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A note on two linear forms

by

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1. Diophantine exponents. Let θ_1, θ_2 be real numbers such that

$$1, \theta_1, \theta_2 \text{ are linearly independent over } \mathbb{Z}. \tag{1.1}$$

We consider the linear form

$$L(\mathbf{x}) = x_0 + x_1\theta_1 + x_2\theta_2, \qquad \mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3.$$

By $|\mathbf{z}|$ we denote the Euclidean length of a vector $\mathbf{z} = (z_0, z_1, z_2) \in \mathbb{R}^3$. Let

$$\hat{\omega} = \hat{\omega}(\theta_1, \theta_2) = \sup\Big\{\gamma : \limsup_{t\to\infty}\Big(t^\gamma \min_{0<|\mathbf{x}|\le t} |L(\mathbf{x})|\Big) < \infty\Big\} \tag{1.2}$$

be the uniform Diophantine exponent for the linear form L.

We consider another linear form $P(\mathbf{x})$. The main result of the present paper is as follows.

Theorem 1. *Suppose that the linear forms $L(x)$ and $P(x)$ are independent and the exponent $\hat{\omega}$ for the form L is defined in* (1.2). *Then for the Diophantine exponent*

$$\omega_{LP} = \sup\{\gamma : \textit{there exist infinitely many } \mathbf{x} \in \mathbb{Z}^3 \textit{ such that } |L(\mathbf{x})| \le |P(\mathbf{x})| \cdot |\mathbf{x}|^{-\gamma}\}$$

we have the lower bound

$$\omega_{LP} \ge \hat{\omega}^2 - \hat{\omega} + 1.$$

Remark. Of course in the definition (1.2) and in Theorem 1 instead of the Euclidean norm $|\mathbf{x}|$ we may consider the value $\max_{j=1,2}|x_j|$, as done by most authors.

Consider a real θ which is not a rational number and not a quadratic irrationality. Define

$$\omega_* = \omega_*(\theta) = \sup\{\gamma : \text{there exist infinitely many algebraic numbers } \xi \text{ of degree} \le 2 \text{ such that } |\theta - \xi| \le H(\xi)^{-\gamma-1}\}$$

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(here $H(\xi)$ is the maximal value of the absolute values of the coefficients for the canonical polynomial to ξ). Then for the linear forms

$$L(\mathbf{x}) = x_0 + x_1\theta + x_2\theta^2, \qquad P(\mathbf{x}) = x_1 + 2x_2\theta$$

one has

$$\omega_* \geq \omega_{LP} - 1. \tag{1.3}$$

This inequality follows immediately from the argument from [2]; see also [1, Lemma A.5].

So Theorem 1 immediately leads to the following corollary.

THEOREM 2. *For a real θ which is not a rational number or a quadratic irrationality, one has*

$$\omega_* \geq \hat{\omega}(\hat{\omega} - 1) \tag{1.4}$$

with $\hat{\omega} = \hat{\omega}(\theta, \theta^2)$.

2. Some history. In 1967 H. Davenport and W. Schmidt [2] (see also Ch. 8 from Schmidt's book [11]) proved that for any two independent linear forms L, P there exist infinitely many integer points $\mathbf{x}$ such that

$$|L(\mathbf{x})| \leq C|P(\mathbf{x})|\,|\mathbf{x}|^{-3},$$

with a positive constant C depending on the coefficients of L, P. From this result they deduced that for any real θ which is not a rational number or a quadratic irrationality, the inequality

$$|\theta - \xi| \leq C_1 H(\xi)^{-3}$$

has infinitely many solutions in algebraic ξ of degree ≤ 2.

We see that for any two pairs of forms one has $\omega_{LP} \geq 3$. But from the Minkowski convex body theorem it follows that under the condition (1.1) one has $\hat{\omega} \geq 2$. Moreover

$$\min_{\hat{\omega} \geq 2}(\hat{\omega}^2 - \hat{\omega} + 1) = 3.$$

So our Theorems 1 and 2 may be considered as generalizations of Davenport–Schmidt's results.

Later Davenport and Schmidt generalized their theorems to the case of several linear forms [3]. In the next paper [4] they showed that the value of the uniform exponent for *simultaneous* approximations to any point (θ, θ^2) is not greater than $(\sqrt{5} - 1)/2$. This together with Jarník's transference equality (see [5]) leads to the bound $\hat{\omega} \leq (3 + \sqrt{5})/2$ which holds for all linear forms with coefficients of the form θ, θ^2. So for a linear form with coefficients θ, θ^2 one has

$$2 \leq \hat{\omega} \leq \frac{3 + \sqrt{5}}{2}. \tag{2.1}$$

D. Roy [9, 10] showed that the set of values $\hat{\omega}$ for linear forms under consideration form a dense set in the interval (2.1). Moreover he constructed a countable set of numbers θ such that

$$\hat{\omega}(\theta,\theta^2)=\frac{3+\sqrt{5}}{2}\quad\text{and}\quad \omega_*(\theta)=3+\sqrt{5}.$$

This shows that our bound (1.4) from Theorem 2 is optimal in the right endpoint of the segment (2.1), namely for $\hat{\omega}=(3+\sqrt{5})/2$.

Our Theorem 2 may be compared with Jarník's inequality between the exponent $\hat{\omega}$ and the ordinary exponent

$$\omega=\omega(\theta_1,\theta_2)=\sup\Big\{\gamma:\liminf_{t\to\infty}\Big(t^{\gamma}\min_{0<|\mathbf{x}|\le t}|L(\mathbf{x})|\Big)<\infty\Big\}.$$

For numbers $1,\theta_1,\theta_2$ linearly independent over $\mathbb{Z}$ Jarník [6, 7] proved the inequality

$$\omega\ge\hat{\omega}(\hat{\omega}-1).$$

Other results on approximation by algebraic numbers are discussed in W. Schmidt's book [11], in the wonderful book by Y. Bugeaud [1] and in M. Waldschmidt's survey [12].

Our proof of Theorem 1 generalizes ideas from [2, 3, 4] and uses Jarník's inequalities [6, 7].

3. Minimal points. In the following we may suppose that $\hat{\omega}>2$, as the case $\hat{\omega}=2$ follows from Davenport–Schmidt's theorem (in this case our Theorem 1 claims that $\omega_{LP}\ge 3$). We take $\alpha<\hat{\omega}$ close to $\hat{\omega}$ so that $\alpha>2$.

A vector $\mathbf{x}=(x_0,x_1,x_2)\in\mathbb{Z}^3\setminus\{\mathbf{0}\}$ is defined to be a *minimal point* (or *best approximation*) if

$$\min_{\mathbf{x}':\,0<|\mathbf{x}'|\le|\mathbf{x}|}|L(\mathbf{x}')|=L(\mathbf{x}).$$

As $1,\theta_1,\theta_2$ are linearly independent, all the minimal points form a sequence $\mathbf{x}_\nu=(x_{0,\nu},x_{1,\nu},x_{2,\nu})$, $\nu=1,2,\ldots$, such that for $X_\nu=|\mathbf{x}_\nu|$ and $L_\nu=L(\mathbf{x}_\nu)$ one has

$$X_1<X_2<\cdots,\qquad L_1>L_2>\cdots.$$

Here we should note that

$$L_j\le X_{j+1}^{-\alpha}\tag{3.1}$$

for all j large enough. Of course each vector $\mathbf{x}_j$ is primitive and each couple $\mathbf{x}_j,\mathbf{x}_{j+1}$ forms a basis of the two-dimensional lattice $\mathbb{Z}^3\cap\operatorname{span}(\mathbf{x}_j,\mathbf{x}_{j+1})$.

Let $F(\mathbf{x})$ be a linear form linearly independent of L and P. Then

$$\max\{|L(\mathbf{x})|,|P(\mathbf{x})|,|F(\mathbf{x})|\}\asymp|\mathbf{x}|.\tag{3.2}$$

We also use the notation $P_\nu = P(\mathbf{x}_\nu), F_\nu = F(\mathbf{x}_\nu)$. We will need the determinants

$$\Delta_j = \begin{vmatrix} L_{j-1} & P_{j-1} & F_{j-1} \\ L_j & P_j & F_j \\ L_{j+1} & P_{j+1} & F_{j+1} \end{vmatrix} = A \begin{vmatrix} x_{0,j-1} & x_{1,j-1} & x_{2,j-1} \\ x_{0,j} & x_{1,j} & x_{2,j} \\ x_{0,j+1} & x_{1,j+1} & x_{2,j+1} \end{vmatrix},$$

where A is a non-zero constant depending on the coefficients of the linear forms L, P, F. We take into account (3.2), (3.1) and the inequality $\alpha > 2$ to see that

$$\begin{aligned} \Delta_j &= L_{j-1}P_jF_{j+1} - L_{j-1}P_{j+1}F_j + O(L_jX_{j+1}^2) \\ &= L_{j-1}P_jF_{j+1} - L_{j-1}P_{j+1}F_j + o(1), \quad j \to \infty. \end{aligned} \tag{3.3}$$

The statement below is a variant of Davenport-Schmidt's lemma. We give it without proof. It deals with three consecutive minimal points $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$ lying in a two-dimensional linear subspace, say π. We should note that our definition of minimal points differs from those in [2, 3, 11]. However the main argument is the same. It is discussed in our survey [8]. One may look for the approximation of the one-dimensional subspace $\ell = \pi \cap \{\mathbf{z} : L(\mathbf{z}) = 0\}$ by the points of the two-dimensional lattice $\Lambda_j = \langle x_{j-1}, x_j \rangle$ Then the points $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1} \in \Lambda_j$ are the consecutive best approximations to ℓ with respect to the *induced* norm on π (see [8, Section 5.5]).

LEMMA 1. *If for some j the points $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$ are linearly dependent then*

$$\mathbf{x}_{j+1} = t\mathbf{x}_j + \mathbf{x}_{j-1} \quad \textit{for some integer } t.$$

The next statement has been known for a long time. It comes from Jarník's papers [6, 7]. It was rediscovered by Davenport and Schmidt [4] and discussed in our survey [8].

LEMMA 2. *There exist infinitely many indices j such that the vectors $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$ are linearly independent.*

The following lemma is due to Jarník [6, 7] (see also [8, Section 5.3]).

LEMMA 3. *Suppose that j is large enough and the points $\mathbf{x}_{j-1}, \mathbf{x}_j, \mathbf{x}_{j+1}$ are linearly independent. Then*

$$X_{j+1} \gg X_j^{\alpha-1}, \tag{3.4}$$

$$L_j \ll X_j^{-\alpha(\alpha-1)}. \tag{3.5}$$

Now we take large ν and $k \geq \nu + 1$ such that

- $\mathbf{x}_{\nu-1}, \mathbf{x}_\nu, \mathbf{x}_{\nu+1}$ are linearly independent;
- $\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}$ are linearly independent;

- $\mathbf{x}_j$, $\nu \le j \le k$, belong to the two-dimensional lattice $\Lambda_\nu = \mathbb{Z}^3 \cap \operatorname{span}(\mathbf{x}_\nu, \mathbf{x}_{\nu+1})$.

From Lemma 1 it follows that for $\nu \le j \le k-1$ one has

$$L_{j+1} = t_{j+1}L_j + L_{j-1}, \qquad P_{j+1} = t_{j+1}P_j + P_{j-1},$$

with some integers t_{j+1}, and hence

$$L_\nu P_{\nu+1} - L_{\nu+1}P_\nu = \pm(L_{k-1}P_k + L_kP_{k-1}). \tag{3.6}$$

Lemma 4. *Suppose that*

$$0 < r < \alpha^2 - \alpha + 1 < \hat{\omega}^2 - \hat{\omega} + 1, \tag{3.7}$$

$$|P_\nu| \le L_\nu X_\nu^r \tag{3.8}$$

for ν is large. Then

$$|P_{\nu+1}| \gg X_\nu^{\alpha-1}. \tag{3.9}$$

Proof. For $j = \nu$ consider the second term on the r.h.s. of (3.3). From (3.1), (3.4), (3.7), (3.8) we have

$$|L_{\nu-1}P_\nu F_{\nu+1}| \ll |L_{\nu-1}L_\nu X_\nu^r|\, X_{\nu+1} \ll X_\nu^{r-\alpha}X_{\nu+1}^{1-\alpha} \ll X_\nu^{r-\alpha^2+\alpha-1} = o(1).$$

As $\Delta_\nu \ne 0$ we see that

$$1 \ll |L_{\nu-1}P_{\nu+1}F_\nu| \ll L_{\nu-1}|P_{\nu+1}|\, X_\nu \ll X_\nu^{1-\alpha}|P_{\nu+1}|$$

(in the last inequalities we use (3.2) and (3.1)). ∎

4. The main estimate. The following lemma presents our main argument.

Lemma 5. *Suppose that r satisfies* (3.7) *and $\beta_0 > 0$. Suppose that there are arbitrarily large values of ν satisfying the following conditions:*

(i) *the inequality* (3.1) *holds for all indices $j \ge \nu$ and*

$$L_\nu \gg X_\nu^{-\beta_0}; \tag{4.1}$$

(ii) *we have simultaneously*

$$|P_\nu| \le L_\nu X_\nu^r, \tag{4.2}$$

$$|P_{k-1}| \le L_{k-1}X_{k-1}^r, \tag{4.3}$$

$$|P_k| \le L_k X_\nu^r. \tag{4.4}$$

Then

$$r \ge \alpha^2 + 1 - \frac{\beta_0}{\alpha - 1}, \tag{4.5}$$

$$L_k \gg X_k^{-\beta'} \quad \textit{with} \quad \beta' = r - \alpha - 1 + \frac{\beta_0}{\alpha-1} < \beta_0. \tag{4.6}$$

Proof. First of all we note that

$$L_{\nu+1}|P_\nu| \le L_\nu L_{\nu+1} X_\nu^r \ll L_\nu X_{\nu+2}^{-\alpha} X_\nu^r \ll L_\nu X_{\nu+1}^{-\alpha} X_\nu^r \\ \ll L_\nu X_\nu^{r-\alpha(\alpha-1)} = o(L_\nu X_\nu^{\alpha-1}).$$

Here the first inequality comes from (4.2). The second inequality is (3.1) with $j = \nu + 1$. The third one is simply $X_{\nu+2} \ge X_{\nu+1}$. The fourth one is (3.4) for $j = \nu$. The last inequality follows from (3.7) as $r < \alpha^2 - \alpha + 1 < \alpha^2 - 1$ (because $\alpha > 2$). We see that the conditions of Lemma 4 are satisfied and by this lemma we see that

$$L_\nu |P_{\nu+1}| \gg L_\nu X_\nu^{\alpha-1}.$$

So on the l.h.s. of (3.6) the first summand is larger than the second. Now from (3.6) we have

$$L_\nu X_\nu^{\alpha-1} \ll L_{k-1}|P_k| + L_k|P_{k-1}|. \tag{4.7}$$

We apply (4.3) and (4.4) to see that

$$\max(L_{k-1}|P_k|, L_k|P_{k-1}|) \le L_{k-1}L_k X_k^r \ll X_k^{r-\alpha} X_{k+1}^{-\alpha} \le X_k^{r-\alpha^2} \\ \le X_{\nu+1}^{r-\alpha^2} \ll X_\nu^{(r-\alpha^2)(\alpha-1)}. \tag{4.8}$$

Here the second inequality comes from (3.6) for $j = k - 1$ and $j = k$. The third inequality is Lemma 3 with $j = k$. The fourth one is just $X_k \ge X_{\nu+1}$. The fifth one is Lemma 3 for $j = \nu$.

Now from estimates (4.7), (4.8) and (4.1) we have

$$X_\nu^{-\beta_0+\alpha-1} \ll X_\nu^{(r-\alpha^2)(\alpha-1)}.$$

As ν can be taken arbitrarily large, this gives

$$r \ge \alpha^2 + 1 - \frac{\beta_0}{\alpha - 1}.$$

So (4.5) is proved.

To get (4.6) we combine the estimate (4.7) with the left inequality of (4.8), the bound (4.1) for $j = \nu$ and the bound (3.1) for $j = k - 1$. This gives

$$X_\nu^{\alpha-1-\beta_0} \le L_\nu X_\nu^{\alpha-1} \ll L_{k-1} L_k X_k^r \ll L_k X_k^{r-\alpha},$$

or

$$L_k \gg X_k^{\alpha-r} X_\nu^{\alpha-1-\beta_0}.$$

But $\beta_0 > \alpha(\alpha - 1) \ge \alpha - 1$ by (3.5), and $X_k \ge X_{\nu+1} \gg X_\nu^{\alpha-1}$ by (3.4). So

$$L_k \gg X_k^{\alpha-r+\frac{\alpha-1-\beta_0}{\alpha-1}},$$

and this is the first inequality in (4.6).

Moreover, as $\beta_0 > \alpha(\alpha - 1)$, we deduce from (3.7) that $\beta' < \beta_0$. ∎

5. Proof of Theorem 1. One may suppose that there exist positive c, β_0 such that for every ν we have $L_\nu \geq cX_\nu^{-\beta_0}$ (otherwise $\omega_{LP} = \infty$).

Suppose that r satisfies (3.7). We take an infinite sequence $\nu_0 < \nu_1 < \cdots$ such that

- for every $i = 1, 2, \ldots$ the vectors $\mathbf{x}_{\nu_i-1}, \mathbf{x}_{\nu_i}, \mathbf{x}_{\nu_i+1}$ are linearly independent;
- for $i = 0, 1, 2, \ldots$ the vectors $\mathbf{x}_j$, $\nu_i \leq j \leq \nu_{i+1}$, belong to the two-dimensional lattice $\Lambda_{\nu_i} = \mathbb{Z}^3 \cap \mathrm{span}(\mathbf{x}_{\nu_i}, \mathbf{x}_{\nu_i+1})$.

Note that we can take as ν_0 an arbitrarily large number.

Now we suppose that the three inequalities (4.2)–(4.4) hold for all triples $(\nu, k-1, k) = (\nu_i, \nu_{i+1} - 1, \nu_{i+1})$ for all $i \geq 0$.

We define recursively

$$\beta_{i+1} = r - \alpha - 1 + \frac{\beta_i}{\alpha - 1}.$$

Then

$$\beta_i \leq \alpha(\alpha - 1) + \frac{\beta_0}{(\alpha - 1)^i} \to \alpha(\alpha - 1), \quad i \to \infty.$$

Now we take an arbitrarily large integer w. We show that (4.1) is satisfied for $\nu = \nu_i$ with β_i instead of β_0, by induction on i from the range $0 \leq i \leq w$. This follows from Lemma 5. One should keep in mind that the constant in $\gg$ in (4.6) will depend on w. However ν_0 can be taken large enough. So (4.5) gives

$$r \geq \alpha^2 + 1 - \frac{\beta_w}{\alpha - 1}.$$

We let $w \to \infty$ to see that

$$r \geq \alpha^2 - \alpha + 1.$$

This contradicts (3.7). So there exists j such that $L_j \leq |P_j| X_j^{-r}$. ∎

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