1)}(d_0, H_{2r+1,\xi}^{(1)}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-2)/2}(p^{-2})}.$$

(2) *For* $1 \le r \le (m-2)/2$ *and* $d \in \mathcal{U}(m-1, m-2r-1, d_0)$ *we have*

$$Q^{(1)}(d_0, d, H_{m-1,\xi}^{(0)}, 2r, \varepsilon^l, t) = \frac{Q^{(0)}(d_0 d, 1, H_{2r,\xi\chi(d)}^{(0)}, 2r, \varepsilon^l, t)}{2\phi_{(m-2r-2)/2}(p^{-2})}$$

if $l\nu(d_0) = 0$, *and*

$$Q^{(1)}(d_0, d, H_{m-1,\xi}^{(0)}, 2r, \varepsilon^l, t) = 0$$

otherwise.

Proof. We may suppose that $r < (m-2)/2$. First suppose that $p \neq 2$. As in the proof of Proposition 4.3.3(1), we have a bijection $p^{-1}S_{2r+1,p}(d_0 d) \cap$

$S_{2r+1,p} \ni B \mapsto dB \in p^{-1}S_{2r+1,p}(d_0) \cap S_{2r+1,p}$. We also note that $\varepsilon(dB) = \varepsilon(B)$ and $\alpha_p(dB) = \alpha_p(B)$. Hence, by (H-$p$-2), Lemmas 4.1.3 and 4.3.2, similarly to Proposition 4.3.3(2), we have

$$\begin{aligned} Q^{(1)}(d_0, H^{(1)}_{m,\xi}, 2r+1, \varepsilon^l, t) &= p^{(m/2-1)l\nu(d_0)}((-1)^{m/2}d_0, (-1)^{lm/2})_p \\ &\times \sum_{B \in p^{-1}S_{2r+1,p}(d_0) \cap S_{2r+1,p}} \frac{H^{(1)}_{2r+1,\xi}(pB)\varepsilon(pB)^l}{2\phi_{(m-2r-2)/2}(p^{-2})\alpha_p(pB)} t^{\nu(\det(pB))} \\ &\times \sum_{d \in \mathcal{U}} (1 + p^{-(m-2r-2)/2}\chi(d))((-1)^{(m-2r-2)/2}d, (-1)^{r+1}d_0 d)^l_p. \end{aligned}$$

Thus (1) clearly holds if $l\nu(d_0) = 0$. Suppose that $l = 1$ and $\nu(d_0) = 1$. Then

$$\begin{aligned} &((-1)^{(m-2r-2)/2}d, (-1)^{r+1}d_0 d)_p \\ &\qquad = \chi(d)((-1)^{r+1}, (-1)^{r+1}d_0 d)_p((-1)^{m/2}, (-1)^{m/2}d_0)_p, \end{aligned}$$

and therefore

$$\begin{aligned} &\sum_{d \in \mathcal{U}} (1 + p^{-(m-2r-2)/2}\chi(d))((-1)^{(m-2r-2)/2}d, (-1)^{r+1}d_0)_p \\ &\qquad = 2p^{-(m-2r-2)/2}((-1)^{r+1}, (-1)^{r+1}d_0 d)_p((-1)^{m/2}, (-1)^{m/2}d_0)_p. \end{aligned}$$

This completes the proof of (1).

By (H-p-4) and by Lemmas 4.1.3 and 4.3.2, we have

$$\begin{aligned} &Q^{(1)}(d_0, d, H^{(1)}_{m-1,\xi}, 2r, \varepsilon^l, t) \\ &\qquad = \frac{Q^{(0)}(d_0 d, 1, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t)}{2\phi_{(m-2r-2)/2}(p^{-2})}((-1)^{(m-2r)/2}d, d_0)^l_p. \end{aligned}$$

Thus (2) follows immediately if $l\nu(d_0) = 0$; and for $l = 1$ and $\nu(d_0) = 1$ it follows from Proposition 4.3.3(2).

Next suppose that $p = 2$. We have

$$\begin{aligned} \varepsilon(2\Theta_{m-2r-2,d} \perp 4B) &= (-1)^{m(m-2)/8}(-1)^{r(r+1)/2}((-1)^{m/2}, (-1)^{m/2}d_0)_2 \\ &\quad \times ((-1)^{r+1}, (-1)^{r+1}d_0 d)_2 (d_0, d)_2\, \varepsilon(4B) \end{aligned}$$

for $d \in \mathcal{U}$ and $B \in S_{2r+1,2}(dd_0)$. Thus, similarly to (1) for $p \neq 2$, we obtain

$$\begin{aligned} &Q^{(11)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = (-1)^{r(r+1)l/2}t^{-2r}((-1)^{r+1}, (-1)^{r+1}d_0)^l_2 \\ &\times 2^{(m/2-1)l\nu(d_0)} \sum_{B \in S_{2r+1,2}(d_0)_e} \frac{2^{r(2r+1)}H^{(1)}_{2r+1,\xi}(4B)\varepsilon(4B)^l}{2 \cdot 2^{m-2r-2}\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B)} t^{\nu(\det(4B))} \\ &\times \sum_{d \in \mathcal{U}} (1 + 2^{-(m-2r-2)/2}\chi(d))(d, d_0)^l_2 \end{aligned}$$

$$= \sum_{d \in \mathcal{U}} (1+2^{-(m-2r-2)/2}\chi(d))(d,d_0)_2^l \frac{Q^{(11)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{2^{1+(m-2r-2)(1-l\nu(d_0)/2)}\phi_{(m-2r-2)/2}(2^{-2})}.$$

In the same manner as above, we obtain

$$Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = (-1)^{r(r+1)l/2} t^{-2r}((-1)^{r+1}, (-1)^{r+1}d_0)_2^l$$
$$\times 2^{(m/2-1)l\nu(d_0)} \sum_{B \in S_{2r+1,2}(d_0)_o} \frac{2^{r(2r+1)} H^{(1)}_{2r+1,\xi}(4B)\varepsilon(4B)^l}{2^{m-2r-2}\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B)} t^{\nu(\det(4B))}$$
$$= \frac{Q^{(11)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{2^{(m-2r-2)(1-l\nu(d_0)/2)}\phi_{(m-2r-2)/2}(2^{-2})}.$$

Furthermore we have

$$\varepsilon(-1 \perp 2\Theta_{m-2r-4} \perp 4B) = (-1)^{m(m-2)/8}(-1)^{r(r+1)/2}((-1)^{m/2}, (-1)^{m/2}d_0)_2$$
$$\times ((-1)^{r+1}, (-1)^{r+1}d_0)_2(2, d_0)_2\varepsilon(2B)$$

for $d \in \mathcal{U}$ and $B \in S_{2r+2,2}(dd_0)_o$. Hence

$$Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = (-1)^{r(r+1)l/2} t^{-2r-2}((-1)^{r+1}, (-1)^{r+1}d_0)_2^l$$
$$\times (2, d_0)_2^l \; 2^{(m/2-1)l\nu(d_0)} \sum_{B \in S_{2r+2,2}(d_0)_o} \frac{H^{(0)}_{2r+2,\xi}(2B)\varepsilon(4B)^l}{\phi_{(m-2r-4)/2}(2^{-2})\alpha_2(2B)} t^{\nu(\det(4B))}$$
$$= \big(((-1)^{r+1}2, d_0)_2(-1)^{(r+1)(r+2)/2}\big)^l 2^{(m/2-1)l\nu(d_0)}$$
$$\times \sum_{B \in S_{2r+2,2}(d_0)_o} \frac{H^{(0)}_{2r+2,\xi}(2B)\varepsilon(2B)^l}{\phi_{(m-2r-4)/2}(2^{-2})\alpha_2(2B)} t^{\nu(\det(2B))}$$
$$= \frac{Q^{(0)}(d_0, H^{(0)}_{2r+2,\xi}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-4)/2}(2^{-2})} 2^{(m/2-1)l\nu(d_0)}.$$

First suppose that $l=0$ or $\nu(d_0)$ is even. Then $(d,d_0)_2^l = 1$. Hence

$$Q^{(11)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) + Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t)$$
$$= \frac{Q^{(1)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{2^{(m-2r-2)(1-\nu(d_0)l/2)}\phi_{(m-2r-2)/2}(2^{-2})}.$$

Furthermore by Proposition 4.3.3(2), we have

$$Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = \frac{Q^{(1)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-4)/2}(2^{-2})}$$

if $l\nu(d_0) = 0$, and

$$Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon, t) = 0$$

if $4^{-1}d_0 \equiv -1 \bmod 4$. Thus summing up these two quantities, we prove (1) in this case.

Next suppose that $l = 1$ and $\nu(d_0) = 3$. Then

$$Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon, t) = 0.$$

We prove

$$Q^{(12)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon, t) = 0.$$

If $r = 0$, then clearly $S_{2r+1,2}(d_0)_o$ is empty. Suppose $r \geq 1$. Then for $B \in 4S_{2r+1,2;o}$ take a canonical form $4c_{01} \perp 4B_1$ with $c_{01} \in \mathbb{Z}_2^*$, $B_1 \in S_{2r,2}$, and put $B' = 4c_{01} \perp 4 \cdot 5B_1$. Then, by Corollary to Lemma 4.3.1, the mapping $B \mapsto B'$ induces a bijection from $4S_{2r+1,2}(d_0)_o/{\sim}$ to itself, and $\varepsilon(B') = -\varepsilon(B)$. Then, by (H-2-6), and Lemma 4.1.4(3), we can prove the above equality in the same way as in the proof of (1) for $p \neq 2$. We also note that $\sum_{d \in \mathcal{U}}(1 + 2^{-(m-2r-2)/2}\chi(d))(d, d_0)_2 = 2^{1-(m-2r-2)/2}$. This proves (1).

The assertion (2) for $p = 2$ can be proved in the same manner as (2) for $p \neq 2$. ■

4.4. Proof of the main result. In this section, we prove our main result. First we give an explicit formula for the power series of Koecher–Maass type.

THEOREM 4.4.1. *Let $d_0 \in \mathcal{F}_p$, and put $\xi_0 = \chi(d_0)$. Then we have the following formulas:*

(1) $$P^{(1)}_{n-1}(d_0, \iota, X, t) = \frac{(p^{-1}t)^{\nu(d_0)}(1 - \xi_0 t^2 p^{-5/2})}{\phi_{(n-2)/2}(p^{-2})(1 - t^2p^{-2}X)(1 - t^2p^{-2}X^{-1})} \times \frac{1}{\prod_{i=1}^{(n-2)/2}(1 - t^2p^{-2i-1}X)(1 - t^2p^{-2i-1}X^{-1})}.$$

(2) $$P^{(1)}_{n-1}(d_0, \varepsilon, X, t) = \frac{(p^{-1}t)^{\nu(d_0)}(1 - \xi_0 t^2 p^{-1/2-n})}{\phi_{(n-2)/2}(p^{-2})} \times \frac{1}{\prod_{i=1}^{n/2}(1 - t^2p^{-2i}X)(1 - t^2p^{-2i}X^{-1})}.$$

To prove the above theorem, we define another formal power series. Namely, for $l = 0, 1$ we set

$$K^{(1)}_{n-1}(d_0, \varepsilon^l, X, t) = \kappa(d_0, n-1, l)^{-1} t^{\delta_{2,p}(2-n)} \times \sum_{B' \in \mathcal{L}^{(1)}_{n-1,p}(d_0)} \frac{G^{(1)}_p(B', p^{-(n+1)/2}X)\varepsilon(B')^l}{\alpha_p(B')} X^{-\mathfrak{e}^{(1)}(B')} t^{\nu(\det B')}.$$

Proposition 4.4.2. *Let d_0 be as above. Then*
$$P^{(1)}_{n-1}(d_0,\omega,X,t)=\prod_{i=1}^{n-1}(1-t^2Xp^{i-n-1})^{-1}K^{(1)}_{n-1}(d_0,\omega,X,t).$$

Proof. We note that B' belongs to $\mathcal{L}^{(1)}_{n-1,p}(d_0)$ if B belongs to $\mathcal{L}^{(1)}_{n-1,p}(d_0)$ and $\alpha_p(B',B)\neq 0$. Hence by Lemma 4.2.2 for $\omega=\varepsilon^l$ with $l=0,1$ we have
$$\begin{aligned}
P^{(1)}_{n-1}(d_0,\omega,X,t)&=\kappa(d_0,n-1,l)^{-1}t^{\delta_{2,p}(2-n)}\\
&\times\sum_{B\in\mathcal{L}^{(1)}_{n-1,p}(d_0)}\frac{1}{\alpha_p(B)}\sum_{B'}\frac{G^{(1)}_p(B',p^{-(n+1)/2}X)X^{-\mathfrak{e}^{(1)}(B')}\alpha_p(B',B)\omega(B')}{\alpha_p(B')}\\
&\qquad\times(p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2}t^{\nu(\det B)}\\
&=\kappa(d_0,n-1,l)^{-1}t^{\delta_{2,p}(2-n)}\sum_{B'\in\mathcal{L}^{(1)}_{n-1,p}(d_0)}\frac{G^{(1)}_p(B',p^{-(n+1)/2}X)\omega(B')}{\alpha_p(B')}X^{-\mathfrak{e}^{(1)}(B')}\\
&\qquad\times\sum_{B\in\mathcal{L}^{(1)}_{n-1,p}(d_0)}\frac{\alpha_p(B',B)}{\alpha_p(B)}(p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2}t^{\nu(\det B)}.
\end{aligned}$$
Hence by [4, Theorem 5], and by Lemma 4.1.1(1), we have
$$\begin{aligned}
\sum_{B}\frac{\alpha_p(B',B)}{\alpha_p(B)}&(p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2}t^{\nu(\det B)}\\
&=\sum_{W\in M_{n-1}(\mathbb{Z}_p)^\times/GL_{n-1}(\mathbb{Z}_p)}(t^2Xp^{-1}p^{-n+1})^{\nu(\det W)}t^{\nu(\det B')}\\
&=\prod_{i=1}^{n-1}(1-t^2Xp^{i-n-1})^{-1}t^{\nu(\det B')}.
\end{aligned}$$
Thus the assertion holds. ∎

For a variable X we introduce the symbol $X^{1/2}$ so that $(X^{1/2})^2=X$, and for an integer a we write $X^{a/2}=(X^{1/2})^a$. Under this convention, we can write $X^{-\mathfrak{e}^{(1)}(T)}t^{\nu(\det T)}$ as $X^{\delta_{2,p}(n-2)/2}X^{\nu(d_0)/2}(X^{-1/2}t)^{\nu(\det T)}$ if $T\in\mathcal{L}'_{n-1,p}(d_0)$, and hence we can write $K^{(1)}_{n-1}(d_0,\varepsilon^l,X,t)$ as
$$\begin{aligned}
K^{(1)}_{n-1}(d_0,\varepsilon^l,X,t)&=\kappa(d_0,n-1,l)^{-1}(tX^{-1/2})^{\delta_{2,p}(2-n)}X^{\nu(d_0)/2}\\
&\times\sum_{B'\in\mathcal{L}^{(1)}_{n-1,p}(d_0)}\frac{G^{(1)}_p(B',p^{-(n+1)/2}X)\varepsilon(B')^l}{\alpha_p(B')}(tX^{-1/2})^{\nu(\det B')}.
\end{aligned}$$
In order to prove Theorem 4.4.1, we introduce some power series. Let m be an integer and $l=0$ or 1. Then for $d_0\in\mathbb{Z}_p^\times$ put

$$\zeta_m(d_0, \varepsilon^l, u) = \sum_{T \in S_{m,p}(d_0)/\sim} \frac{\varepsilon(T)^l}{\alpha_p(T)} u^{\nu(\det T)},$$

and for $d_0 \in \mathbb{Z}_2^\times$ put

$$\zeta_m^*(d_0, \varepsilon^l, u) = \sum_{T \in S_{m,2}(d_0)_e/\sim} \frac{\varepsilon(T)^l}{\alpha_2(T)} u^{\nu(\det T)}.$$

We make the convention that $\zeta_0(d_0, \varepsilon^l, u) = \zeta_m^*(d_0, \varepsilon^l, u) = 1$ or 0 according as $d_0 \in \mathbb{Z}_p^*$ or not. Now for $d \in \mathbb{Z}_p^\times$, let $Z_m(u, \varepsilon^l, d)$ and $Z_m^*(u, \varepsilon^l, d)$ be the formal power series in Theorems 5.1, 5.2, and 5.3 of [11], which are given by

$$Z_m(u, \varepsilon^l, d) = 2^{-\delta_{2,p} m} \sum_{i=0}^\infty \sum_{T \in \mathbf{S}_m(\mathbb{Z}_p, p^i d)/\sim} \frac{\varepsilon(T)^l}{\alpha_p(T)} (\eta_m^l p^{(m+1)/2} u)^i,$$

$$Z_m^*(u, \varepsilon^l, d) = 2^{-m} \sum_{i=0}^\infty \sum_{T \in \mathbf{S}_m(\mathbb{Z}_2, 2^i d)_e/\sim} \frac{\varepsilon(T)^l}{\alpha_2(T)} (\eta_m^l 2^{(m+1)/2} u)^i,$$

where $\mathbf{S}_m(\mathbb{Z}_p, a) = \{T \in S_m(\mathbb{Z}_p) \mid \det T = a \bmod \mathbb{Z}_p^{*\square}\}$, $\mathbf{S}_m(\mathbb{Z}_p, a)_e = \mathbf{S}_m(\mathbb{Z}_p, a) \cap S_m(\mathbb{Z}_p)_e$, and $\eta_m = ((-1)^{(m+1)/2}, p)_p$ or 1 according as m is odd or even. Here we recall that the local density for $T \in S_m(\mathbb{Z}_p)$ in our paper is $2^{-\delta_{2,p} m}$ times that in [11]. Put

$$Z_{m,e}(u, \varepsilon^l, d) = \tfrac{1}{2}(Z_m(u, \varepsilon^l, d) + Z_m(-u, \varepsilon^l, d)),$$
$$Z_{m,o}(u, \varepsilon^l, d) = \tfrac{1}{2}(Z_m(u, \varepsilon^l, d) - Z_m(-u, \varepsilon^l, d)).$$

We also define $Z_{m,e}^*(u, \varepsilon^l, d)$ and $Z_{m,o}^*(u, \varepsilon^l, d)$ in the same way. Furthermore put $x(i) = e$ or o according as i is even or odd. Let $d_0 \in \mathcal{F}_p$. Let $p \neq 2$. Then

$$\begin{aligned}&\zeta_m(d_0, \varepsilon^l, u) \\ &\quad = Z_{m,x(\nu(d_0))}\big(p^{-(m+1)/2}((-1)^{(m+1)/2}, p)_p u, \varepsilon^l, p^{-\nu(d_0)}(-1)^{(m+1)/2} d_0\big)\end{aligned}$$

or

$$\zeta_m(d_0, \varepsilon^l, u) = Z_{m,x(\nu(d_0))}(p^{-(m+1)/2} u, \varepsilon^l, p^{-\nu(d_0)}(-1)^{[(m+1)/2]} d_0)$$

according as m is odd and $l = 1$, or not. Let $p = 2$ and suppose m is odd. Then

$$\zeta_m(d_0, \varepsilon^l, u) = 2^m Z_{m,x(\nu(d_0))}(2^{-(m+1)/2} u, \varepsilon^l, 2^{-\nu(d_0)}(-1)^{(m+1)/2} d_0).$$

Let $p = 2$ and suppose m is even. Then

$$\zeta_m^*(d_0, \varepsilon^l, u) = 2^m Z_{m,x(\nu(d_0))}^*(2^{-(m+1)/2} u, \varepsilon^l, (-1)^{m/2} 2^{-\nu(d_0)} d_0).$$

Proposition 4.4.3. *Let $d_0 \in \mathcal{F}_p$. For a positive even integer r and $d \in \mathcal{U}$ put*

$$c(r, d_0, d, X) = (1 - \chi(d_0)p^{-1/2}X) \prod_{i=1}^{r/2-1} (1 - p^{2i-1}X^2)(1 + \chi(d)p^{r/2-1/2}X),$$

and put $c(0, d_0, d, X) = 1$. *Furthermore, for a positive odd integer* r *put*

$$c(r, d_0, X) = (1 - \chi(d_0)p^{-1/2}X) \prod_{i=1}^{(r-1)/2} (1 - p^{2i-1}X^2).$$

(1) *Suppose that* $p \neq 2$.

(1.1) *Let* $l = 0$ *or* $\nu(d_0) = 0$. *Then*

$$\begin{aligned}
&K^{(1)}_{n-1}(d_0, \varepsilon^l, X, t) \\
&= X^{\nu(d_0)/2}\Bigg\{ \sum_{r=0}^{(n-2)/2} \sum_{d \in \mathcal{U}(n-1, n-2r-1, d_0)} \frac{p^{-r(2r+1)}(tX^{-1/2})^{2r} c(2r, d_0, d, X)}{2^{1-\delta_{0,r}} \phi_{(n-2r-2)/2}(p^{-2})} \\
&\qquad\qquad\qquad \times (p, d_0 d)_p^l \, \zeta_{2r}(d_0 d, \varepsilon^l, tX^{-1/2}) \\
&+ \sum_{r=0}^{(n-2)/2} \frac{p^{-(r+1)(2r+1)}(tX^{-1/2})^{2r+1} c(2r+1, d_0, X)}{\phi_{(n-2r-2)/2}(p^{-2})} \zeta_{2r+1}(p^* d_0, \varepsilon^l, tX^{-1/2}) \Bigg\},
\end{aligned}$$

where $p^* d_0 = pd_0$ *or* $p^{-1}d_0$ *according as* $\nu(d_0) = 0$ *or* $\nu(d_0) = 1$.

(1.2) *Let* $\nu(d_0) = 1$. *Then*

$$\begin{aligned}
&K^{(1)}_{n-1}(d_0, \varepsilon, X, t) \\
&\quad = X^{\nu(d_0)/2} \sum_{r=0}^{(n-2)/2} \frac{p^{-(r+1)(2r+1)-r}(tX^{-1/2})^{2r+1} c(2r+1, d_0, X)}{\phi_{(n-2r-2)/2}(p^{-2})} \\
&\qquad\qquad \times \zeta_{2r+1}(p^{-1}d_0, \varepsilon, tX^{-1/2}).
\end{aligned}$$

(2) *Suppose that* $p = 2$.

(2.1) *Let* $l = 0$ *or* $d_0 \equiv 1 \bmod 4$. *Then*

$$\begin{aligned}
&K^{(1)}_{n-1}(d_0, \varepsilon^l, X, t) = X^{\nu(d_0)/2} \\
&\times \Bigg\{ \sum_{r=0}^{(n-2)/2} \sum_{d \in \mathcal{U}(n-1, n-2r-1, d_0)} (tX^{-1})^{2r} 2^{-r(2r+1)} \frac{c(2r, d_0, d, X)}{2^{1-\delta_{0,r}} \phi_{(n-2r-2)/2}(2^{-2})} \\
&\qquad\qquad \times \big((-1)^{(r+1)r/2}(2, d_0 d)_2\big)^l \zeta^*_{2r}(d_0 d, \varepsilon, tX^{-1/2}) \\
&+ \sum_{r=0}^{(n-2)/2} (tX^{-1/2})^{2r+1} 2^{-(r+1)(2r+1)} \frac{c(2r+1, d_0, X)}{\phi_{(n-2r-2)/2}(2^{-2})} \\
&\qquad \times \big((-1)^{(r+1)r/2}((-1)^{r+1}, (-1)^{r+1}d_0)_2\big)^l \zeta_{2r+1}(d_0, \varepsilon^l, tX^{-1/2}) \Bigg\}.
\end{aligned}$$

(2.2) *Suppose that* $4^{-1}d_0 \equiv -1 \bmod 4$ *or* $8^{-1}d_0 \in \mathbb{Z}_2^*$. *Then*

$$\begin{aligned} &K_{n-1}^{(1)}(d_0, \varepsilon, X, t) \\ &= X^{\nu(d_0)/2} \sum_{r=0}^{(n-2)/2} (tX^{-1/2})^{2r+1} 2^{-(r+1)(2r+1)-r\nu(d_0)} \frac{c(2r+1, d_0, X)}{\phi_{(n-2r-2)/2}(2^{-2})} \\ &\qquad \times (-1)^{(r+1)r/2}((-1)^{r+1}, (-1)^{r+1}d_0)_2 \zeta_{2r+1}(d_0, \varepsilon, tX^{-1/2}). \end{aligned}$$

Proof. Put $H_{2r+j,\xi}^{(j)}(B) = 1$ for $j \in \{0,1\}$, $1-j \le r \le m/2 - j$, $\xi = \pm 1$, and $B \in S_{2r+j,p}$. Then clearly the set $\{H_{2r+j,\xi}^{(j)} \mid j \in \{0,1\},\ 1-j \le r \le n/2 - j,\ \xi = \pm 1\}$ satisfies the conditions (H-p-0)–(H-p-5) in Subsection 4.3 for any positive even integer $m \le n$. Hence by Lemma 4.2.1 and Proposition 4.3.4, and by using the same argument as in [10, Lemma 3.1(1)], we have

$$\begin{aligned} &K_{n-1}^{(1)}(d_0, \varepsilon^l, X, t) \\ &= \gamma_{l,d_0} X^{\nu(d_0)/2} \sum_{r=0}^{(n-2)/2} \sum_{d \in \mathcal{U}(n-1, n-2r-1, d_0)} \frac{c(2r, d_0, d, X)}{2^{1-\delta_{0,r}} \phi_{(n-2r-2)/2}(p^{-2})} \\ &\qquad \times \sum_{B \in S_{2r,p}(d_0 d)} \frac{\varepsilon(pB)^l}{\alpha_p(pB)} (tX^{-1/2})^{\nu(\det(pB))} \\ &\quad + X^{\nu(d_0)/2} \sum_{r=0}^{(n-2)/2} \frac{c(2r+1, d_0, d, X)}{\phi_{(n-2r-2)/2}(p^{-2})} \\ &\quad \times \sum_{B \in p^{-1}S_{2r+1,p}(d_0) \cap S_{2r+1,p}} \frac{((-1)^{(r+1)/2}, (-1)^{(r+1)/2}d_0)_p^l p^{-l\nu(d_0)} \varepsilon(pB)^l}{\alpha_p(pB)} \\ &\qquad \times (tX^{-1/2})^{\nu(\det(pB))}, \end{aligned}$$

where $\gamma_{l,d_0} = 1$ or 0 according as $\nu(d_0)l = 0$ or 1. Thus (1.1) follows from Lemmas 4.1.3 and 4.3.2 by noting that $p^{-1}S_{2r+1,p}(d_0) \cap S_{2r+1,p} = S_{2r+1}(p^* d_0)$. Similarly (1.2) can be proved by observing that $\varepsilon(pB) = ((-1)^{r+1}, p)\varepsilon(B)$ for $B \in p^{-1}S_{2r+1,p}(d_0) \cap S_{2r+1,p}$. The assertion for $p = 2$ can be proved in the same manner. ∎

Remark. As seen above, to prove Proposition 4.4.3, we have only to prove Propositions 4.3.2 and 4.3.3 for the simplest case where $\{H_{2r+j,\xi}^{(j)}\}$ are constant functions. However a similar statement for more general $\{H_{2r+j,\xi}^{(j)}\}$ will be necessary to give an explicit formula for the Rankin–Selberg series of $\sigma_{n-1}(\phi_{I_n(h),1})$ (cf. [15]). Indeed, the proofs are essentially the same as those for the simplest case. This is why we formulate and prove those propositions in more general settings.

Proof of Theorem 4.4.1 in case $p \neq 2$. (1) First let $d_0 \in \mathbb{Z}_p^*$. Then by Proposition 4.4.3(1.1), we have

$$\begin{aligned}K_{n-1}^{(1)}(d_0,\iota,X,t) = \frac{1}{\phi_{(n-2)/2}(p^{-2})} &+ \sum_{r=1}^{(n-2)/2}\sum_{d\in\mathcal{U}} \frac{p^{-r(2r+1)}(t^2X^{-1})^r\prod_{i=1}^{r-1}(1-p^{2i-1}X^2)}{2\phi_{(n-2r-2)/2}(p^{-2})} \\ &\quad \times (1-p^{-1/2}\xi_0X)(1+\eta_d p^{r-1/2}X)\zeta_{2r}(d_0d,\iota,tX^{-1/2}) \\ &+ \sum_{r=0}^{(n-2)/2} \frac{p^{-(2r+1)(r+1)}(t^2X^{-1})^{r+1/2}\prod_{i=1}^{r}(1-p^{2i-1}X^2)}{\phi_{(n-2r-2)/2}(p^{-2})} \\ &\quad \times (1-p^{-1/2}\xi_0X)\zeta_{2r+1}(pd_0,\iota,tX^{-1/2}).\end{aligned}$$

Here we put $\eta_d = \chi(d)$ for $d \in \mathcal{U}$. By [11, Theorem 5.1], we have

$$\zeta_{2r+1}(pd_0,\iota,tX^{-1/2}) = \frac{p^{-1}tX^{-1/2}}{\phi_r(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{r}(1-p^{2i-3-2r}t^2X^{-1})},$$

$$\zeta_{2r}(d_0d,\iota,tX^{-1/2}) = \frac{(1+\xi_0\eta_d p^{-r})(1-\xi_0\eta_d p^{-r-2}t^2X^{-1})}{\phi_r(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{r}(1-p^{2i-3-2r}t^2X^{-1})}.$$

Hence the assertion for $n=2$ can be proved by a direct calculation. Suppose that $n \geq 4$. Then $K_{n-1}^{(1)}(d_0,\iota,X,t)$ can be expressed as

$$K_{n-1}^{(1)}(d_0,\iota,X,t) = \frac{S(d_0,\iota,X,t)}{\phi_{(n-2)/2}(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{(n-2)/2}(1-p^{2i-n-1}t^2X^{-1})},$$

where $S(d_0,\iota,X,t)$ is a polynomial in t of degree n. We have

$$\begin{aligned}2^{-1}(1-p^{-1/2}\xi_0X)\sum_{\eta=\pm1}&(1+\eta p^{(n-2)/2-1/2}X)(1+\xi_0\eta p^{-(n-2)/2}) \\ &\times(1-\xi_0\eta p^{-(n-2)/2-2}t^2X^{-1}) \\ &= (1-\xi_0p^{-1/2}X)(1+\xi_0p^{-1/2}X-\xi_0p^{-5/2}t^2-p^{-n}t^2X^{-1}).\end{aligned}$$

Hence

$$\begin{aligned}2^{-1}\sum_{d\in\mathcal{U}} p^{(n-1)(-n+2)/2}(t^2X^{-1})^{(n-2)/2}&\prod_{i=1}^{(n-2)/2-1}(1-p^{2i-1}X^2) \\ &\times(1-p^{-1/2}\xi_0X)(1+\eta_d p^{(n-2)/2-1/2}X)\zeta_{n-2}(d_0d,\iota,tX^{-1/2}) \\ + p^{-(n-1)n/2}(t^2X^{-1})^{(n-2)/2+1/2}&\prod_{i=1}^{(n-2)/2}(1-p^{2i-1}X^2)(p^{-2})^{-1} \\ &\times(1-p^{-1/2}\xi_0X)\zeta_{n-1}(pd_0,\iota,tX^{-1/2})\end{aligned}$$

$$= \frac{(p^{-(n-1)}X^{-1}t^2)^{(n-2)/2}(1-\xi_0 p^{-5/2}t^2)\prod_{i=0}^{(n-2)/2-1}(1-p^{2i-1}X^2)}{\phi_{(n-2)/2}(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{(n-2)/2}(1-p^{2i-n-1}t^2X^{-1})},$$

and therefore $S(d_0,\iota,X,t)$ can be expressed as

(A) $$S(d_0,\iota,X,t) = (p^{-(n-1)}X^{-1}t^2)^{(n-2)/2}\prod_{i=0}^{(n-2)/2-1}(1-p^{2i-1}X^2)(1-p^{-5/2}\xi_0 t^2) + (1-p^{-n+1}t^2X^{-1})U(X,t),$$

where $U(X,t)$ is a polynomial in X, X^{-1} and t. Now by Proposition 4.4.2, we have

$$P_{n-1}^{(1)}(d_0,\iota,X,t) = \frac{S(d_0,\iota,X,t)}{\phi_{(n-2)/2}(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{(n-2)/2}(1-p^{2i-n-1}t^2X^{-1})\prod_{i=1}^{n-1}(1-p^{i-n-1}Xt^2)}.$$

Hence the power series $P_{n-1}^{(1)}(d_0,\iota,X,t)$ is a rational function in X and t. Since $\widetilde{F}_p^{(1)}(T,X^{-1}) = \widetilde{F}_p^{(1)}(T,X)$ for any $T \in \mathcal{L}_{n-1,p}^{(1)}$, it follows that $P_{n-1}^{(1)}(d_0,\iota,X^{-1},t) = P_{n-1}^{(1)}(d_0,\iota,X,t)$. This implies that the reduced denominator of the rational function $P_{n-1}^{(1)}(d_0,\iota,X,t)$ in t is at most

$$(1-p^{-2}t^2X^{-1})(1-p^{-2}t^2X)\prod_{i=1}^{(n-2)/2}\{(1-p^{2i-n-1}t^2X^{-1})(1-p^{2i-n-1}t^2X)\}.$$

Hence we have

(B) $$S(d_0,\iota,X,t) = \prod_{i=1}^{(n-2)/2}(1-p^{2i-n-2}t^2X)(a_0(X)+a_1(X)t^2)$$

with some polynomials $a_0(X)$, $a_1(X)$ in $X+X^{-1}$. We easily see $a_0(X)=1$. By substituting $p^{(n-1)/2}X^{1/2}$ for t in (A) and (B), and comparing them we see that $a_1(X) = -p^{-5/2}\xi_0$. This proves the assertion.

Next let $d_0 \in p\mathbb{Z}_p^*$. Then by Proposition 4.4.3(1.1), we have

$$K_{n-1}^{(1)}(d_0,\iota,X,t) = X^{1/2}\Big\{2^{-1}\sum_{r=1}^{(n-2)/2}\sum_{d\in\mathcal{U}}\frac{p^{-r(2r+1)}(t^2X^{-1})^r\prod_{i=1}^{r-1}(1-p^{2i-1}X^2)}{\phi_{(n-2r-2)/2}(p^{-2})} \times (1+\eta_d p^{r-1/2}X)\zeta_{2r}(d_0d,\iota,tX^{-1/2})$$

$$+\sum_{r=0}^{(n-2)/2}\frac{p^{-(2r+1)(r+1)}(t^2X^{-1})^{r+1/2}\prod_{i=1}^{r}(1-p^{2i-1}X^2)}{\phi_{(n-2r-2)/2}(p^{-2})} \times \zeta_{2r+1}(p^{-1}d_0,\iota,tX^{-1/2})\Big\}.$$

By [11, Theorem 5.1], we have

$$\zeta_{2r+1}(p^{-1}d_0,\iota,tX^{-1/2})=\frac{1}{\phi_r(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{r}(1-p^{2i-3-2r}t^2X^{-1})},$$

$$\zeta_{2r}(d_0d,\iota,tX^{-1/2})=\frac{p^{-1}tX^{-1/2}}{\phi_{r-1}(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{r}(1-p^{2i-3-2r}t^2X^{-1})}.$$

Thus the assertion can be proved in the same manner as above.

(2) First let $d_0\in\mathbb{Z}_p^*$. Then by Proposition 4.4.3(1.1), we have

$$\begin{aligned}K_{n-1}^{(1)}(d_0,\varepsilon,X,t)=&\frac{1}{\phi_{(n-2)/2}(p^{-2})}\\&+\sum_{r=1}^{(n-2)/2}\sum_{d\in\mathcal{U}}\frac{p^{-r(2r+1)}(t^2X^{-1})^r\prod_{i=1}^{r-1}(1-p^{2i-1}X^2)}{2\phi_{(n-2r-2)/2}(p^{-2})}\\&\times(1-p^{-1/2}\xi_0X)(1+\eta_dp^{r-1/2}X)\xi_0\eta_d\zeta_{2r}(d_0d,\varepsilon,tX^{-1/2})\\&+\sum_{r=0}^{(n-2)/2}\frac{p^{-(2r+1)(r+1)}(t^2X^{-1})^{r+1/2}\prod_{i=1}^{r}(1-p^{2i-1}X^2)}{\phi_{(n-2r-2)/2}(p^{-2})}\\&\times(1-p^{-1/2}\xi_0X)\zeta_{2r+1}(pd_0,\varepsilon,tX^{-1/2}).\end{aligned}$$

By [11, Theorem 5.2],

$$\zeta_{2r}(d_0d,\varepsilon,tX^{-1/2})=\frac{1+\xi_0\eta_dp^{-r}}{\phi_r(p^{-2})\prod_{i=1}^{r}(1-p^{-2i}t^2X^{-1})},$$

$$\zeta_{2r+1}(pd_0,\varepsilon,tX^{-1/2})=\frac{p^{-r-1}tX^{-1/2}}{\phi_r(p^{-2})\prod_{i=1}^{r+1}(1-p^{-2i}t^2X^{-1})}.$$

Hence $K_{n-1}^{(1)}(d_0,\varepsilon,X,t)$ can be expressed as

$$K_{n-1}^{(1)}(d_0,\varepsilon,X,t)=\frac{T(d_0,\varepsilon,X,t)}{\phi_{(n-2)/2}(p^{-2})\prod_{i=1}^{n/2}(1-p^{-2i}t^2X^{-1})},$$

where $T(d_0,\iota,X,t)$ is a polynomial in t of degree n, expressed as

$$\text{(C)}\quad \begin{aligned}T(d_0,\iota,X,t)=&(p^{-n}X^{-1}t^2)^{n/2}(1-\xi_0p^{-1/2}X)\prod_{i=1}^{(n-2)/2}(1-p^{2i-1}X^2)\\&+(1-p^{-n}t^2X^{-1})V(X,t),\end{aligned}$$

with a polynomial $V(X,t)$ in X, X^{-1} and t. On the other hand, by using the same argument as in (1), we can show that

$$\text{(D)} \qquad T(d_0,\varepsilon,X,t) = \prod_{i=1}^{(n-2)/2} (1-p^{-2i-1}t^2X)(1+b_1(X)t^2)$$

with $b_1(X)$ a polynomial in $X+X^{-1}$. Thus, by substituting $p^{n/2}X^{1/2}$ for t in (C) and (D), and comparing them, we prove the assertion.

Next let $d_0 \in p\mathbb{Z}_p^*$. Then by Proposition 4.4.3(1.2), we have

$$\begin{aligned} &K_{n-1}^{(1)}(d_0,\varepsilon,X,t) \\ &\quad = X^{1/2} \sum_{r=0}^{(n-2)/2} \frac{p^{-(2r+1)(r+1)-r}(t^2X^{-1})^{r+1/2}\prod_{i=1}^{r}(1-p^{2i-1}X^2)}{\phi_{(n-2r-2)/2}(p^{-2})} \\ &\qquad\qquad \times \zeta_{2r+1}(p^{-1}d_0,\varepsilon,tX^{-1/2}). \end{aligned}$$

By [11, Theorem 5.2],

$$\zeta_{2r+1}(p^{-1}d_0,\varepsilon,tX^{-1/2}) = \frac{1}{\phi_r(p^{-2})\prod_{i=1}^{r}(1-p^{-2i}t^2X^{-1})}.$$

Hence

$$\begin{aligned} K_{n-1}^{(1)}(d_0,\varepsilon,X,t) = p^{-1}t \sum_{r=0}^{(n-2)/2} &\frac{p^{-(2r+1)r}(p^{-2}t^2X^{-1})^{r}\prod_{i=1}^{r}(1-p^{2i-1}X^2)}{\phi_{(n-2r-2)/2}(p^{-2})} \\ &\times \frac{1}{\phi_r(p^{-2})\prod_{i=1}^{r}(1-p^{-2i}t^2X^{-1})}. \end{aligned}$$

Thus the assertion can be proved in the same way as above. ∎

Proof of Theorem 4.4.1 in case $p=2$. The assertion can also be proved by using Proposition 4.4.3(2) as above. ∎

PROPOSITION 4.4.4. *Let k and n be positive even integers. Given a Hecke eigenform $h \in \mathfrak{S}_{k-n/2+1/2}^{+}(\Gamma_0(4))$, let $f \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$ be the primitive form as in Section 2. Then*

$$L(s,h) = L(2s,f) \sum_{d_0 \in \mathcal{F}((-1)^{n/2})} c_h(|d_0|)|d_0|^{-s} L\biggl(2s-k+n/2+1, \biggl(\frac{d_0}{*}\biggr)\biggr)^{-1},$$

where $L\bigl(s,\bigl(\frac{d_0}{*}\bigr)\bigr)$ is Dirichlet's L-function for the character $\bigl(\frac{d_0}{*}\bigr)$.

Proof. The assertion can be proved immediately by noting that

$$\sum_{m=1}^{\infty} c_h(|d_0|m^2)m^{-2s} = c_h(|d_0|)L\left(2s-k+n/2+1, \left(\frac{d_0}{*}\right)\right)^{-1} L(2s,f)$$

for $d_0 \in \mathcal{F}^{((-1)^{n/2})}$. ∎

Proof of Theorem 2.1. By Theorem 4.4.1, we have

$$\begin{aligned}\prod_p P^{(1)}_{n-1,p}(d_0,\iota_p,\alpha_p,p^{-s+k/2+n/4-1/4}) &= |d_0|^{-s+k/2+n/4-5/4} \\ &\times \prod_{i=1}^{(n-2)/2} \zeta(2i)L\left(2s-k-n/2+3, \left(\frac{d_0}{*}\right)\right)^{-1} L(2s-n+2,f) \\ &\times \prod_{i=1}^{(n-2)/2} L(2s-n+2i+1,f)\end{aligned}$$

and

$$\begin{aligned}\prod_p P^{(1)}_{n-1,p}(d_0,\varepsilon_p,\alpha_p,p^{-s+k/2+n/4-1/4}) &= |d_0|^{-s+k/2+n/4-5/4} \\ &\times \prod_{i=1}^{(n-2)/2} \zeta(2i)L\left(2s-k+n/2+1, \left(\frac{d_0}{*}\right)\right)^{-1} \prod_{i=1}^{n/2} L(2s-n+2i,f).\end{aligned}$$

Thus the assertion follows from Theorem 3.2 and Proposition 4.4.4. ∎

Remark. Let m be a nonnegative integer, and let k be a positive integer such that $k > m+2$. Let $E^{(m+1)}_k$ be the Siegel–Eisenstein series of weight k and of degree $m+1$. (For the definition of the latter, see, for example, [6].) Suppose that $m > 0$, and let $e^{(m+1)}_{k,1}$ be the first Fourier–Jacobi coefficient of $E^{(m+1)}_k$. Then $e^{(m+1)}_{k,1}$ belongs to $J_{k,1}(\Gamma^{(m)}_J)$. In [6], Hayashida defined the generalized Cohen–Eisenstein series $E^{(m)}_{k-1/2}$ as $E^{(m)}_{k-1/2} = \sigma_m(e^{(m+1)}_{k,1})$, where σ_m is the Ibukiyama isomorphism. It turns out that $E^{(m)}_{k-1/2}$ belongs to $\mathfrak{M}^+_{k-1/2}(\Gamma^{(m)}_0(4))$, and in particular, $E^{(1)}_{k-1/2}$ coincides with the Cohen–Eisenstein series defined in [5]. Let k and n be positive even integers such that $k > n+1$. Then $E^{(1)}_{2k-n}$ is the Hecke eigenform corresponding to $E^{(1)}_{k-n/2+1/2}$ under the Shimura correspondence, and $E^{(n)}_k$ can be regarded as a noncuspidal version of the Duke–Imamoḡlu–Ikeda lift of $E^{(1)}_{k-n/2+1/2}$. Therefore, by using the same method as in the proof of Theorem 2.1, we can express the Koecher–Maass series of $E^{(n-1)}_{k-1/2}$ explicitly in terms of $L(s, E^{(1)}_{k-n/2+1/2})$ and $L(s, E^{(1)}_{2k-n})$.

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