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Cuspidal groups, ordinary Eisenstein series, and Kubota–Leopoldt p-adic L-functions

by

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Throughout this paper p will be a fixed prime ≥ 5 and N a positive integer prime to p. Our main result is an analogue of Stickelberger's theorem for ordinary subgroups of the cuspidal divisor class groups attached to the congruence groups $\Gamma_1(p^n N)$.

Denoting the ring $\mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^*$ by $\mathbb{Z}_{p,N}^*$, let $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$ be the completed group ring of $\mathbb{Z}_{p,N}^*$ over $\mathbb{Z}_p$ which we identify with $\mathrm{Dist}(\mathbb{Z}_{p,N}^*, \mathbb{Z}_p)$. Often we will refer to the elements of $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$ simply as measures. Regarding $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$ as $\Lambda[(\mathbb{Z}/pN\mathbb{Z})^*]$, where Λ is the Iwasawa algebra, the p-primary parts of the ideal class groups of the cyclotomic fields $\mathbb{Q}(\mu_{p^n N})$ are $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$-modules. One may derive from Stickelberger's theorem [20] that these groups are all annihilated by a certain ideal of the ring $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$. This ideal is generated by the "weight one" Bernoulli measures E_c where $c \in \mathbb{Z}_p^*, c^2 \neq 1$. As shown by Iwasawa and Mazur any of these measures may be used to construct the Kubota–Leopoldt p-adic L-function $L_p(s, \chi)$.

The p-primary parts of the cuspidal divisor class groups attached to the congruence groups $\Gamma_1(p^n N)$, which we denote by $\mathcal{C}_n$, are also naturally $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$-modules. In Proposition 5.10 we show that with respect to Hida's idempotent e we have direct sums

$$\text{(i)} \qquad \mathcal{C}_n = \mathcal{C}_n^{\mathrm{nil}} \oplus \mathcal{C}_n^0, \qquad n \geq 1,$$

where $\mathcal{C}_n^0 = e\mathcal{C}_n$ is the ordinary part of $\mathcal{C}_n$.

Now suppose that $N > 1$ is squarefree. Our main result, Theorem 5.14, is the construction of a canonical "weight two" Bernoulli measure β which has the property that

$$\text{(ii)} \qquad \beta \text{ annihilates } \mathcal{C}_n^0 \text{ for all } n \geq 1.$$

Moreover, the measure β leads naturally to the construction of an everywhere analytic p-adic L-function $L_p^*(s, \xi)$ which turns out to be a modification of the Kubota–Leopoldt p-adic L-function.

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Subgroups of the cuspidal groups $\mathcal{C}_n$ have been analyzed in detail when $N = 1$ [9]. These results have been extended to the case $N > 1$ by Yu [25]. In both cases these groups have properties similar to the groups $\mathcal{C}_n^0$ but are themselves not ordinary. The Kubert–Lang theory played a major role in Mazur and Wiles' proof of the Main Conjecture of Iwasawa theory [12].

Let $\mathcal{O}$ be the ring formed by adjoining to $\mathbb{Z}_p$ the values of even primitive Dirichlet characters of conductor dividing pN. In [12] Mazur and Wiles analyze certain cuspidal groups for general N, $\mathcal{C}_\chi^{(n)}$, defined in terms of Igusa curves of characteristic p. For $n \geq 1$, if $N > 1$ the union of these groups is isomorphic to a proper subset of $\mathcal{C}_n^0 \otimes_{\mathbb{Z}_p} \mathcal{O}$. If $N = 1$ the union of these groups is isomorphic to $\mathcal{C}_n^0$. Although the author was unaware of this at the time the work in this article was carried out, for each group $\mathcal{C}_\chi^{(n)}$ Mazur and Wiles construct a corresponding element in the group ring $\mathbb{Q}[(\mathbb{Z}/p^nN\mathbb{Z})^*]$ which annihilates it. These elements are similar to those that may be fit together to construct the measure β. Thus if $N > 1$ is squarefree our results extend some of those of Mazur and Wiles to the groups $\mathcal{C}_n^0$.

Stevens has shown that the cuspidal divisor class group associated with any $\Gamma_1(M)$ may be analyzed in terms of weight two Eisenstein series. A major part of this analysis relies on a theorem, Theorem 5.7, which relates the periods of Eisenstein series to special values of L-functions [19]. To analyze cuspidal divisor class groups Stevens constructs a basis of Eisenstein series which consist of eigenfunctions for the associated Nebentype and Galois groups. These are then modified by Euler factors for each prime l dividing M leading to a particularly canonical basis.

We now assume for the rest of this introduction that N is squarefree. In Section 1 we give the construction of the measure β and the corresponding p-adic L-function $L_p^*(s,\xi)$ when $N > 1$. The exact relation between $L_p^*(s,\xi)$ and $L_p(s,\chi)$ is given in Theorem 1.13. In Section 2, following Hecke [3], we construct a generating set for the space of weight k Eisenstein series for each $k \geq 2$. In Section 3 we specialize to design a basis for each space of generalized weight k Eisenstein series for the group $\Gamma_1(M)$. These series are well known, but are presented here in a form by which the actions of the Nebentype and Galois groups are determined by simple transformations. Essentially these are weight k analogues of weight two Eisenstein series defined by Stevens [19], and many of our proofs are simple generalizations.

In connection with his study of the Tate module $\mathrm{Ta}_p(J_\infty)$, and its ordinary part, Hida defines spaces of ordinary p-adic Eisenstein series $\mathcal{E}_k(\Gamma_1(p^nN); \overline{\mathbb{Q}}_p)^0$, for each weight k and level $\Gamma_1(p^nN)$, and gives a basis for each [5]. In Section 4 we present a different basis for these series, all of which are related to the p-adic L-function $L_p^*(s,\xi)$ (Proposition 4.8).

In Section 5 we use our results for $k = 2$, and Theorem 5.7, to establish Theorem 5.14. The measure β is constructed by smoothing the unbounded "weight two" Bernoulli measure by each prime $l \mid N$ for $N > 1$. In Theorem 5.15 we present some evidence that this smoothing for each l is necessary to annihilate generalized cuspidal groups. Finally we consider the case $N = 1$ and give a different proof of a theorem of Mazur–Wiles (Theorem 5.16).

We may take a direct limit of the ordinary cuspidal groups, setting $\mathcal{C}^0 = \varinjlim \mathcal{C}_n^0$. Let $\mathcal{C}^*$ be the Pontryagin dual of $\mathcal{C}^0$, equipped with a natural action of $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$. As shown in [13], $\mathcal{C}^*$ is a finitely generated torsion Λ-module. Thus our results here may be a step in the direction of a structure theorem for cuspidal groups analogous to the Main Conjecture. It is worth mentioning that we may also form an inverse limit of the $\mathcal{C}_n^0$—this is almost certain to be without Λ-torsion. For an illustration of this in the nonordinary case the reader should consult [9], Chapter 6.

As shown in [13] $\mathcal{C}^*$ is isomorphic to a Λ-adic group $\mathcal{C}_\Gamma(\mathbf{D})$ which is an example of a generalized cuspidal divisor class group for Λ-adic modular symbols for the group $\Gamma = \Gamma_1(N)$. It turns out that we may also express $\mathcal{C}_\Gamma(\mathbf{D})$ via

$$\text{(iii)} \qquad \mathcal{C}_\Gamma(\mathbf{D}) = \mathrm{Ta}_p(J_\infty)/S,$$

where S is a submodule invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The author's thesis problem in part was to analyze the group $\mathcal{C}_\Gamma(\mathbf{D})$ and using this determine a bound on the poles of a two variable p-adic L-function designed by Greenberg and Stevens [2]. I wish to express my great thanks to my thesis advisor Glenn Stevens, who posed this beautiful problem, and who was always a source of inspiration and encouragement. I am also indebted to Robert Sczech and Jacob Sturm for many valuable discussions and comments throughout the writing of this paper.

Notation. For $M \in \mathbb{Z}^+$, i.e., for a positive integer M, we write $\mathbb{Z}(M)^*$ for the group $(\mathbb{Z}/M\mathbb{Z})^*$. The divisors of M are understood to be restricted to positive divisors; in particular this applies to sums indexed by $d \mid M$. We let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in $\mathbb{C}$, and let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of the p-adic rationals $\mathbb{Q}_p$. We also let $\mathbb{C}_p$ denote the p-adic completion of $\overline{\mathbb{Q}}_p$, and fix once and for all an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. For $z \in \mathbb{C}$ we will write $e(z) = e^{2\pi i z}$. Given a ring $R \subseteq \overline{\mathbb{Q}}_p$, and a primitive Dirichlet character ξ, the ring formed by adjoining the values of ξ to R will be denoted $R[\xi]$.

1. Kubota–Leopoldt p-adic L-functions. Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a Dirichlet character, not necessarily primitive. If M is the smallest positive period of χ, we indicate this by writing $m(\chi) = M$. Note that χ is a map from

$\mathbb{Z}$ to $\overline{\mathbb{Q}}$, hence to $\overline{\mathbb{Q}}_p$ via our fixed embedding. If χ is primitive we denote its conductor by $\mathrm{cond}(\chi)$. In general a Dirichlet character is not assumed to be primitive, and we will only define conductors for primitive χ. If χ and χ' are two Dirichlet characters, we define their product simply by $\chi\chi'(a) = \chi(a)\chi'(a)$, $a \in \mathbb{Z}$. This is not standard and in general $\chi\chi'$ is not primitive. An exception to this convention occurs as explained in the remarks preceding (1.8). For a Dirichlet character χ, and $M \in \mathbb{Z}^+$, we define a Dirichlet character χ_M by

$$\chi_M(a) = \begin{cases} \chi(a) & \text{if } (a,M)=1, \\ 0 & \text{if } (a,M)>1. \end{cases}$$

Any Dirichlet character is then equal to χ_M for a unique primitive Dirichlet character χ and some $M \in \mathbb{Z}^+$. In general M is not unique.

For $k \in \mathbb{Z}^+$ we define the Bernoulli polynomials $B_k(x)$ by

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}.$$

Then in particular $B_1(x) = x - 1/2$, and in general

$$B_k(x) = x^k - \frac{k}{2}x^{k-1} + \text{ lower order terms.}$$

For $x \in \mathbb{R}$ we let $[x]$ denote the largest integer less than or equal to x, and set $\{x\} = x - [x]$. We define the Bernoulli functions $\mathbf{B}_k : \mathbb{R} \to \mathbb{R}$ by setting $\mathbf{B}_k(x) = B_k(\{x\})$. The Bernoulli functions satisfy the distribution law

$$\mathbf{B}_k(x) = M^{k-1}\sum_{a=0}^{M-1}\mathbf{B}_k\left(\frac{x+a}{M}\right), \qquad M \in \mathbb{Z}^+;$$

thus if $c \mid M$ we have for $x \in \mathbb{Z}$,

$$c^{k-1}\mathbf{B}_k\left(\frac{x}{c}\right) = M^{k-1}\sum_{\substack{y\equiv x \ (\mathrm{mod}\ c)\\ 0\le y\le M-1}}\mathbf{B}_k\left(\frac{y}{M}\right). \tag{1.1}$$

Suppose that χ is a Dirichlet character such that $m(\chi) \mid A$. For $k \geq 1$ we define the character sum $\mathbf{B}_k(\chi)$ by

$$\mathbf{B}_k(\chi) = A^{k-1}\sum_{x=0}^{A-1}\mathbf{B}_k\left(\frac{x}{A}\right)\chi(x). \tag{1.2}$$

By (1.1) this definition does not depend on A. We define the Möbius function $\mu : \mathbb{Z}^+ \to \mathbb{Z}$ as usual by (i) $\mu(1) = 1$ and (ii) $\mu(d) = 0$, if d is not squarefree, and $\mu(d) = (-1)^f$ otherwise, where f is the number of distinct prime factors of d. Let R be a ring. For any function $f:\mathbb{Z}^+\to R$

$$\sum_{\substack{x=0\\(x,A)=1}}^{A-1} f(x)=\sum_{d|A}\mu(d)\sum_{x=0}^{A/d-1} f(xd). \tag{1.3}$$

From this it follows easily, letting $A=m(\chi)p$, that

$$\mathbf{B}_k(\chi_p)=(1-\chi(p)p^{k-1})\mathbf{B}_k(\chi),\qquad k\geq 1. \tag{1.4}$$

DEFINITION 1.5. For each integer $n\geq 0$ we denote the set of primitive Dirichlet characters of conductor dividing p^nN by $K(Np^n)$, and set $K(Np^\infty)=\bigcup_{n\geq 0}K(Np^n)$. Generally we will denote an element of $K(Np^\infty)$ by ξ.

To construct the measure β we first need a general result.

LEMMA 1.6. *Let $N>1$ be squarefree, and let $n\geq 0$. For each $d\,|\,N$ let $d'\in\mathbb{Z}$ be such that $dd'\equiv 1 \pmod{p^nN/d}$. Let R be a $\overline{\mathbb{Q}}$-module. Then for any function $f:\mathbb{Q}/\mathbb{Z}\to R$, and $\xi\in K(Np^n)$,*

$$\sum_{d|N}\mu(d)\sum_{\substack{x=0\\(x,p^nN)=1}}^{p^nN-1} f\left(\frac{xd'}{p^nN/d}\right)\xi(x)$$
$$=\sum_{d|N}\mu(d)\xi(d)d\sum_{x=0}^{p^nN/d-1} f\left(\frac{x}{p^nN/d}\right)\xi_{p^n}(x).$$

Proof. We may write the left hand side of the above as

$$\sum_{d|N}\mu(d)\sum_{x\in\mathbb{Z}(p^nN)^*} f\left(\frac{xdd'}{p^nN}\right)\xi(x),$$

where $f(xdd'/(p^nN))$ has the obvious meaning for $x\in\mathbb{Z}(p^nN)^*$. Suppose that ξ has conductor p^rc where $c\,|\,N$. If $d\,|\,N$ and $l\,|\,(d,c)$ for some prime l, via the canonical decomposition $\mathbb{Z}(p^nN)^*=\mathbb{Z}(p^nN/l)^*\times\mathbb{Z}(l)^*$ we have

$$\sum_{x\in\mathbb{Z}(p^nN)^*} f\left(\frac{xdd'}{p^nN}\right)\xi(x)=\sum_{y\in\mathbb{Z}(p^nN/l)^*}\sum_{z\in\mathbb{Z}(l)^*} f\left(\frac{yzdd'}{p^nN}\right)\xi(yz)=0,$$

for if y is fixed,

$$\sum_{z\in\mathbb{Z}(l)^*} f\left(\frac{yzdd'}{p^nN}\right)\xi(yz)=\xi(y)f\left(\frac{ydd'}{p^nN}\right)\sum_{z\in\mathbb{Z}(l)^*}\xi(z)=0,$$

as $l\,|\,d$ and ξ is primitive. Then the above becomes

$$\sum_{d|(N/c)} \mu(d)\phi(d) \sum_{x\in\mathbb{Z}(p^nN/d)^*} f\Big(\frac{xd'}{p^nN/d}\Big)\xi(x)$$
$$= \sum_{d|N} \mu(d)\xi(d)\phi(d) \sum_{x\in\mathbb{Z}(p^nN/d)^*} f\Big(\frac{x}{p^nN/d}\Big)\xi_{p^n}(x)$$
$$= \sum_{d|N} \mu(d)\xi(d)\phi(d) \sum_{m|(N/d)} \mu(m) \sum_{x=0}^{p^nN/dm-1} f\Big(\frac{xm}{p^nN/d}\Big)\xi_{p^n}(xm),$$

using (1.3), or letting $D = dm$,

$$= \sum_{D|N} \mu(D)\xi(D)\Big(\sum_{d|D}\phi(d)\Big) \sum_{x=0}^{p^nN/D-1} f\Big(\frac{x}{p^nN/D}\Big)\xi_{p^n}(x)$$
$$= \sum_{D|N} \mu(D)\xi(D)D \sum_{x=0}^{p^nN/D-1} f\Big(\frac{x}{p^nN/D}\Big)\xi_{p^n}(x).$$

Corollary 1.7. *Let $N > 1$ be squarefree. Then for $k \geq 1$, $n \geq 0$, and $\xi \in K(Np^n)$,*

$$(p^nN)^{k-1}\sum_{d|N}\mu(d) \sum_{x\in\mathbb{Z}(p^nN)^*} \mathbf{B}_k\Big(\frac{xd'}{p^nN/d}\Big)\xi(x)$$
$$= \Big(\prod_{l|N}(1-\xi(l)l^k)\Big)\mathbf{B}_k(\xi_{p^n}).$$

Proof. Letting $\mathbf{B}_k = f$, and applying the lemma, the left hand side of the above

$$= (p^nN)^{k-1}\sum_{d|N}\mu(d)\xi(d)d \sum_{x=0}^{p^nN/d-1} \mathbf{B}_k\Big(\frac{x}{p^nN/d}\Big)\xi_{p^n}(x)$$
$$= \sum_{d|N}\mu(d)\xi(d)d^k\Big[\Big(\frac{p^nN}{d}\Big)^{k-1} \sum_{x=0}^{p^nN/d-1} \mathbf{B}_k\Big(\frac{x}{p^nN/d}\Big)\xi_{p^n}(x)\Big]$$
$$= \sum_{d|N}\mu(d)\xi(d)d^k\cdot\mathbf{B}_k(\xi_{p^n}).$$

We assume that the reader is familiar with the theory of p-adic distributions as described, for example in [11] or [22]. Let d be a positive integer prime to p, and $\mathcal{O}$ a subring of $\mathbb{C}_p$. For integers $m \geq n$ the natural maps $\mathbb{Z}(p^md)^* \to \mathbb{Z}(p^nd)^*$ induce trace maps on the group rings $\mathcal{O}[\mathbb{Z}(p^md)^*] \to \mathcal{O}[\mathbb{Z}(p^nd)^*]$; accordingly we set $\mathcal{O}[[\mathbb{Z}_{p,d}^*]] = \varprojlim \mathcal{O}[\mathbb{Z}(p^nd)^*]$, which we identify with the ring of $\mathcal{O}$-valued distributions on $\mathbb{Z}_{p,d}^*$, $\mathrm{Dist}(\mathbb{Z}_{p,d}^*,\mathcal{O})$. Given a distribution $\nu \in \mathbb{Z}_p[[\mathbb{Z}_{p,d}^*]]$, for any continuous $f : \mathbb{Z}_{p,d}^* \to \mathbb{C}_p$, the integral $\nu(f) = \int f\, d\nu$ is defined. For $a \in \mathbb{Z}_{p,d}^*$ we set $\nu(a) = \int \delta_a\, d\nu$, where δ_a is the Dirac measure at a.

If R is a $\overline{\mathbb{Q}}$-module and $f : \mathbb{Q}/\mathbb{Z} \to R$ a function then for $n \geq 0$ the function $y \mapsto f(y/p^n N)$ for $y \in \mathbb{Z}(p^n N)^*$ has a natural extension to $\mathbb{Z}_{p,N}^*$. Via our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ we may regard Corollary 1.7 as an equality in either $\mathbb{C}$ or $\overline{\mathbb{Q}}_p$, and we will use this to construct elements of $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$ which are related to the Kubota–Leopoldt p-adic L-function.

For any Dirichlet character χ we define the complex Dirichlet L-function $L(s,\chi)$ by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}, \qquad \mathrm{Re}(s) > 1.$$

By a classical result $L(s,\chi)$ has an analytic continuation to the complex plane with a possible simple pole at $s = 1$; denoting this extension also by $L(s,\chi)$, we have the identities

$$L(1-k,\chi) = -\frac{1}{k}\mathbf{B}_k(\chi), \qquad k \in \mathbb{Z}^+.$$

A p-adic analogue of the L-function $L(s,\chi)$, which interpolates these values, was first constructed by Kubota and Leopoldt [10]. We denote this p-adic L-function by $L_p(s,\chi)$.

Suppose χ is a primitive Dirichlet character of conductor $p^r d$ where $(d,p) = 1$. Following Iwasawa, suppose that f_χ is a continuous function on $\mathbb{Z}_p$ (or on $\mathbb{Z}_p \backslash \{1\}$ if χ is trivial) that satisfies

$$f_\chi(1-k) = L(1-k,\chi_p)$$

for all $k \in \mathbb{Z}^+$ such that $k \equiv 0 \pmod{p-1}$. Since the set of such k is dense in $\mathbb{Z}_p$, f_χ is unique. The existence of such a function follows from congruences similar to the Kummer congruences. The Kubota–Leopoldt p-adic L-function is then defined by

$$L_p(s,\chi) = f_\chi(s), \qquad s \in \mathbb{Z}_p.$$

The Kubota–Leopoldt p-adic L-function is described in terms of power series by Iwasawa [6, 7].

Let $\omega : \mathbb{Z}_p^* \to \mu_{p-1}$ be the Teichmüller character, and $\langle\ \rangle : \mathbb{Z}_p^* \to 1 + p\mathbb{Z}_p$ the projection onto the principal units. Then $a = \omega(a)\langle a \rangle$ for $a \in \mathbb{Z}_p^*$. We also define projections $\omega_p : \mathbb{Z}_{p,d}^* \to \mu_{p-1}$ and $\langle\ \rangle_p : \mathbb{Z}_{p,d}^* \to 1+p\mathbb{Z}_p$ as induced by the natural map $\mathbb{Z}_{p,d}^* \to \mathbb{Z}_p^*$, and set $a_p = \omega_p(a)\langle a \rangle_p$ for $a \in \mathbb{Z}_{p,d}^*$. If χ may be written as $\chi = \eta\omega_p^j$ for some η of conductor prime to p, $j \in \mathbb{Z}$, we define $\chi\omega_p^k = \eta$ if $j + k \equiv 0 \pmod{p-1}$. In particular ω_p^k is trivial if $k \equiv 0 \pmod{p-1}$. This does not agree with our prior conventions, but will simplify our notation.

Iwasawa and Mazur have determined the relation between $L_p(s,\chi)$ and certain measures E_c in $\mathbb{Z}_p[[\mathbb{Z}^*_{p,d}]]$ ([22], Theorem 12.2). In general we have

$$L_p(1-k,\chi\omega_p^k) = -\frac{\mathbf{B}_k(\chi_p)}{k} = L(1-k,\chi_p), \qquad k \geq 1 \tag{1.8}$$

([22], Theorem 5.11 and (1.4)). We now use the distributive property (1.1) to define a series of Bernoulli distributions.

Definition 1.9. Let N be squarefree, and for each $d \mid N$ let d' be the inverse of d in $\mathbb{Z}^*_{p,N/d}$. For $k \geq 1$ we define $\beta_{k,N} \in \overline{\mathbb{Q}}_p[[\mathbb{Z}^*_{p,N}]]$ via

$$\beta_{k,N} = -\frac{1}{k}\varprojlim (p^nN)^{k-1}\sum_{d\mid N}\mu(d)\sum_{x\in\mathbb{Z}(p^nN)^*}\mathbf{B}_k\left(\frac{xd'}{p^nN/d}\right)\cdot[x].$$

If N is understood we simply write $\beta_k = \beta_{k,N}$. Note that if $N=1$ each β_k is unbounded.

Lemma 1.10. *Let $N > 1$ be squarefree, and for $n \geq 1$ let $\beta_1^{(n)} \in \mathbb{Q}[\mathbb{Z}(p^nN)^*]$ be defined by*

$$\beta_1^{(n)} = \sum_{d\mid N}\mu(d)\sum_{x\in\mathbb{Z}(p^nN)^*}\mathbf{B}_1\left(\frac{xd'}{p^nN/d}\right)\cdot[x],$$

where for each $d \mid N$ d' is an integer such that $dd' \equiv 1 \pmod{p^nN/d}$. Then $\beta_1^{(n)} \in (2N)^{-1}\mathbb{Z}[[\mathbb{Z}^*_{p,N}]]$. Thus β_1 is a measure.

Proof. Fix $n \geq 1$. We have

$$\begin{aligned}2N\sum_{d\mid N}\mu(d)&\sum_{x\in\mathbb{Z}(p^nN)^*}\mathbf{B}_1\left(\frac{xd'}{p^nN/d}\right)\cdot[x]\\ &\equiv 2N\sum_{d\mid N}\mu(d)\sum_{x\in\mathbb{Z}(p^nN)^*}\left(\frac{\overline{x}d'}{p^nN/d}\right)\cdot[x] \pmod{\mathbb{Z}[\mathbb{Z}(p^nN)^*]},\end{aligned}$$

where for each x, $\overline{x}$ is a fixed representative in $\mathbb{Z}$. Fix a prime $l \mid N$. Then the coefficient of any $[x]$ in the above

$$= \frac{2}{p^n}\sum_{d\mid(N/l)}\mu(d)[\overline{x}dd' - \overline{x}dl(dl)'] \in \mathbb{Z},$$

by the definitions, and this proves the claim.

The next lemma is analogous to a result of Lang for the weight k analogues of the measure E_c ([11], Chapter 2, Theorem 2.2). Its proof is similar, and is also based on the proof presented by Washington for Theorem 12.2 of [22].

LEMMA 1.11. *Let $N > 1$ be squarefree. For each integer $k > 1$ let $\beta^{(k)} \in \mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$ be defined by $\beta^{(k)}(a) = a_p^{k-1}\beta_1(a)$ for $a \in \mathbb{Z}_{p,N}^*$. Then $\beta_k = \beta^{(k)}$ for every $k > 1$.*

Proof. For $n \geq 1$ and $k > 1$ we define a step function $f_{k,n} : \mathbb{Z}_{p,N}^* \to \mathbb{Z}_p$ given by $f_{k,n}(a) = b^{k-1}$, where $0 \leq b < p^n N$ and $a \equiv b \pmod{p^n N}$. Let $f_k : \mathbb{Z}_{p,N}^* \to \mathbb{Z}_p$ be defined by $f_k(a) = a_p^{k-1}$. Then with respect to the topology given by the sup norm, $f_{k,n} \to f_k$ uniformly, and $\int f_{k,n}\, d\beta_1$ converges to $\int f_k\, d\beta_1$. Given $m \in \mathbb{Z}^+$, fix $k > 1$ and $a \in \mathbb{Z}_{p,N}^*$, and choose $n \geq m$ such that

$$\begin{aligned} A_k = &-\frac{1}{k}(p^n N)^{k-1}\sum_{d|N}\mu(d)\left[\left\{\frac{bd'}{p^n N/d}\right\}^k - \frac{k}{2}\left\{\frac{bd'}{p^n N/d}\right\}^{k-1}\right] \\ &\equiv \beta_k(a) \pmod{p^m\mathbb{Z}_p}, \end{aligned}$$

and

$$A_1 = -b^{k-1}\sum_{d|N}\mu(d)\left[\left\{\frac{bd'}{p^n N/d}\right\} - \frac{1}{2}\right] \equiv \beta_1^{k-1}(a) \pmod{p^m\mathbb{Z}_p},$$

where $f_{1,n}(a) = b$.

Fix $d \,|\, N$ and for convenience assume that d' is an integer satisfying $dd' \equiv 1 \pmod{p^{2n}N/d}$. Let $d'b = b_1 + (p^n N/d)b_2$, where $0 \leq b_1 < p^n N/d$. Then

$$\left\{\frac{bd'}{p^n N/d}\right\} = \frac{db_1}{p^n N} \quad \text{and} \quad b \equiv db_1 + p^n N b_2 \pmod{p^{2n}}.$$

Thus

$$A_k \equiv -\frac{(db_1)^k}{kp^n N} + \frac{(db_1)^{k-1}}{2} \pmod{p^m\mathbb{Z}_p}$$

and

$$\begin{aligned} A_1 &\equiv -[(db_1)^{k-1} + (k-1)(db_1)^{k-2}p^n N b_2]\left[\frac{db_1}{p^n N} - \frac{1}{2}\right] \\ &\equiv -\frac{(db_1)^k}{p^n N} + \frac{(db_1)^{k-1}}{2} - (k-1)(db_1)^{k-1}b_2 \pmod{p^m\mathbb{Z}_p}. \end{aligned}$$

Note that $b^k \equiv (db_1)^k + k(db_1)^{k-1}p^n N b_2 \pmod{p^m\mathbb{Z}_p}$. Using this to solve for $(k-1)(db_1)^{k-1}b_2 \pmod{p^m\mathbb{Z}_p}$, and substituting this into the last congruence for A_1, we obtain

$$A_k - A_1 \equiv \sum_{d|N}\mu(d)\frac{(k-1)b^k}{kp^n N} = 0 \pmod{p^m\mathbb{Z}_p}.$$

Since β_2 is a "weight two" measure, to define the corresponding p-adic L-function, we twist by the character ω_p^{-2}, where recall that $\xi\omega_p^{-2}$ is the primitive character corresponding to the product of ξ and ω_p^{-2} (see the remarks preceding (1.8)).

DEFINITION 1.12. Suppose $N > 1$ is squarefree, and let $\beta = \beta_2$ be as in Definition 1.9. For $\xi \in K(Np^\infty)$ we define a *p-adic L-function* $L_p^*(s, \xi)$ by

$$L_p^*(s, \xi) = \int_{\mathbb{Z}_{p,N}^*} \xi\omega_p^{-2}(a)\langle a\rangle_p^{-s}\, d\beta, \qquad s \in \mathbb{Z}_p.$$

The relation between $L_p^*(s, \xi)$ and $L_p(s, \xi)$ is an easy consequence of Lemma 1.11.

THEOREM 1.13. *Let $N > 1$ be squarefree. Then for every $\xi \in K(Np^\infty)$, $s \in \mathbb{Z}_p$,*

$$L_p^*(2 - s, \xi) = \Big(\prod_{\substack{l \mid N \\ l \text{ prime}}} (1 - \xi(l)\langle l\rangle^s)\Big) L_p(1 - s, \xi).$$

Proof. Let $\xi \in K(Np^\infty)$. Both sides of the identity to be shown are continuous in s, except possibly at $s = 1$. Thus it suffices to prove the above for $k \in \mathbb{Z}^+$ such that $k \equiv 0 \pmod{p - 1}$. Fix such a k. Then by (1.8) we must show that

$$\int_{\mathbb{Z}_{p,N}^*} \xi(a)a_p^{k-2}\, d\beta = \Big(\prod_{l \mid N} (1 - \xi(l)l^k)\Big)\Big(-\frac{\mathbf{B}_k(\xi_p)}{k}\Big).$$

This follows at once from Lemma 1.11 and Corollary 1.7.

We may regard the p-adic L-function $L_p^*(s, \xi)$ as a two-variable function $L_p^* : \mathbb{Z}_p \times K(Np^\infty) \to \overline{\mathbb{Q}}_p$. As suggested by an idea of Greenberg [1], we may extend the domain of such an L-function in a natural way. Note that each continuous ring homomorphism $\overline{\kappa} : \mathbb{Z}_{p,N}^* \to \overline{\mathbb{Q}}_p$ extends uniquely to a nonzero continuous $\mathbb{Z}_p$-algebra homomorphism $\kappa : \mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]] \to \overline{\mathbb{Q}}_p$. For simplicity we will simply write κ for the restriction $\overline{\kappa}$. The homomorphism is given by $\nu \mapsto \nu(\kappa)$ where, recall, $\nu(\kappa) = \int_{\mathbb{Z}_{p,N}^*} \kappa\, d\nu$.

DEFINITION 1.14. Following [2], Definition 1.11, we denote the $\mathbb{Z}_p$-module of nonzero continuous $\mathbb{Z}_p$-algebra homomorphisms from $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$ to $\overline{\mathbb{Q}}_p$ by $\mathcal{X}_N$. $\kappa \in \mathcal{X}_N$ is defined to be an arithmetic character if its restriction to $\mathbb{Z}_{p,N}^*$ is of the form $\kappa(a) = \xi(a)a_p^{k-2}$ for an integer $k \ge 2$ and some $\xi \in K(Np^\infty)$. κ is then said to have *signature* (ξ, k) and *level* r if $m(\xi_p) = p^r d$ for some $d \mid N$. We denote the set of arithmetic characters by $\mathcal{X}_N^{\mathrm{arith}}$. For either choice of $\pm$ sign we also let $\mathcal{X}_N^{\pm}$ be the subset of $\mathcal{X}_N$ consisting of characters κ such that $\kappa : \mathbb{Z}_{p,N}^* \to \overline{\mathbb{Q}}_p^*$ satisfies $\kappa(-1) = \pm 1$, and define $\mathcal{X}_N^{\mathrm{arith},\pm}$ similarly. Now if $N > 1$ is squarefree we extend the domain of the p-adic L-function $L_p^*(s,\xi)$ by defining

$$L_p^*(\kappa) = \beta(\kappa) = \int\limits_{\mathbb{Z}_{p,N}^*} \kappa \, d\beta, \qquad \kappa \in \mathcal{X}_N.$$

If $N = 1$ and $\kappa \in \mathcal{X}_N^{\mathrm{arith}}$ has signature (ξ, k) we also define $\beta(\kappa) = -\mathbf{B}_k(\xi_p)/k$.

We now suppose that $N > 1$ is squarefree so that the p-adic L-function $\beta(\kappa)$ is defined. Note that $\beta(\kappa) = 0$ for all $\kappa \in \mathcal{X}_N^-$. Now from Definition 1.12, if $\kappa \in \mathcal{X}_N^{\mathrm{arith},+}$ has signature (ξ, k), then as shown in the proof of Theorem 1.13,

$$\beta(\kappa) = \int\limits_{\mathbb{Z}_{p,N}^*} \xi(a) a_p^{k-2} \, d\beta = -\Big(\prod_{l \mid N} (1 - \xi(l) l^k)\Big) \frac{\mathbf{B}_k(\xi_p)}{k}. \tag{1.15}$$

By the above definition this also holds for $N = 1$ where the empty product is regarded as 1.

2. Weight k Eisenstein series. Let Γ be a congruence group, i.e., a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ which contains some $\Gamma(M)$. For an integer $k \geq 2$ we denote the space of weight k Eisenstein series for Γ by $\mathcal{E}_k(\Gamma)$. The space of weight k Eisenstein series of all levels is then given by

$$\mathcal{E}_k = \bigcup_{M \geq 1} \mathcal{E}_k(\Gamma(M)).$$

We will use the methods of Hecke to construct a generating set for $\mathcal{E}_k$ for each $k \geq 2$. Our approach, in particular, is based on [3]. For a discussion of Eisenstein series in general, we refer the reader to [14] or [15].

Let $\mathbf{H}$ be the upper half plane, and let $\mathbf{S} = M_2(\mathbb{Z}) \cap \mathrm{GL}^+(\mathbb{Q})$. Then $\mathbf{S}$ acts on $\mathbf{H}$ by fractional linear transformations. For an integer $k \geq 2$, and a function $f : \mathbf{H} \to \mathbb{C}$, we define the weight k action of $\mathbf{S}$ by

$$(f|[\alpha]_k)(z) = (\det \alpha)^{k-1} (cz + d)^{-k} f(\alpha z) \quad \text{for} \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{S}. \tag{2.1}$$

If it is clear that $f \in \mathcal{E}_k$, we will simply write $f|\alpha$ for $f|[\alpha]_k$. We caution the reader that this definition differs from the weight k action defined, for example, by Shimura. Here we put $(\det \alpha)^{k-1}$ rather than $(\det \alpha)^{k/2}$; this has the advantage of simplifying most of our remarks. Of course if $k = 2$, or $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, the two definitions agree.

First suppose $k > 2$, and let $\underline{a} = (a_1, a_2) \in \mathbb{Z}^2$. For an integer $M \geq 1$ we define an Eisenstein series $G_{M,k,\underline{a}}(z)$ via

$$G_{M,k,\underline{a}}(z) = \sum_{\substack{m_1 \equiv a_1 \ (\mathrm{mod}\ M) \\ m_2 \equiv a_2 \ (\mathrm{mod}\ M) \\ \underline{m} \neq 0}} (m_1 z + m_2)^{-k}, \qquad z \in \mathbf{H}. \tag{2.2}$$

This series converges absolutely and uniformly on bounded vertical strips, hence defines a holomorphic function on $\mathbf{H}$. Let $\delta(a_1/M) = 1$ if $a_1/M \in \mathbb{Z}$, and $= 0$ otherwise. The Fourier expansion of $G_{M,k,\underline{a}}$ is given by

$$G_{M,k,\underline{a}}(z) = \alpha_0(M,k,\underline{a}) + \frac{(-2\pi i)^k}{M^k(k-1)!} \sum_{m_1 \equiv a_1\,(M)} \sum_{\substack{tm_1>0 \\ t\in\mathbb{Z}}} t^{k-1}\,\mathrm{sgn}(t) e\left(\frac{a_2 t + ztm_1}{M}\right),$$

where

$$\alpha_0(M,k,\underline{a}) = \delta\left(\frac{a_1}{M}\right) \sum_{\substack{m_2 \equiv a_2\,(M) \\ m_2 \neq 0}} m_2^{-k}$$

([15], p. 156). Let

$$A_{k,\underline{a}} = \sum_{\substack{m_1 \equiv a_1\,(M) \\ m_1 \in \mathbb{Z}^+}} \sum_{t=1}^{\infty} t^{k-1} e\left(\frac{a_2 t + ztm_1}{M}\right);$$

then

$$G_{M,k,\underline{a}}(z) = \alpha_0(M,k,\underline{a}) + \frac{(-2\pi i)^k}{M^k(k-1)!}(A_{k,\underline{a}} + (-1)^k A_{k,-\underline{a}}).$$

Now suppose $k = 2$. For $\underline{a} \in \mathbb{Z}^2$ the corresponding function on $\mathbf{H}$ is not holomorphic. Hecke has shown that the function

$$G_{M,2,\underline{a}}(z,s) = \sum_{\substack{\underline{m} \equiv \underline{a}\,(M) \\ \underline{m} \neq 0}} (m_1 z + m_2)^{-2} |m_1 z + m_2|^{-s},$$

defined for $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$, has an analytic continuation, for fixed $z \in \mathbf{H}$, to a holomorphic function on $\mathbb{C}$. Let $G_{M,2,\underline{a}}(z) = G_{M,2,\underline{a}}(z,0)$. The Fourier expansion of $G_{M,2,\underline{a}}(z)$ is given by

$$G_{M,2,\underline{a}}(z) = \alpha_0(M,2,\underline{a}) + \frac{(-2\pi i)^2}{M^2}(A_{2,\underline{a}} + A_{2,-\underline{a}}) - \frac{2\pi i}{M^2(z-\bar{z})},$$

with notation analogous to that of higher weight ([15], p. 167). Let $V = \mathbb{Q}^2/\mathbb{Z}^2$; then $\mathbf{S}$ acts on V by right matrix multiplication. For $k > 2$, in the case of absolute convergence, one verifies easily that for $\underline{a} \in V$ and $M \geq 1$,

$$G_{M,k,\underline{a}}|[\alpha]_k = G_{M,k,\underline{a}\alpha}, \qquad \alpha \in \mathrm{SL}_2(\mathbb{Z}).$$

Hecke has shown that the corresponding property also holds for $k = 2$; thus the $G_{M,k,\underline{a}}$ are invariant with respect to $\Gamma(M)$.

Now suppose $\underline{x} = (x_1, x_2) \in V$, and that $\underline{x} \in \left(\frac{1}{M}\mathbb{Z}/\mathbb{Z}\right)^2$ for some $M \geq 1$. Writing $\underline{x} \equiv (y_1/M, y_2/M) \pmod{\mathbb{Z}^2}$ for $(y_1, y_2) \in \mathbb{Z}^2$, as suggested by

Hecke for the case $k=2$, we define for $k \geq 2$ a $\Gamma(M)$-invariant function on $\mathbf{H}$, $\phi_k\underline{x}$, via

$$\phi_k\underline{x}(z) = \frac{(k-1)!}{(2\pi i)^k} \sum_{\underline{a} \bmod M} e\left(\frac{a_2y_1 - a_1y_2}{M}\right) G_{M,k,\underline{a}}(z), \tag{2.3}$$

where the sum ranges over elements $\underline{a} = (a_1, a_2) \in \mathbb{Z}^2 \bmod M$. Let

$$\delta(\underline{x}) = \begin{cases} 0 & \text{if } \underline{x} \notin \mathbb{Z}^2, \text{ or } k > 2, \\ \dfrac{i}{2\pi(z-\overline{z})} & \text{if } \underline{x} \in \mathbb{Z}^2 \text{ and } k = 2. \end{cases}$$

Then

$$\begin{aligned} \phi_k\underline{x} = \delta(\underline{x}) + &\sum_{\underline{a} \bmod M} e\left(\frac{a_2y_1 - a_1y_2}{M}\right) \\ &\times \left[\frac{(k-1)!}{(2\pi i)^k}\alpha_0(M,k,\underline{a}) + \frac{(-1)^k}{M^k}(A_{k,\underline{a}} + (-1)^k A_{k,-\underline{a}})\right]. \end{aligned}$$

For $x \in \mathbb{R}$, and $k \geq 2$, we have the classical identity

$$\mathbf{B}_k(x) = -\sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{k!}{(2\pi i m)^k} e(mx);$$

thus

$$\begin{aligned} &\sum_{\underline{a} \bmod M} e\left(\frac{a_2y_1 - a_1y_2}{M}\right) \frac{(k-1)!}{(2\pi i)^k}\alpha_0(M,k,\underline{a}) \\ &\qquad = \sum_{\underline{a} \bmod M} e\left(\frac{a_2y_1 - a_1y_2}{M}\right) \frac{(k-1)!}{(2\pi i)^k}\delta\left(\frac{a_1}{M}\right) \sum_{\substack{m_2 \equiv a_2\, (M) \\ m_2 \neq 0}} m_2^{-k} \\ &\qquad = \sum_{a_2 \bmod M} e\left(\frac{a_2y_1}{M}\right) \frac{(k-1)!}{(2\pi i)^k} \sum_{\substack{m_2 \equiv a_2\, (M) \\ m_2 \neq 0}} m_2^{-k} \\ &\qquad = -\frac{1}{k}\mathbf{B}_k(x_1). \end{aligned}$$

We have

$$\begin{aligned} &M^{-k} \sum_{\underline{a} \bmod M} e\left(\frac{a_2y_1 - a_1y_2}{M}\right) A_{k,\underline{a}} \\ &\quad = M^{-k} \sum_{\underline{a} \bmod M} e\left(\frac{a_2y_1 - a_1y_2}{M}\right) \sum_{\substack{m_1 \equiv a_1\, (M) \\ m_1 \in \mathbb{Z}^+}} \sum_{t=1}^{\infty} t^{k-1} e\left(\frac{a_2t + ztm_1}{M}\right) \end{aligned}$$

$$= M^{-k}\sum_{t=1}^{\infty} t^{k-1}\Bigg[\Bigg(\sum_{a_1 \bmod M}\ \sum_{\substack{m_1\equiv a_1\,(M)\\ m_1\in\mathbb{Z}^+}} e\Big(\frac{ztm_1 - m_1y_2}{M}\Big)\Bigg)$$

$$\times\Bigg(\sum_{a_2 \bmod M} e\Big(\frac{a_2(y_1+t)}{M}\Big)\Bigg)\Bigg]$$

$$= M^{1-k}\sum_{\substack{t\equiv -y_1\,(M)\\ t\in\mathbb{Z}^+}} t^{k-1}\Bigg(\sum_{a_1 \bmod M}\ \sum_{\substack{m_1\equiv a_1\,(M)\\ m_1\in\mathbb{Z}^+}} e\Big(\frac{m_1(zt-y_2)}{M}\Big)\Bigg)$$

$$= \sum_{\substack{t\equiv -x_1\,(1)\\ t\in\mathbb{Q}^+}} t^{k-1}\sum_{m=1}^{\infty} e(m(zt-x_2)).$$

Now letting

$$P_{k,\underline{x}}(z) = \sum_{\substack{t\equiv x_1\,(1)\\ t\in\mathbb{Q}^+}} t^{k-1}\sum_{m=1}^{\infty} e(m(zt+x_2)), \tag{2.4}$$

we obtain

$$\phi_k\underline{x} = \delta(\underline{x}) - \frac{1}{k}\mathbf{B}_k(x_1) + P_{k,\underline{x}} + (-1)^k P_{k,-\underline{x}}. \tag{2.5}$$

The definition of $\phi_k\underline{x}$ is therefore independent of M, and if $k>2$, or $\underline{x}\in V\backslash\{0\}$, then $\phi_k\underline{x}$ is an element of $\mathcal{E}_k(\mathit{\Gamma}(M))$.

For $\alpha\in\mathbf{S}$, let $\alpha^* = (\det\alpha)\alpha^{-1}$. The $\phi_k\underline{x}$ satisfy a distribution law given below. This has been shown in the weight two case by Stevens, and the proof for general k is essentially the same.

Lemma 2.6 (Stevens) (cf. [18], Proposition 2.4.2(b)). *Let $\alpha\in\mathbf{S}$ and $\underline{x}\in V$. For $k\geq 2$ we have*

(a) $\phi_k\underline{x} = \sum_{\substack{\underline{y}\in V\\ \underline{y}\alpha=\underline{x}}} \phi_k\underline{y}|\alpha.$

(b) $(\det\alpha)^{2-k}\phi_k\underline{x}|\alpha = \sum_{\substack{\underline{y}\in V\\ \underline{y}\alpha^*=\underline{x}}} \phi_k\underline{y}.$

(c) $\phi_k(x_1,x_2) = n^{k-2}\sum_{a=0}^{n-1}\sum_{b=0}^{n-1}\phi_k\Big(\frac{x_1+a}{n},\frac{x_2+b}{n}\Big)$ *for $n\in\mathbb{Z}^+$.*

Proof. For $\alpha\in\mathrm{GL}_2^+(\mathbb{Q})$ we may write $\alpha=\gamma\tau\gamma'$ with $\gamma,\gamma'\in\mathrm{SL}_2(\mathbb{Z})$, and $\tau\in T=\left\{\left(\begin{smallmatrix} * & 0\\ 0 & *\end{smallmatrix}\right)\right\}$. Now $\alpha\in\mathbf{S}$, so that τ has integral coefficients. Since (a) holds for $\alpha\in\mathrm{SL}_2(\mathbb{Z})$, we may assume $\alpha\in T$. First note that by (2.5) $\phi_k\underline{x} = (-1)^k\phi_k{-\underline{x}}$; thus (a) holds for $\alpha=-\mathbf{I}$. Now it suffices to show that

(a) holds if $\alpha = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)$ for $n \in \mathbb{Z}^+$. Both cases follow from (2.4), (2.5), and a simple calculation. The identity (b) follows from (a) and the fact that $\phi_k\underline{x}\big|\left(\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}\right) = n^{k-2}\phi_k\underline{x}$. (c) then follows by using (b) with $\alpha = \left(\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}\right)$.

The function $\phi_2\underline{0}$ is not holomorphic, but is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. Accordingly if Γ is a congruence group, for $k \geq 2$ we set

$$\mathcal{E}_k^*(\Gamma) = \mathcal{E}_k(\Gamma) + \mathbb{C} \cdot \phi_k\underline{0}, \qquad \mathcal{E}_k^* = \mathcal{E}_k + \mathbb{C} \cdot \phi_k\underline{0}.$$

Let M be a fixed positive integer and let Γ be a congruence group containing $\Gamma(M)$. We set $\mathbf{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, and $\overline{\mathbf{H}} = \mathbf{H} \cup \mathbf{P}^1(\mathbb{Q})$. The modular curve $X(\Gamma) = \Gamma\backslash\overline{\mathbf{H}}$ is a compact Riemann surface which has a canonical model over $\mathbb{Q}$ as described by Shimura [16, Chapter 6]. $X(\Gamma)$ is the union of the affine curve $Y(\Gamma) = \Gamma\backslash\mathbf{H}$ and the cusps of Γ. The latter corresponds to the finite set of orbit classes of $\mathbf{P}^1(\mathbb{Q})$ under the action of Γ by fractional linear transformations. We denote this set by $\mathrm{cusps}(\Gamma)$. For each $x \in \mathrm{cusps}(\Gamma)$ we let $\mathcal{O}_x$ denote the set of $y \in \mathbf{P}^1(\mathbb{Q})$ in the orbit class of x. For $y \in \mathcal{O}_x$, we define the sets S_y and S_x by

$$S_y = \{\gamma_y \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma_y \cdot i\infty = y\}, \qquad S_x = \bigcup_{y \in \mathcal{O}_x} S_y;$$

then S_x is the set of elements of $\mathrm{SL}_2(\mathbb{Z})$ which send $i\infty$ to the Γ-orbit of x. For $x \in \mathrm{cusps}(\Gamma)$ let $e_\Gamma(x)$ denote the ramification index of x over the modular curve $X(\mathrm{SL}_2(\mathbb{Z}))$, and for $y \in \mathcal{O}_x$, write $e_\Gamma(y) = e_\Gamma(x)$. Let $\mathcal{D}_\Gamma$ be the group of divisors with support in $\mathrm{cusps}(\Gamma)$ and let $\mathcal{D}'_\Gamma \subseteq \mathcal{D}_\Gamma$ be the subgroup of divisors of degree zero. For each $x \in \mathrm{cusps}(\Gamma)$ fix $\gamma_x \in S_x$, and for $M \geq 1$ let $V_M = \left(\frac{1}{M}\mathbb{Z}/\mathbb{Z}\right)^2$.

For $k \geq 2$ the $G_{M,k,\underline{a}}$ and $\phi_k\underline{x}$ for $\underline{x} \in V_M$ span the same vector space over $\mathbb{C}$. For a subfield K of $\mathbb{C}$ let $\phi_k(M;K)$ denote the K-span of the $\phi_k\underline{x}$ for $\underline{x} \in V_M$. Then via the map $E \mapsto \sum_{x \in \mathrm{cusps}(\Gamma)} e_\Gamma(x) a_0(E|\gamma_x) \cdot (x)$ we have isomorphisms

$$\phi_k(M;K) \to \mathcal{D}_{\Gamma(M)} \otimes_{\mathbb{Z}} K, \tag{2.7}$$

when $\Gamma = \Gamma(M)$ and $K = \mathbb{C}$ ([14], Proposition 18; [15], Chapter VII, Theorem 8). Since the maps $\mathbf{B}_k : \mathbb{R} \to \mathbb{R}$ map $\mathbb{Q}$ into $\mathbb{Q}$ by Lemma 2.6(a) the image of $\phi_k(M;\mathbb{Q})$ is contained in $\mathcal{D}_\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ when $\Gamma = \Gamma(M)$. Thus (2.7) is an isomorphism for any subfield K of $\mathbb{C}$. Now let $V_k(K)$ be the K-vector space spanned by the $\phi_k\underline{x}$ for $\underline{x} \in V$, and let $\mathcal{E}_k^*(\Gamma;K)$ denote the K-submodule of $\mathcal{E}_k^*(\Gamma)$ generated by $\phi_k\underline{0}$ and Eisenstein series E such that $a_0(E|\gamma) \in K$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We define $\mathcal{E}_k(\Gamma;K)$ similarly.

Setting $\mathcal{E}_k^*(K) = \bigcup_{M \geq 1} \mathcal{E}_k^*(\Gamma(M);K)$ we then have isomorphisms

$$V_k(K) \to \mathcal{E}_k^*(K), \qquad k \geq 2.$$

We have a natural injection $\mathcal{D}_\Gamma \to \mathcal{D}_{\Gamma(M)}$; thus the maps

$$\mathcal{E}_k^*(\Gamma;K) \to \mathcal{D}_\Gamma \otimes_{\mathbb{Z}} K, \qquad k \geq 2,$$

are also isomorphisms.

3. Weight k Eisenstein series for $\Gamma_1(M)$. We now fix a positive integer M, and set $\Gamma = \Gamma_1(M)$. The multiplicative group $\mathbb{Z}(M)^*$ acts on the space $\mathcal{E}_k^*(\Gamma)$, $k \geq 2$, in two natural ways. First we may identify $\mathbb{Z}(M)^*$ with the Galois group $G_M = \mathrm{Gal}(\mathbb{Q}(e(1/M))/\mathbb{Q})$. Fix $k \geq 2$. By the remarks concluding Section 2 we have an isomorphism

$$\phi_k(M;K) \to \mathcal{D}_{\Gamma(M)} \otimes_{\mathbb{Z}} K,$$

where $K = \mathbb{Q}$ or $\mathbb{C}$. Thus if $\sum_i c_i\phi_k(x_i,y_i) = 0$ for some $c_i \in \mathbb{C}, (x_i,y_i) \in V_M$, we also have a relation $\sum_i q_i\phi_k(x_i,y_i) = 0$ for some $q_i \in \mathbb{Q}$, whence $\sum_i q_i\phi_k(x_i,jy_i) = 0$ for any $j \in \mathbb{Z}(M)^*$ as well. Thus since $\phi_k(x,jy)$ for $(x,y) \in V_M$ is clearly invariant under the action of $\Gamma(M)$, we may define the action of G_M on $\mathcal{E}_k^*(\Gamma(M))$ by

$$\Big(\sum_i c_i\phi_k(x_i,y_i)\Big)\Big|\tau_j = \sum_i c_i\phi_k(x_i,jy_i) \tag{3.1}$$

for $c_i \in \mathbb{C}$ and $(x_i,y_i) \in V_M$. Suppose now that $E = \sum_i c_i\phi_k(x_i,y_i) \in \mathcal{E}_k^*(\Gamma)$. Let $j \in \mathbb{Z}(M)^*$, and choose $j' \in \mathbb{Z}$ such that $jj' \equiv 1 \pmod M$. By a calculation it follows easily that

$$(E|\tau_j)\left|\begin{pmatrix}1&1\\0&1\end{pmatrix}\right. = E\left|\begin{pmatrix}1&j'\\0&1\end{pmatrix}\right|\tau_j = E|\tau_j.$$

Thus since Γ is generated by $\Gamma(M)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, we obtain an action of G_M on $\mathcal{E}_k^*(\Gamma)$ as well.

We also let $\mathbb{Z}(M)^*$ act on $\mathcal{E}_k^*(\Gamma)$ via the Nebentype. Thus for $j \in \mathbb{Z}(M)^*$ we choose $\gamma_j = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Gamma_0(M)$ such that $d \equiv j \pmod M$, and define the Nebentype automorphism by

$$E|\langle j\rangle = E|\gamma_j, \qquad E \in \mathcal{E}_k^*(\Gamma). \tag{3.2}$$

We denote this automorphism group by N_M.

For $g \in \mathbf{S}$ we define the Hecke action in the usual way. Writing $\Gamma g\Gamma = \bigcup_i \Gamma g_i$ as a disjoint union, the action of $T(g)$ on $\mathcal{E}_k^*(\Gamma)$ is given by $E|T(g) = \sum_i E|g_i$. In the special case $g = \left(\begin{smallmatrix}1&0\\0&m\end{smallmatrix}\right)$, $m \in \mathbb{Z}^+$, we write $T_m = T(g)$.

Lemma 3.3 (Stevens). (a) *Let $E \in \mathcal{E}_k^*(\Gamma)$. Then for any prime $l \nmid M$,*

$$E|T_l = E|(\tau_l + l^{k-1}\langle l\rangle\tau_l^{-1}).$$

(b) *Suppose that c and d are positive integers such that $c\,|\,d$ and $(d,p)=1$.*

Then for any integer s prime to p and $n \geq 1$,

$$\sum_{i=0}^{p-1} \phi_k\left(\frac{s}{p^n c}, y\right)\Big| \begin{pmatrix} 1 & p^n di \\ 0 & p \end{pmatrix} = \phi_k\left(\frac{s}{p^n c}, py\right), \qquad y \in \mathbb{Q}.$$

Proof. The proofs of (a) and (b) for $k = 2$ are given in Proposition 2.4.7 of [18]. The proof of (a) for $k > 2$ follows similarly from Lemma 2.6; as this is not central to our results we omit the details. For the second claim we have

$$\sum_{i=0}^{p-1} \phi_k\left(\frac{s}{p^n c}, y\right)\Big| \begin{pmatrix} 1 & p^n di \\ 0 & p \end{pmatrix} = \sum_{i=0}^{p-1} \phi_k\left(\frac{s}{p^n c}, y\right)\Big| \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & p^n di \\ 0 & 1 \end{pmatrix}.$$

Using Lemma 2.6 this becomes

$$\begin{aligned} p^{k-2}\sum_{a=0}^{p-1}\sum_{i=0}^{p-1} \phi_k\left(\frac{s}{p^{n+1}c} + \frac{a}{p}, y\right)\Big| \begin{pmatrix} 1 & p^n di \\ 0 & 1 \end{pmatrix} \\ = p^{k-2}\sum_{a=0}^{p-1}\sum_{i=0}^{p-1} \phi_k\left(\frac{s}{p^{n+1}c} + \frac{a}{p}, \frac{sid/c}{p} + y\right) \\ = \phi_k\left(\frac{s}{p^n c}, py\right). \end{aligned}$$

To describe the cusps of Γ we will use Shimura's method. Given $x \in \mathbf{P}^1(\mathbb{Q})$ we may write $x = r/s$, where r and s are integers which are always assumed to be relatively prime. We represent the Γ-orbit of x by $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_\Gamma$ where a and b are integers such that $(a, b) \equiv (r, s) \pmod{M}$, and where the triple (r, s, M) is relatively prime. The following relations characterize cusps(Γ).

(i) $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma = \begin{bmatrix} c \\ d \end{bmatrix}_\Gamma$ if $(a, b) \equiv (c, d) \pmod{M}$;

(ii) $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma = \begin{bmatrix} -a \\ -b \end{bmatrix}_\Gamma$;

(iii) $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma = \begin{bmatrix} c \\ b \end{bmatrix}_\Gamma$ if $a \equiv c \pmod{b}$.

The integer $d = \gcd(b, M)$ is independent of the cusp $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_\Gamma$; following Stevens we refer to d as the "divisor" of $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_\Gamma$, denoted $\operatorname{div}\left(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_\Gamma\right)$. If our choice of Γ is clear we will omit the subscript Γ. Via the automorphism groups G_M and N_M the group

$$A_M^* = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}_M \,\Big|\, r, s \in \mathbb{Z}(M)^* \right\} \subseteq \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z}) \tag{3.4}$$

acts on $\mathcal{E}_k^*(\Gamma)$. A_M^* also acts on cusps(Γ) as defined in [19] by

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}_M \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ra \\ sb \end{bmatrix}.$$

The Nebentype and Galois groups also act on cusps(Γ). We define the action of N_M via the correspondence

$$\langle j \rangle \to \begin{pmatrix} j & 0 \\ 0 & j^{-1} \end{pmatrix}_M, \qquad j \in \mathbb{Z}(M)^*.$$

In Shimura's model for $X(\Gamma)$ over $\mathbb{Q}$ the group $\mathrm{Gal}(\mathbb{Q}(e(1/M))/\mathbb{Q})$ acts on cusps(Γ). The correspondence for G_M is given by

$$\tau_j \to \begin{pmatrix} 1 & 0 \\ 0 & j^{-1} \end{pmatrix}_M$$

([18], Theorem 1.3.1).

We will now assume as in the applications that $M > 4$. For a divisor d of M let $\mathcal{D}_{\Gamma,d}$ denote the group of divisors supported on cusps of Γ of divisor d. Then A_M^* preserves $\mathcal{D}_{\Gamma,d}$. The rank of $\mathcal{D}_{\Gamma,d}$ is given by

$$\operatorname{rank}_{\mathbb{Z}} \mathcal{D}_{\Gamma,d} = \frac{1}{2}\phi(d)\phi(M/d). \tag{3.5}$$

Let η be a primitive Dirichlet character of conductor f. The Gauss sum of η is given by

$$\tau(\eta) = \sum_{a=0}^{f-1} \eta(a)e(a/f).$$

We let $\widehat{\eta} : \mathbb{Z} \to \overline{\mathbb{Q}}$ denote the Fourier transform of η, i.e.,

$$\widehat{\eta}(n) = \sum_{a=0}^{f-1} \eta(a)e(an/f).$$

Since η is primitive $\widehat{\eta}(n) = \tau(\eta)\overline{\eta}(n)$. Any $E \in \mathcal{E}_k^*(\Gamma)$ has a Fourier expansion of the form

$$E(z) = \frac{a_{-1}}{z - \overline{z}} + \sum_{n=0}^{\infty} a_n(E)e(nz),$$

and we set $\widetilde{E(z)} = \sum_{n\geq 1} a_n(E)e(nz)$. The L-series of E is given by

$$L(E,s) = \sum_{n\geq 1} a_n(E)n^{-s}, \qquad \mathrm{Re}(s) > k,$$

and extends to a meromorphic function on $\mathbb{C}$ which has possible simple poles at $s = 1$ and $s = k$. We let $L(E,s)$ denote the analytic continuation as well.

Definition 3.6. Let $k \geq 2$ and suppose η and χ are Dirichlet characters with η primitive such that (i) $\eta\chi(-1) = (-1)^k$, and (ii) $M_1M_2 \mid M$, where

$\mathrm{cond}(\eta) = M_1$ and $m(\chi) = M_2$. For a divisor d of M such that $M_1 \mid (M/d)$ and $M_2 \mid d$, we define an *Eisenstein series* $E_k(\eta, \chi, d)$, as suggested in [19] for the case $k = 2$, by setting

$$E_k(\eta, \chi) = \sum_{r=0}^{M_1-1} \sum_{s=0}^{M_2-1} \overline{\eta}(r)\chi(s)\phi_k\left(\frac{s}{M_2}, \frac{r}{M_1}\right)$$

and

$$E_k(\eta, \chi, d) = E_k(\eta, \chi) \,\Big|\, \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}.$$

By inspection, given (η, χ, d) as above, the groups G_M and N_M commute with the action of $\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $E_k(\eta, \chi)$ and act on $E_k(\eta, \chi, d)$ by η and $\eta\chi$ respectively. It is easily verified that $E_k(\eta, \chi, d) \in \mathcal{E}_k^*(\Gamma)$, and by Lemma 3.3(a) $E_k(\eta, \chi, d)$ is an eigenform for T_l such that $l \nmid M$. In [23] Weisinger proves that the set of $E_k(\eta, \chi, d)$ with χ primitive is a basis for $\mathcal{E}_k(\Gamma)$ if $k > 2$. We will give a modified proof of this.

PROPOSITION 3.7. *Let $k \geq 2$, and let η, χ and d be as in Definition* 3.6. *Let $E = E_k(\eta, \chi, d)$. Then*

(a) *We have*

$$\widetilde{E(z)} = 2\left(\frac{d}{M_2}\right)^{k-1} \sum_{n \geq 1} a_n e(nzd/M_2),$$

where $a_n = \sum_{m \mid n} \widehat{\overline{\eta}}(m)\chi(n/m)(n/m)^{k-1}$.

(b) *Suppose $d = M_2$. Then $L(E, s) = 2\tau(\overline{\eta})L(s, \eta)L(s - k + 1, \chi)$.*

(c) *The set of $E_k(\eta, \chi, d)$ as above, where each χ is primitive, is a basis for $\mathcal{E}_k^*(\Gamma)$.*

Proof. Using (2.4) and (2.5), and the fact that $\eta\chi(-1) = (-1)^k$, we have

$$\begin{aligned}
\widetilde{E(z)} &= 2 \sum_{r=0}^{M_1-1} \sum_{s=0}^{M_2-1} \overline{\eta}(r)\chi(s)P_{k,(s/M_2, r/M_1)} \,\Big|\, \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \\
&= 2 \sum_{m \geq 1} \Big(\sum_{r=0}^{M_1-1} \overline{\eta}(r)e(mr/M_1)\Big) \\
&\qquad \times \Big(\sum_{s=0}^{M_2-1} \chi(s) \sum_{\substack{t \equiv s/M_2\,(1) \\ t \in \mathbb{Q}^+}} t^{k-1}e(mzt)\Big) \,\Big|\, \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

$$= 2\left(\frac{d}{M_2}\right)^{k-1} \sum_{m\geq 1} \widehat{\overline{\eta}}(m) \sum_{s=0}^{M_2-1} \chi(s) \sum_{\substack{t\equiv s\ (M_2)\\ t\in\mathbb{Z}^+}} t^{k-1} e(mztd/M_2)$$

$$= 2\left(\frac{d}{M_2}\right)^{k-1} \sum_{m\geq 1} \widehat{\overline{\eta}}(m) \sum_{t\geq 1} \chi(t) t^{k-1} e(mztd/M_2).$$

Letting $n = mt$ proves (a). Now suppose $d = M_2$. Then

$$L(E,s) = 2L(s,\widehat{\overline{\eta}})L(s-k+1,\chi) = 2\tau(\overline{\eta})L(s,\eta)L(s-k+1,\chi).$$

To prove (c) note that for a fixed $d \mid M$ the number of triples (η,χ,d) is equal to the number of cusps of Γ of divisor d ((3.5)). Thus it suffices to show the $E_k(\eta,\chi,d)$ are linearly independent. Suppose that

$$\sum_{i=1}^{g} c_i E_k(\eta_i,\chi_i,d_i) = 0, \qquad c_i \in \mathbb{C},$$

where g is minimal. By considering the action of G_M and N_M it follows easily that the η_i are all the same, and similarly for the χ_i. Let $d = \min_i\{d_i\}$. By (a) the coefficient of $e(zd/M_2)$ in the Fourier expansion of any $E_k(\eta_i,\chi_i,d_i)$ is nonzero only if $d = d_i$. Since the d_i are distinct, this is a contradiction.

We now focus our attention on the weight two case. The remainder of this section is based on [19].

For each $y \in \mathbf{P}^1(\mathbb{Q})$ let Γ_y be the stabilizer group of y, i.e., $\Gamma_y = \{\gamma \in \Gamma \mid \gamma y = y\}$. Since $M > 4$, Γ_y is cyclic. For a choice of generator π_y, choose $\gamma_y \in S_y$ and set $\pi_y = \gamma_y \left(\begin{smallmatrix}1 & e\\ 0 & 1\end{smallmatrix}\right)\gamma_y^{-1}$, where e is the smallest positive integer such that $\pi_y \in \Gamma$. Then $e = e_\Gamma(x)$ where x is the cusp corresponding to y. If $y = r/s$, we have

$$\pi_y = \begin{pmatrix} 1-ers & er^2\\ -es^2 & 1+ers\end{pmatrix};$$

thus if $\operatorname{div}(x) = d = (s,M)$, it follows that $e_\Gamma(x) = M/d$.

Let $E \in \mathcal{E}_2(\Gamma)$. Then $E(z)dz$ is a Γ-invariant differential form on $\mathbf{H}$; thus there is a 1-form on $X(\Gamma)$, $\omega_\Gamma(E)$, whose pullback to $\mathbf{H}$ is $E(z)dz$. We set

$$r_{\Gamma,E}(x) = 2\pi i \operatorname{Res}_x \omega_\Gamma(E), \qquad x \in \operatorname{cusps}(\Gamma), \tag{3.8}$$

and define the residual divisor of E in $\mathcal{D}_\Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ by

$$\delta_\Gamma(E) = \sum_{x\in \operatorname{cusps}(\Gamma)} r_{\Gamma,E}(x)\cdot(x). \tag{3.9}$$

For $x \in \operatorname{cusps}(\Gamma)$ the value of $r_{\Gamma,E}(x)$ is given by

$$r_{\Gamma,E}(x) = e_\Gamma(x) a_0(E|\gamma_x) \tag{3.10}$$

for any $\gamma_x \in S_x$ ([19], Theorem 1.3(a)). Indeed if $y \in \mathcal{O}_x$, $\gamma_y \in S_y$ then for any $z_0 \in \mathbf{H}$,

$$r_{\Gamma,E}(x) = \int_{z_0}^{\pi_y z_0} E(z)\,dz = \int_{z_0}^{\gamma_y\left(\begin{smallmatrix}1 & e\\ 0 & 1\end{smallmatrix}\right)\gamma_y^{-1}z_0} E(z)\,dz,$$

where $e = e_\Gamma(x)$. Now since $E|\gamma_y\left(\begin{smallmatrix}1 & e\\ 0 & 1\end{smallmatrix}\right) = E|\gamma_y$, $E|\gamma_y$ has a Fourier expansion of the form

$$(E|\gamma_y)(z) = \sum_{n=0}^{\infty} a_n(E|\gamma_y)e(nz/e).$$

Replacing z_0 by $\gamma_y z_0$ we obtain

$$\int_{z_0}^{\left(\begin{smallmatrix}1 & e\\ 0 & 1\end{smallmatrix}\right)z_0} (E|\gamma_y)(z)\,dz = e_\Gamma(x)a_0(E|\gamma_y).$$

We may extend the map δ_Γ to $\mathcal{E}_2^*(\Gamma)$ by defining

$$\delta_\Gamma(\phi_2\underline{0}) = -\frac{1}{12}\sum_x e_\Gamma(x)\cdot(x).$$

For $j \in \mathbb{Z}(M)^*$ and $E \in \mathcal{E}_2^*(\Gamma)$ it now follows easily from the definitions that

$$\text{(3.11)}\qquad \begin{aligned} &\text{(i)}\quad \delta_\Gamma(E|\langle j\rangle) = \langle j\rangle\cdot\delta_\Gamma(E);\\ &\text{(ii)}\quad \delta_\Gamma(E|\tau_j) = \tau_j\cdot\delta_\Gamma(E)\end{aligned}$$

([18], Proposition 3.2.1). Now choose some $E = E_k(\eta,\chi,d)$ as in Definition 3.6, χ primitive. Let $\delta_{\Gamma,d}(E)$ be the projection of $\delta_\Gamma(E)$ onto $\mathcal{D}_{\Gamma,d}\otimes\mathbb{C}$. By the remarks following Definition 3.6, and the above, it follows that

$$\delta_{\Gamma,d}(E) = c(d)\sum \eta(b)\overline{\chi}(a)\begin{bmatrix} a\\ bd\end{bmatrix},$$

for some $c(d) \in \mathbb{C}$, and where the sum is over cusps of divisor d. In general, however, $\delta_\Gamma(E) \neq \delta_{\Gamma,d}(E)$. In [19] Stevens introduces for each $l\,|\,M$, and associated with the pair (η,χ), an Euler factor e_l in the group ring $\overline{\mathbb{Q}}[\mathbf{S}]$ with the property that $\delta_{\Gamma,d}(E') = \delta_\Gamma(E)$ where $E' = E|(\prod_l e_l)$. This leads to a canonical basis for $\mathcal{E}_2^*(\Gamma)$.

4. Ordinary Eisenstein series. For $n \geq 1$ set $\Gamma_n = \Gamma_1(p^nN)$. Let $J_{n/\mathbb{Q}}$ be the Jacobian of the modular curve $X(\Gamma_n)$ with Shimura's canonical model [16] associated with the adelic group

$$\left\{g \in \mathrm{GL}_2(\mathbf{A}_f)\;\middle|\; g \equiv \begin{pmatrix}1 & *\\ 0 & *\end{pmatrix} \pmod{p^nN}\right\}.$$

Let $\mathrm{Ta}_p(J_n)$ be the p-adic Tate module of $J_{n/\mathbb{Q}}$ for $n \geq 1$. Then we may form the projective limit

$$\mathrm{Ta}_p(J_\infty) = \varprojlim \mathrm{Ta}_p(J_{n/\mathbb{Q}})$$

on which we have an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and an action of a Hecke algebra [2].

Let A be a profinite abelian group, and $T_p : A \to A$ a continuous homomorphism. Under suitable conditions Hida has shown that, taking the limit as $n \to \infty$, $\lim \Phi | T_p^{n!}$ exists for all $\Phi \in A$ and so the operator $e = \lim T_p^{n!}$ defines an idempotent on A. The submodule eA is referred to as the *ordinary part* of A and denoted by A^0; we then have a direct sum

$$A = A^{\mathrm{nil}} \oplus A^0.$$

In particular this applies to the Tate module $\mathrm{Ta}_p(J_\infty)$. The factor $\mathrm{Ta}_p(J_\infty)^0$ has been analyzed by Hida and related to ordinary p-adic modular forms and their Hecke algebras [4, 5]. This work in part relies on an analysis of ordinary Eisenstein series.

Recall that for $k \geq 2$, $n \geq 1$, the $\overline{\mathbb{Q}}$-module $\mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}})$ is the set of all $E \in \mathcal{E}_k(\Gamma_n)$ such that $e_n(x) a_0(E|\gamma_x) \in \overline{\mathbb{Q}}$, $x \in \mathrm{cusps}(\Gamma_n)$, $\gamma_x \in \mathcal{O}_x$, where we set $e_n(x) = e_{\Gamma_n}(x)$. Equivalently $\mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}})$ is the set of $E \in \mathcal{E}_k(\Gamma_n)$ with Fourier coefficients in $\overline{\mathbb{Q}}$. We now set

$$\mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p) = \mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_p.$$

As shown in [5] the idempotent e acts on this space of p-adic Eisenstein series.

Definition 4.1. For $n \geq 1$ the subset of ordinary cusps of Γ_n, denoted $\mathrm{cusps}(\Gamma_n)^0$, is the set of $x \in \mathrm{cusps}(\Gamma_n)$ such that $p^n \mid \mathrm{div}(x)$, where $\mathrm{div}(x)$ is as defined before (3.4). We also let $\mathrm{cusps}(\Gamma_n)^{\mathrm{nil}} \subseteq \mathrm{cusps}(\Gamma_n)$ be the complement of the set of ordinary cusps.

Since $p \geq 5$, by (3.5) the order of $\mathrm{cusps}(\Gamma_n)^0$ is given by

$$\#\{\mathrm{cusps}(\Gamma_n)^0\} = \frac{1}{2}\phi(p^n) \sum_{d|N} \phi(d)\phi(N/d). \tag{4.2}$$

For $x \in \mathrm{cusps}(\Gamma_n)$ let x_N be the image of x under the natural projection $\mathrm{cusps}(\Gamma_n) \to \mathrm{cusps}(\Gamma_1(N))$. A cusp $x \in \mathrm{cusps}(\Gamma_n)$ is said to be *unramified* over $\Gamma_1(N)$ if $e_n(x) = e_{\Gamma_1(N)}(x_N)$. As a special case of [5], Lemma 5.1, it follows that the ordinary cusps of Γ_n are those that are unramified over $\Gamma_1(N)$. Indeed this is clear from Section 3. As a consequence, for any $n \geq 1$ there are p elements of $\mathrm{cusps}(\Gamma_{n+1})^0$ which lie over a given $x \in \mathrm{cusps}(\Gamma_n)^0$. We will show that the elements of $\mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p)^0$ may be characterized in terms of $\mathrm{cusps}(\Gamma_n)^0$. Before doing this we need a simple lemma.

Lemma 4.3. *Let $k \geq 2$ be an integer and suppose $E \in \mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p)$ for some $n \geq 1$. Let $v \in \mathbb{Z}^+$ and for $0 \leq i \leq p^v - 1$ set $\tau_i = \left(\begin{smallmatrix} 1 & Ni \\ 0 & p^v \end{smallmatrix}\right)$. Given $y = r/s \in \mathbf{P}^1(\mathbb{Q})$ let $g_i = \gcd(r + sNi, p^v)$ for each i. Then for $\gamma_y \in S_y$ there exists $\gamma_{\tau_i y} \in S_{\tau_i y}$ such that*

$$a_0((E|T_p^v)|\gamma_y) = \sum_{i=0}^{p^v-1} a_0(E|\tau_i\gamma_y) = \sum_{i=0}^{p^v-1} \frac{g_i^k}{p^v} a_0(E|\gamma_{\tau_i y}).$$

If k is even then the result holds for any $\gamma_{\tau_i y} \in S_{\tau_i y}$.

Proof. Fix $v \geq 1$, $n \geq 1$ and $k \geq 2$. The first equality is clear. Choose $\gamma_y = \left(\begin{smallmatrix} r & b \\ s & d \end{smallmatrix}\right) \in S_y$, and set

$$\gamma_{\tau_i y} = \begin{pmatrix} (r + sNi)/g_i & f_i \\ sp^v/g_i & h_i \end{pmatrix} \in S_{\tau_i y} \quad \text{for each } i;\ f_i, h_i \in \mathbb{Z}.$$

Then for any i, by a calculation, we have

$$\tau_i \gamma_y = \gamma_{\tau_i y} \begin{pmatrix} g_i & m_i \\ 0 & p^v/g_i \end{pmatrix} \quad \text{for } m_i \in \mathbb{Z}.$$

Thus

$$a_0(E|\tau_i\gamma_y) = a_0\left((E|\gamma_{\tau_i y}) \,\middle|\, \begin{pmatrix} g_i & m_i \\ 0 & p^v/g_i \end{pmatrix} \right).$$

The result for the sum now follows from the definition of the weight k action. The restriction for odd k is necessary, as in this case $a_0(E|\gamma_x)$ for $x \in \operatorname{cusps}(\Gamma_n)$ depends, up to sign, on a choice of $\gamma_x \in S_x$.

Lemma 4.4. *Let $n \geq 1$, $k \geq 2$, and let $E \in \mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p)$. Then*

(a) *For any $x \in \operatorname{cusps}(\Gamma_n)^0$, $\gamma_x \in S_x$, $a_0(eE|\gamma_x) = a_0(E|\gamma_x)$, where e is Hida's idempotent, i.e., $eE = E|\lim T_p^{n!}$.*

(b) *Suppose in addition $E \in \mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p)^0$ and that $a_0(E|\gamma_x) = 0$ for every $x \in \operatorname{cusps}(\Gamma_n)^0$, $\gamma_x \in S_x$. Then $E = 0$.*

Proof. Let $x \in \operatorname{cusps}(\Gamma_n)^0$, $y = r/s \in \mathcal{O}_x$, where $p^n \mid s$, and let $\gamma_y = \left(\begin{smallmatrix} r & b \\ s & d \end{smallmatrix}\right) \in S_y$. Choose an integer $v \geq n$ satisfying $p^v \equiv 1 \pmod{N}$, and let $\tau_i = \left(\begin{smallmatrix} 1 & Ni \\ 0 & p^v \end{smallmatrix}\right)$ for $0 \leq i \leq p^v - 1$. Fixing i we have $\tau_i\gamma_y = \left(\begin{smallmatrix} r+sNi & f \\ sp^v & h \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & * \\ 0 & p^v \end{smallmatrix}\right)$ for some $f, h \in \mathbb{Z}$, since $(r + sNi, p^v) = 1$. Now since the form of $\left(\begin{smallmatrix} 1 & * \\ 0 & p^v \end{smallmatrix}\right)$ is unchanged by multiplication on the left by $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$, $m \in \mathbb{Z}$, and $h \equiv d \pmod{p^n}$, we may assume that $h \equiv d \pmod{p^n N}$. Then $(r, s, d) \equiv (r + sNi, sp^v, h) \pmod{p^n N}$ so that there exists $\alpha_i \in \Gamma_n$ such that $\left(\begin{smallmatrix} r+sNi & f \\ sp^v & h \end{smallmatrix}\right) = \alpha_i\gamma_y$. Putting this all together, and using Lemma 4.3, we obtain

$$a_0((E|T_p^v)|\gamma_y) = \sum_i a_0\left((E|\alpha_i\gamma_y) \,\middle|\, \begin{pmatrix} 1 & * \\ 0 & p^v \end{pmatrix} \right) = a_0(E|\gamma_y),$$

proving (a).

Now suppose $E\in\mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)^0$ satisfies $a_0(E|\gamma_x)=0$ for all $x\in\mathrm{cusps}(\Gamma_n)^0$, $\gamma_x \in S_x$. First suppose $x \in \mathrm{cusps}(\Gamma_n)$ is such that $p\,|\,\mathrm{div}(x)$. Let $y = r/s \in \mathcal{O}_x$, $\gamma_y \in S_y$. We have

$$a_0((E|T_p^v)|\gamma_y) = \sum_i a_0(E|\tau_i\gamma_y) = \sum_i a_0\biggl((E|\gamma_{\tau_i y})\biggm| \begin{pmatrix} 1 & * \\ 0 & p^v \end{pmatrix}\biggr),$$

for τ_i as above and appropriate $\gamma_{\tau_i y} \in S_{\tau_i y}$. Since the cusps corresponding to the $\tau_i y$ are all ordinary it follows that $a_0((E|T_p^v)|\gamma_y) = 0$, whence $a_0(E|\gamma_y) = 0$. Now let $y = r/s \in \mathbf{P}^1(\mathbb{Q})$ where $(p, s) = 1$. Then by the preceding $a_0((E|T_p^v)|\gamma_y) = a_0(E|\tau_i\gamma_y)$, where $0 \le i \le p^v - 1$ satisfies $p^v\,|\,(r + sNi)$. Choosing $\gamma_{\tau_i y} \in S_{\tau_i y}$ and using Lemma 4.3 with $g_i = p^v$ we obtain $a_0((E|T_p^v)|\gamma_y) = p^{v(k-1)}a_0(E|\gamma_{\tau_i y})$. Since v is arbitrary, this establishes (b).

The next lemma, due to Hida, follows immediately from Lemma 4.4 and (4.2).

Lemma 4.5 (Hida) (cf. [5], Lemma 5.3, Theorem 5.8). *Let $n \ge 1$ and $k \ge 2$. Then*

(a) *The dimension of $\mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)^0$ over $\overline{\mathbb{Q}}_p$ is given by*

$$\frac{1}{2}\phi(p^n)\sum_{d|N}\phi(d)\phi(N/d).$$

(b) *$E \in \mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)$ is an element of $\mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)^{\mathrm{nil}}$ if and only if $a_0(E|\gamma_x) = 0$ for all $x \in \mathrm{cusps}(\Gamma_n)^0$, $\gamma_x \in S_x$.*

Note that, in general, $E \in \mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)^0$ does not satisfy $a_0(E|\gamma_x) = 0$ for all $x \in \mathrm{cusps}(\Gamma_n)^{\mathrm{nil}}$. This is obvious if $k = 2$.

Hida proves Lemma 4.5(a) by constructing a basis for $\mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)^0$. Essentially the method is to consider triples of the form (η, χ, d) satisfying

(i) η and χ are primitive Dirichlet characters such that $\eta\chi(-1) = (-1)^k$;
(ii) if $\mathrm{cond}(\eta) = N_1$ and $m(\chi_p) = N_2p^m$, then $N_1N_2\,|\,N$ and $m \le n$;
(iii) d is a divisor of N such that $N_1\,|\,(N/d)$ and $N_2\,|\,d$.

The set of $E_k(\eta, \chi_p, p^m d)$ is then a basis for $\mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)^0$.

Given $E \in \mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)$ one may verify that $E|T_p\langle j\rangle = E|\langle j\rangle T_p$ for $j \in \mathbb{Z}(p^nN)^*$, so that $N_n = N_{p^nN}$ acts on $\mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)^0$. The same is true for $G_n = G_{p^nN}$ since this group acts only on the Fourier coefficients. Referring to (3.4) let $A^* = \varprojlim A^*_{p^nN}$, and set $\mathcal{E}_k(\overline{\mathbb{Q}}_p)^0 = \bigcup_{n\ge1}\mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)^0$. In general, for a fixed k, we identify A^* with its image in $\mathrm{Aut}(\mathcal{E}_k(\overline{\mathbb{Q}}_p)^0)$, and define $\mathcal{H}$ to be the algebra generated over $\mathbb{Z}_p$ by the images of the T_m, $m \ge 1$, in $\mathrm{Aut}(\mathcal{E}_k(\overline{\mathbb{Q}}_p)^0)$.

We may also define an action of $\mathbb{Z}_{p,N}^*$ on $\mathcal{E}_k(\overline{\mathbb{Q}}_p)^0$ by identifying $\mathbb{Z}_{p,N}^*$ with either $\varprojlim G_n$ or $\varprojlim N_n$. In order to work with $\beta \in \mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$ we must identify $\mathbb{Z}_{p,N}^*$ differently.

DEFINITION 4.6. For $n \geq 1$ and $k \geq 2$ let $[j] \in \mathrm{Aut}(\mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p)^0)$ be defined as the "mixed" operator

$$[j] = \langle j \rangle \cdot \tau_j^{-2}, \qquad j \in \mathbb{Z}(p^n N)^*.$$

Denoting the set of such automorphisms by D_n, we define the action of $\mathbb{Z}_{p,N}^* = \varprojlim \mathbb{Z}(p^n N)^*$ on $\mathcal{E}_k(\overline{\mathbb{Q}}_p)^0$ by the correspondence $D_n \cong \mathbb{Z}(p^n N)^*$.

Suppose now that N is squarefree. Then given $n \geq 1$, $k \geq 2$, the space $\mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p)^0$ has dimension $\frac{1}{2}\tau(N)\phi(p^n N)$ where $\tau(N)$ is the number of divisors of N. Since $\frac{1}{2}\phi(p^n N)$ is also equal to the number of $\xi \in K(Np^n)$ such that $\xi(-1) = (-1)^k$, this suggests the following definition.

DEFINITION 4.7. Suppose that N is squarefree and that $\kappa \in \mathcal{X}_N^{\mathrm{arith},+}$ has signature (ξ, k) and level m. Then E is said to be an *ordinary weight κ Eisenstein series* if

(i) $E \in \mathcal{E}_k(\Gamma_m; \overline{\mathbb{Q}}_p)^0$;
(ii) $\mathbb{Z}_{p,N}^*$ acts on E via ξ.

For $n \geq 1$ and $d \mid N$ let $I_{\kappa,d,n}$ be the unique element of $\mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p)^0$ such that

(i) $\mathbb{Z}_{p,N}^*$ acts on $I_{\kappa,d,n}$ via ξ;
(ii) $e_n\left(\begin{bmatrix} 1 \\ p^n c \end{bmatrix}\right) a_0\left(I_{\kappa,d,n} \,\middle|\, \begin{pmatrix} 1 & 0 \\ p^n c & 1 \end{pmatrix}\right) = \begin{cases} 1 & \text{if } c = d, \\ 0 & \text{if } c \neq d \end{cases}$

for $c \mid N$.

Note that for squarefree N, given $\xi \in K(Np^n)$ and $d \mid N$, there exist unique primitive characters η and χ of conductors dividing N/d and $p^n d$ respectively such that $\overline{\eta}\chi = \xi$. We will next construct a canonical basis of ordinary weight k Eisenstein series for each $k \geq 2$, $n \geq 1$. Our method turns out to be equivalent to the modification of the basis given by Hida by Euler factors similar to those used in the weight two case as described at the conclusion of Section 3. Having "absorbed" these factors we obtain series which are naturally related to the measure β.

PROPOSITION 4.8. *Suppose N is squarefree. Let $\kappa \in \mathcal{X}_N^{\mathrm{arith},+}$, and suppose κ has signature (ξ, k) and level m. Then for $n \geq m$ and $d \mid N$ we define* *an Eisenstein series* $E_{\kappa,d,n}$ *by*

$$E'_{\kappa,d,n} = \left(\frac{N}{d}\right)^{k-2} \sum_{u|(N/d)} \sum_{v|d} \mu(uv) \times \sum_{r\in\mathbb{Z}(N/d)^*} \sum_{s\in\mathbb{Z}(p^n d)^*} \xi(sN/d + rp^n d)\phi_k\left(\frac{sv'}{p^n d/v}, \frac{ru'}{N/du}\right),$$

and

$$E_{\kappa,d,n} = E'_{\kappa,d,n} \,\Bigg|\, \begin{pmatrix} p^n d & 0 \\ 0 & 1 \end{pmatrix},$$

where u' *and* v' *are the respective inverses of* u *and* v *in* $\mathbb{Z}(N/du)^*$ *and* $\mathbb{Z}^*_{p,d/v}$. *Let* $E = E_{\kappa,d,n}$, *set* $\xi = \overline{\eta}\chi$ *for* η *and* χ *as above, and let* $N_1 = \mathrm{cond}(\eta)$. *Then*

(a) E *is an ordinary weight* κ *Eisenstein series.*

(b) G_m *and* N_m *act on* E *respectively by* η *and* $\eta\chi$, *and*

$$E|T_l = (\eta(l) + \chi(l)l^{k-1}) \cdot E \quad \text{for primes } l \nmid pN.$$

(c) $L(E,s) = 2\tau(\overline{\eta})\overline{\eta}(p^n d)\chi(N/d)(N/d)^{k-2}$

$$\times \Bigg(\prod_{l|(N/N_1 d)} (l^{1-s} - \overline{\eta}(l)l)\Bigg)\Bigg(\prod_{l|d}(1-\chi(l)l^{k-s})\Bigg) \times L(s,\eta)L(s-k+1,\chi_p).$$

(d) $E = \beta(\kappa) \cdot I_{\kappa,d,n}$.

(e) *The set of* $E_{\kappa,d,n}$ *for* κ *of weight* k *and level* $\leq n$ *is a basis for* $\mathcal{E}_k(\Gamma_n;\overline{\mathbb{Q}}_p)^0$.

Proof. First suppose $n = m$. Writing

$$\xi(sN/d + rp^m d) = \overline{\eta}(p^m d)\chi(N/d)\overline{\eta}(r)\chi(s)$$

for $r \in \mathbb{Z}(N/d)^*$ and $s \in \mathbb{Z}(p^m d)^*$, it follows easily that G_m and N_m act on E via η and $\eta\chi$. Thus $\mathbb{Z}^*_{p,N}$ acts by ξ; the claim for T_l follows from Lemma 3.3(a). E is easily shown to be invariant under the action of Γ_m. Using Lemma 3.3(b) yields $E|T_p = \eta(p)\cdot E$, thus E is ordinary. This completes the proof of (a) and (b) if $n = m$.

We now assume that $n \geq m$ and calculate the L-series of E. First set

$$E' = \eta(p^n d)\overline{\chi}(N/d)(d/N)^{k-2} \cdot E,$$

so that

$$E' = \sum_{u,v} \mu(uv) \sum_{r\in\mathbb{Z}(N/d)^*} \sum_{s\in\mathbb{Z}(p^n d)^*} \overline{\eta}(r)\chi(s)\phi_k\left(\frac{sv'}{p^n d/v}, \frac{rv'}{N/du}\right) \Bigg| \begin{pmatrix} p^n d & 0 \\ 0 & 1 \end{pmatrix}.$$

We may now use Lemma 1.6. We obtain

$$E' = \sum_{u,v} \mu(uv)\overline{\eta}(u)\chi(v)uv \times \sum_{r=0}^{N/du-1} \sum_{s=0}^{p^n d/v-1} \overline{\eta}(r)\chi_p(s)\phi_k\left(\frac{s}{p^n d/v}, \frac{r}{N/du}\right) \Big| \begin{pmatrix} p^n d & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $\overline{\eta}(u) = 0$ if $u \nmid (N/N_1 d)$. Now for fixed v and $u \,|\, (N/N_1 d)$ we have by Lemma 2.6(b)

$$\left(\frac{N}{N_1 du}\right)^{2-k} \sum_{r=0}^{N_1-1} \sum_{s=0}^{p^n d/v-1} \phi_k\left(\frac{s}{p^n d/v}, \frac{r}{N_1}\right) \Big| \begin{pmatrix} N/N_1 du & 0 \\ 0 & 1 \end{pmatrix} = \sum_{r=0}^{N/du-1} \sum_{s=0}^{p^n d/v-1} \phi_k\left(\frac{s}{p^n d/v}, \frac{r}{N/du}\right).$$

Thus

$$E' = \sum_{u,v} \mu(uv)\overline{\eta}(u)\chi(v)\left(\frac{N}{N_1 du}\right)^{2-k} \sum_{r=0}^{N_1-1} \sum_{s=0}^{p^n d/v-1} \overline{\eta}(r)\chi_p(s) \times \phi_k\left(\frac{s}{p^n d/v}, \frac{r}{N_1}\right) \Big| \begin{pmatrix} p^n d/v & 0 \\ 0 & 1 \end{pmatrix} \Big| \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N/N_1 du & 0 \\ 0 & 1 \end{pmatrix}.$$

Now using Lemma 2.6 and the fact that

$$L\left(F \Big| \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = t^{k-1-s} L(F,s) \quad \text{for } t \in \mathbb{Z}^+ \text{ and } F \in \mathcal{E}_k(\overline{\mathbb{Q}}_p)^0,$$

we have by Proposition 3.7(b),

$$\begin{aligned} L(E',s) &= \sum_{u,v} \mu(uv)\overline{\eta}(u)\chi(v) \\ &\quad \times uv[2\tau(\overline{\eta})L(s,\eta)L(s-k+1,\chi_p)]v^{k-1-s}\left(\frac{N}{N_1 du}\right)^{1-s} \\ &= \Big(\sum_{u|(N/N_1 d)} \mu(u)\overline{\eta}(u)u^s\Big)\left(\frac{N}{N_1 d}\right)^{1-s}\Big(\sum_{v|d} \mu(v)\chi(v)v^{k-s}\Big) \\ &\quad \times 2\tau(\overline{\eta})L(s,\overline{\eta})L(s-k+1,\chi_p), \end{aligned}$$

from which the claim for $L(E,s)$ follows easily. We also obtain $E = \overline{\eta}(p^{n-m})E_{\kappa,d,m}$ which proves (a) and (b).

Now let $E_0 = E|\left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}$. Then by Lemma 2.6(b),

$$E_0 = N^{k-2} \sum_{x=0}^{d-1} \sum_{u,v} \mu(uv) \sum_{r,s} \xi(sN/d + rp^n d)\phi_k\left(\frac{sv'}{p^n d/v}, \frac{ruu' + xN/d}{N}\right).$$

For each $c \mid N$ set $E_c = E_0 \big| \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$. Then

$$E \left| \begin{pmatrix} 1 & 0 \\ p^n c & 1 \end{pmatrix} = E_0 \right| \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^n c & 1 \end{pmatrix} = E_0 \left| \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \right.$$
$$= E_c \left| \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \right.,$$

so that

$$a_0\left(E \left| \begin{pmatrix} 1 & 0 \\ p^n c & 1 \end{pmatrix} \right.\right) = p^{n(k-1)} a_0(E_c).$$

We will show that $a_0(E_c) = 0$ if $c \neq d$.

Using Lemma 2.6 and the fact that $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

$$a_0(E_c) = -\frac{N^{k-2}}{k} \sum_{u,v} \mu(uv) \sum_{r,s} \xi(sN/d + rp^n d)$$
$$\times \sum_{x=0}^{d-1} \mathbf{B}_k\left(\frac{svv'}{p^n d} + \frac{c(ruu' + xN/d)}{N}\right).$$

First suppose l is a prime dividing $(c, N/d)$. Fix r, s and v, and let

$$L_{r,s,v} = \sum_{u \mid (N/d)} \mu(u) \sum_{x=0}^{d-1} \mathbf{B}_k\left(\frac{svv'}{p^n d} + \frac{c(ruu' + xN/d)}{N}\right).$$

Then

$$L_{r,s,v} = \sum_{u \mid (N/dl)} \mu(u) \Bigg[\sum_{x=0}^{d-1} \Bigg[\mathbf{B}_k\left(\frac{svv'}{p^n d} + \frac{c(ruu' + xN/d)}{N}\right)$$
$$- \mathbf{B}_k\left(\frac{svv'}{p^n d} + \frac{c(rul(ul)' + xN/d)}{N}\right)\Bigg]\Bigg].$$

For a fixed u we may write the sum in the outside set of brackets as

$$\sum_{x=0}^{d-1} \Bigg[\mathbf{B}_k\left(\frac{svv'}{p^n d} + \frac{c(ruu' + xN/d)}{N}\right)$$
$$- \mathbf{B}_k\left(\frac{svv'}{p^n d} + \frac{c(rul(ul)' + (x+y)N/d)}{N}\right)\Bigg]$$

for any integer y. Since $uu' \equiv (ul)(ul)' \equiv 1 \pmod{N/dl}$, we have

$$cr(uu' + (ul)(ul)') - cyN/d \equiv 0 \pmod{N/d} \quad \text{as } l \mid c.$$

Now since $(d, N/d) = 1$ we may assume that y is chosen so that the left hand side of the above is $\equiv 0 \pmod N$ as well. Thus $L_{r,s,v} = 0 = a_0(E_c)$.

Now suppose $c\,|\,d$. Fixing r, s and u, we set

$$L_{r,s,u} = \sum_{v|d} \mu(v) \sum_{x=0}^{d-1} \mathbf{B}_k\left(\frac{svv'}{p^n d} + \frac{c(ruu' + xN/d)}{N}\right).$$

Then if l is a prime dividing d/c, a similar argument shows that $a_0(E_c)$ is also zero. Now since

$$e_n\left(\begin{bmatrix} 1 \\ p^n d \end{bmatrix}\right) = \frac{N}{d} \quad \text{and} \quad \begin{pmatrix} p^n d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p^n d & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p^n d & 0 \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned} &e_n\left(\begin{bmatrix} 1 \\ p^n d \end{bmatrix}\right) a_0\left(E \,\middle|\, \begin{pmatrix} 1 & 0 \\ p^n d & 1 \end{pmatrix}\right) \\ &\qquad = -\frac{(p^n N)^{k-1}}{k} \sum_{u,v}\sum_{r,s} \xi(sN/d + rp^n d)\mathbf{B}_k\left(\frac{sv'}{p^n d/v} + \frac{ru'}{N/du}\right). \end{aligned}$$

Fix r, s, u and v, and let $y = (p^n d/v)ru' + (N/du)sv'$. Then

$$uvy \equiv sN/d \pmod{p^n d/v} \quad \text{and} \quad uvy \equiv rp^n d \pmod{N/du};$$

thus letting $uv = D$ we have $Dy \equiv sN/d + rp^n d \pmod{p^n N/D}$, and

$$\begin{aligned} &e_n\left(\begin{bmatrix} 1 \\ p^n d \end{bmatrix}\right) a_0\left(E \,\middle|\, \begin{pmatrix} 1 & 0 \\ p^n d & 1 \end{pmatrix}\right) \\ &\qquad\qquad = -\frac{(p^n N)^{k-1}}{k} \sum_{D|N} \mu(D) \sum_{x\in\mathbb{Z}(p^n N)^r} \mathbf{B}_k\left(\frac{xD'}{p^n N/D}\right)\xi(x), \end{aligned}$$

where $DD' \equiv 1 \pmod{p^n N/D}$. Part (d) now follows from Corollary 1.7 and (1.15). By Lemma 4.4 the dimension of $\mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p)^0$ is equal to the order of $\mathrm{cusps}(\Gamma_n)^0$; since this also equal to the number of κ of weight k and level $\le n$, (e) follows immediately.

Now suppose $\kappa \in \mathcal{X}_N^{\mathrm{arith},+}$ has signature (ξ, k) and level m. Since $E_{\kappa,d,n} = \overline{\eta}(p^{n-m})E_{\kappa,d,m}$ for $n \ge m$, it follows that for any $\kappa \in \mathcal{X}_N^{\mathrm{arith},+}$ the $\overline{\mathbb{Q}}_p$-vector space of ordinary weight κ Eisenstein series has dimension $\tau(N)$ over $\overline{\mathbb{Q}}_p$. Given $E \in \mathcal{E}_k(\overline{\mathbb{Q}}_p)^0$, $m \ge 1$, $E|T_m$ is a $\mathbb{Z}$-linear combination of elements of the form $E|T_l^t$, l prime, $t \ge 0$. (See [16], Theorem 3.34(1).) Combining this with Proposition 4.8(b), it follows at once that the algebra $\mathcal{H}[A^*] \subseteq \mathrm{Aut}(\mathcal{E}_k(\overline{\mathbb{Q}}_p)^0)$ is commutative.

5. Cuspidal divisor groups. Let $M \ge 1$ and Γ be one of the groups $\Gamma(M)$ or $\Gamma_1(M)$. We let $U(\Gamma)$ denote the group of modular units over Γ. Then $g \in U(\Gamma)$ if g is a modular function for Γ without zeros or poles in $\mathbf{H}$. Thus g may be regarded as a function on the modular curve $X(\Gamma)$ with support in $\mathrm{cusps}(\Gamma)$. Letting $\mathcal{D}'_\Gamma \subseteq \mathcal{D}_\Gamma$ be the subgroup of divisors of degree zero, let $\mathrm{div}_\Gamma(g)$ be the divisor of the associated function on $X(\Gamma)$. The cuspidal divisor class group $\mathcal{C}_\Gamma$ is then the finite group

$$\mathcal{C}_\Gamma = \mathcal{D}'_\Gamma / \mathrm{div}_\Gamma(U(\Gamma)).$$

For simplicity now assume $M > 4$. The cuspidal divisor class group has been analyzed by Kubert and Lang in a series of articles, and in [9]. In the case $\Gamma = \Gamma(M)$ the group $\mathcal{C}_\Gamma$ has a complete description for $M = p^n$, and the case for M in general has been determined up to 2-torsion [8, 9]. When $\Gamma = \Gamma_1(M)$ the situation is more complex. Kubert and Lang have analyzed two subgroups of $\mathcal{C}_\Gamma$ for $M = p^n$, both of which depend on units g which are determined by the part of $\mathrm{div}_\Gamma(g)$ whose support lies on cusps of divisor 1 [9]. This has been generalized to all $M > 4$ by Yu [25], using the Kubert–Lang theory and methods of Sinnott [17].

The appearance of ideals similar to the Stickelberger ideals in this setting suggested the possibility that the theory of modular forms could be applied to the analysis of cyclotomic ideal class groups. This approach led to the proof by Mazur and Wiles of the main conjecture of Iwasawa theory [12], in part by applying the theory of Kubert and Lang to Igusa curves of characteristic p.

In general Stevens has formulated a different approach to the analysis of $\mathcal{C}_\Gamma$, which is based on Eisenstein series [19]. We now give a survey of some of these results as needed in the sequel.

For $E \in \mathcal{E}_2(\Gamma)$, let $\Phi(E) : \Gamma \to \mathbb{C}$ be the map given by

$$\Phi(E)(\gamma) = \int_{z_0}^{\gamma z_0} E(z)\, dz, \qquad \gamma \in \Gamma, \tag{5.1}$$

where $z_0 \in \mathbf{H}$. Let $P_\Gamma = \bigcup_{y \in \mathbf{P}^1(\mathbb{Q})} \Gamma_y$ be the set of parabolic elements of Γ. For a $\mathbb{Z}$-submodule $K \subseteq \mathbb{C}$ we define two subgroups of $\mathcal{E}_2(\Gamma)$ by

$$\begin{aligned} \mathcal{E}_\Gamma(K) &= \{E \in \mathcal{E}_2(\Gamma) \mid \Phi(E)(\pi) \in K,\ \forall \pi \in P_\Gamma\}; \\ \mathcal{E}_\Gamma[K] &= \{E \in \mathcal{E}_2(\Gamma) \mid \Phi(E)(\gamma) \in K,\ \forall \gamma \in \Gamma\}. \end{aligned} \tag{5.2}$$

As discussed in Section 3, for a cusp x, and $y \in \mathcal{O}_x$, we have

$$\Phi(E)(\pi_y) = r_{\Gamma,E}(x) = e_\Gamma(x) a_0(E|\gamma_x), \qquad \gamma_x \in \mathcal{O}_x;$$

thus by (3.9), $\mathcal{E}_\Gamma(K)$ is the set of $E \in \mathcal{E}_2(\Gamma)$ such that $\delta_\Gamma(E) \in \mathcal{D}_\Gamma \otimes K$.

If $g \in U(\Gamma)$, then $(2\pi i)^{-1} g'(z)/g(z) = E_g$ can be shown to be an element of $\mathcal{E}_\Gamma[\mathbb{Z}]$ and we have $\delta_\Gamma(E_g) = \mathrm{div}_\Gamma(g)$. In their analysis of $\mathcal{C}_\Gamma$, Kubert and Lang work with the Siegel units $g_{\underline{a}}$ defined for $\underline{a} \in V \backslash \{0\}$ ([9], p. 29); by a calculation one can verify that $(2\pi i)^{-1}(g_{\underline{a}})'/g_{\underline{a}}$ is equal to the Eisenstein series $-\phi_{2,\underline{a}}$. The map

$$(2\pi i)^{-1} d\log/dz : U(\Gamma)/\mathbb{C}^* \to \mathcal{E}_\Gamma[\mathbb{Z}] \tag{5.3}$$

is an isomorphism. Thus we may express C_Γ in terms of Eisenstein series for Γ by

$$\mathcal{C}_\Gamma \simeq \mathcal{E}_\Gamma(\mathbb{Z})/\mathcal{E}_\Gamma[\mathbb{Z}]. \tag{5.4}$$

We now fix $\Gamma = \Gamma_1(M)$. Suppose that $E \in \mathcal{E}_2(\Gamma)$ and that α is a primitive Dirichlet character whose conductor is prime to M. The Dirichlet series

$$\sum_{n\geq 1} a_n(E)\alpha(n)n^{-s}$$

converges absolutely for $\mathrm{Re}(s) > 2$ and extends to a meromorphic function on $\mathbb{C}$ with a possible simple pole at $s = 2$. We denote this analytic continuation by $L(E, \alpha, s)$ and define the special value $\Lambda(E, \alpha, 1)$ by

$$\Lambda(E,\alpha,1) = \frac{\tau(\overline{\alpha})L(E,\alpha,1)}{2\pi i}. \tag{5.5}$$

Let S_M be the set of primes q such that $q \equiv 3 \pmod 4$ and $q \equiv -1 \pmod M$. We define C_M to be the set of nonquadratic characters α such that $\mathrm{cond}(\alpha) = Q = q^r$ for some $q \in S_M$ and $r \geq 1$. We set

$$C_M^{\pm} = \{\Psi \in C_M \mid \Psi(-1) = \pm 1\}$$

for either choice of $\pm$ sign. For $\alpha \in C_M$ let $\alpha_q = (\cdot/q)$ be the quadratic character of conductor q and let

$$\Lambda_\pm(E,\alpha,1) = \frac{1}{2}(\Lambda(E,\alpha,1) \pm \Lambda(E,\alpha\alpha_q,1)), \tag{5.6}$$

for either choice of $\pm$ sign. For such a character α and a $\mathbb{Z}$-module K of $\mathbb{C}$, let $K[\alpha, 1/Q]$ be the ring generated over K by $1/Q$ and the values of α. Combining Theorem 1.3(b) and Lemma 2.2(1) of [19] we have the following theorem.

THEOREM 5.7 (Stevens). *Let $K \subseteq \mathbb{C}$ be a finitely generated $\mathbb{Z}$-module. Suppose that $E \in \mathcal{E}_2(\Gamma) \cap \mathcal{E}_\Gamma(K)$. Then $E \in \mathcal{E}_\Gamma[K]$ if and only if for all $\alpha \in C_M$, $\Lambda_\pm(E, \alpha, 1) \in K[\alpha, 1/Q]$, where* $\mathrm{cond}(\alpha) = Q$.

With appropriate definitions this also holds for $\Gamma(M)$. We will use this theorem to analyze cuspidal divisor class groups associated with the groups Γ_n. We will also need the following theorem to prove Theorem 5.15.

THEOREM 5.8. *Let $\xi \in K(Np^\infty)$ be even and let $q \in S_{pN}$ be a prime such that $q > L^{2f+1}$ where f is the number of distinct prime divisors of pN and L is the largest. Let $\mathbf{P} \nmid 2qN$ be a prime of $\overline{\mathbb{Q}}$. Then*

(a) (Washington [21]) *There exists $r_q \in \mathbb{Z}^+$ such that for all $r \geq r_q$ and $\alpha \in C_{pN}^-$ with* $\mathrm{cond}(\alpha) = q^r$, $\mathbf{B}_1(\overline{\alpha})$ *and* $\mathbf{B}_1(\alpha\xi)$ *are* $\mathbf{P}$*-units.*

(b) (Stevens [19]) *For every algebraic integer ϵ and $l \mid pN$ there are infinitely many $\alpha \in C_{pN}^-$ such that $\mathbf{P} \nmid (1 - \epsilon\alpha(l))$.*

(c) *For infinitely many* $\alpha \in C_{pN}^{-}$,

$$\mathbf{P} \nmid \Big(\prod_{l|N}(1-\alpha\xi(l)l)\Big)\mathbf{B}_1(\overline{\alpha})\mathbf{B}_1(\alpha\xi_p).$$

Proof. To prove (c) we will use the method used to prove (b). Let q and r_q be as above, and fix $r > r_q$. Let χ be a primitive Dirichlet character of conductor q^r such that the primitive character corresponding to χ^ν is nontrivial unless $\phi(q^r) \mid \nu$. As $q \in S_{pN}$, $\Psi = \chi^{(q-1)/2}$ is odd. Moreover as $q > 2f+1$,

$$(2f+1)\frac{q-1}{2} < \phi(q^r),$$

so that the $f+1$ characters Ψ^t are distinct for t odd, $1 \le t \le 2f+1$. Since the q-power roots are distinct mod $\mathbf{P}$, and $q > l^{2f+1}$ for any prime divisor l of pN, there is a choice of t as above for which

$$\mathbf{P} \nmid \Big(\prod_{l|N}(1-\Psi^t\xi(l)l)\Big)(1-\Psi^t\xi(p)).$$

Now set $\alpha = \Psi^t$.

We set $\mathcal{E}_n(\overline{\mathbb{Q}}_p) = \mathcal{E}_2(\Gamma_n;\overline{\mathbb{Q}}_p)$ for $n \ge 1$. For $x \in \operatorname{cusps}(\Gamma_n)$ we extend our definitions of $r_{\Gamma_n,E}(x)$ and $\delta_{\Gamma_n}(E)$ to $E \in \mathcal{E}_n(\overline{\mathbb{Q}}_p)$, denoting these by $r_{n,E}(x)$ and $\delta_n(E)$ respectively. Let $\mathcal{O}$ be a $\mathbb{Z}_p$-submodule of $\overline{\mathbb{Q}}_p$. We define two $\mathcal{O}$-submodules of $\mathcal{E}_n(\overline{\mathbb{Q}}_p)$ by

$$\mathcal{E}_n(\mathcal{O}) = \mathcal{E}_{\Gamma_n}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}, \qquad \mathcal{E}_n[\mathcal{O}] = \mathcal{E}_{\Gamma_n}[\mathbb{Z}] \otimes_{\mathbb{Z}} \mathcal{O},$$

and set

$$\mathcal{E}_n(\mathcal{O})^0 = \mathcal{E}_n(\mathcal{O}) \cap \mathcal{E}_n(\overline{\mathbb{Q}}_p)^0, \qquad \mathcal{E}_n[\mathcal{O}]^0 = \mathcal{E}_n[\mathcal{O}] \cap \mathcal{E}_n(\overline{\mathbb{Q}}_p)^0,$$

with analogous definitions for $\mathcal{E}_n(\mathcal{O})^{\mathrm{nil}}$ and $\mathcal{E}_n[\mathcal{O}]^{\mathrm{nil}}$.

For $n \ge 1$ we define a cuspidal divisor class group $\mathcal{C}_n(\mathcal{O})$ by

$$\mathcal{C}_n(\mathcal{O}) = \mathcal{E}_n(\mathcal{O})/\mathcal{E}_n[\mathcal{O}],$$

and set

$$\mathcal{C}_n(\mathcal{O})^0 = \mathcal{E}_n(\mathcal{O})^0/\mathcal{E}_n[\mathcal{O}]^0, \qquad \mathcal{C}_n(\mathcal{O})^{\mathrm{nil}} = \mathcal{E}_n(\mathcal{O})^{\mathrm{nil}}/\mathcal{E}_n[\mathcal{O}]^{\mathrm{nil}}.$$

Let $\mathcal{C}_n$ denote the p-primary part of the cuspidal divisor class group associated with Γ_n. Then by (5.4) we have

$$\mathcal{C}_n = \mathcal{C}_n(\mathbb{Z}_p) = \mathcal{E}_n(\mathbb{Z}_p)/\mathcal{E}_n[\mathbb{Z}_p]. \tag{5.9}$$

For simplicity we will write $\mathcal{C}_n^0 = \mathcal{C}_n(\mathbb{Z}_p)^0$ and $\mathcal{C}_n^{\mathrm{nil}} = \mathcal{C}_n(\mathbb{Z}_p)^{\mathrm{nil}}$. As part of the next proposition we show that $\mathcal{H}[A^*]$ preserves each $\mathcal{E}_n[\mathcal{O}]$, so that $\mathcal{C}_n(\mathcal{O}), \mathcal{C}_n(\mathcal{O})^{\mathrm{nil}}$, and $\mathcal{C}_n(\mathcal{O})^0$ are all naturally $\mathcal{H}[A^*]$-modules.

PROPOSITION 5.10. *For any $n \geq 1$, and any $\mathbb{Z}_p$-submodule $\mathcal{O}$ of $\overline{\mathbb{Q}}_p$, we have an isomorphism of $\mathcal{H}[A^*]$-modules*

$$\mathcal{C}_n(\mathcal{O}) \cong \mathcal{C}_n(\mathcal{O})^{\mathrm{nil}} \oplus \mathcal{C}_n(\mathcal{O})^0.$$

Proof. Fix $n \geq 1$. To prove that the canonical map $\mathcal{E}_n(\mathcal{O}) \to \mathcal{E}_n(\mathcal{O})^{\mathrm{nil}} \oplus \mathcal{E}_n(\mathcal{O})^0$ is well defined, it suffices to show that T_p preserves $\mathcal{E}_n(\mathcal{O})$. The isomorphism will then follow provided that in addition T_p preserves $\mathcal{E}_n[\mathcal{O}]$. For each $y = r/s \in \mathbf{P}^1(\mathbb{Q})$ fix $\gamma_y \in S_y$ and let $p(y)$ be the highest power of p dividing s. First suppose $E \in \mathcal{E}_n(\mathcal{O})$; then $e_n(y)a_0(E|\gamma_y) = \Phi(E)(\pi_y) \in \mathcal{O}$ for all $y \in \mathbf{P}^1(\mathbb{Q})$. We must show that

$$e_n(y)a_0((E|T_p)|\gamma_y) = e_n(y)\sum_{i=0}^{p-1} a_0(E|\tau_i\gamma_y) \in \mathcal{O},$$

where $\tau_i = \left(\begin{smallmatrix} 1 & Ni \\ 0 & p \end{smallmatrix}\right)$. Suppose $p(y) < n$. Using Lemma 4.3 and the fact that $e_n(y) = p^n N/\mathrm{div}(y)$, we obtain

$$e_n(y)a_0(E|\tau_i\gamma_y) = \begin{cases} (pe_n(\tau_i y))\left(\dfrac{1}{p}a_0(E|\gamma_{\tau_i y})\right) & \text{if } g_i = (r+siN, p) = 1, \\ (e_n(\tau_i y))(pa_0(E|\gamma_{\tau_i y})) & \text{if } g_i = (r+siN, p) = p. \end{cases}$$

Thus $e_n(y)a_0((E|T_p)|\gamma_y) \in \mathcal{O}$. Now if $p(y) \geq n$, then for $0 \leq i \leq p-1$, $e_n(y)a_0(E|\tau_i\gamma_y) = (1/p)e_n(\tau_i y)a_0(E|\gamma_{\tau_i y})$. Here the cusps $\tau_i y$ are all Γ_n-equivalent so $e_n(y)a_0((E|T_p)|\gamma_y) \in \mathcal{O}$.

The group Γ_n is generated by the group $\Gamma(p^n N)$ and the parabolic element π_∞. For any $E \in \mathcal{E}_n(\overline{\mathbb{Q}}_p)$ and $\gamma, \gamma' \in \Gamma$ we have $\Phi(E)(\gamma\gamma') = \Phi(E)(\gamma) + \Phi(E)(\gamma')$. Thus

$$\mathcal{E}_n[\mathcal{O}] = \mathcal{E}_{\Gamma(p^n N)}[\mathcal{O}] \cap \mathcal{E}_n(\mathcal{O}); \qquad \mathcal{E}_n[\mathcal{O}]^0 = \mathcal{E}_{\Gamma(p^n N)}[\mathcal{O}] \cap \mathcal{E}_n(\mathcal{O})^0, \tag{5.11}$$

where $\mathcal{E}_{\Gamma(p^n N)}[\mathcal{O}]$ is defined analogously. Now suppose $E \in \mathcal{E}_n[\mathcal{O}]$. Then $\Phi(E|T_p)(\pi_\infty) \in \mathcal{O}$. Let $\gamma \in \Gamma(p^n N)$. By a calculation we have $\tau_i\gamma\tau_i^{-1} \in \Gamma_n$ for $0 \leq i \leq p-1$, and so $\Phi(E)(\tau_i\gamma\tau_i^{-1}) = \Phi(E|\tau_i)(\gamma) \in \mathcal{O}$. This shows that e preserves $\mathcal{E}_n[\mathcal{O}]$.

Since $\mathcal{H}[A^*]$ is commutative, to prove the $\mathcal{H}[A^*]$-equivariance it suffices to show that $\mathcal{H}[A^*]$ preserves both $\mathcal{E}_n(\mathcal{O})$ and $\mathcal{E}_n[\mathcal{O}]$. By (3.11) both N_n and G_n preserve these modules. Since $\Gamma_0(p^n N)$ normalizes Γ_n, the Nebentype automorphisms preserve $\mathcal{E}_n[\mathcal{O}]$ as well.

Now suppose we have a product $g = \prod_{\underline{a}} g_{\underline{a}}^{m(\underline{a})}$ where $\underline{a}$ ranges over $V_{p^n N}$ and the $m(\underline{a})$ are integers. As shown by Kubert and Lang $g \in U(\Gamma(p^n N))$ if and only if the $m(\underline{a})$ satisfy certain congruence conditions ([9], Chapter 3; Theorems 5.2, 5.3). One can easily check that the action of G_n given by $g_{(a_1,a_2)}|\tau_j = g_{(a_1,ja_2)}$ respects these congruences, thus by (5.3) and (5.11) G_n preserves $\mathcal{E}_n[\mathcal{O}]$.

This gives the desired result for A^*. Now by the final remarks in Section 4, to conclude the proof we need only show that T_l, l prime, preserves $\mathcal{E}_n(\mathcal{O})$ and $\mathcal{E}_n[\mathcal{O}]$. This has already been shown for $l = p$. With obvious modifications the same argument works for any $l \mid N$. Finally if $l \nmid Np$ then the result follows from Lemma 3.3(a).

In particular Proposition 5.10 demonstrates that $\mathcal{C}_n(\mathcal{O})^0$ is a $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$-module.

For $n \geq 1$ let $\mathcal{D}_n = \mathcal{D}_{\Gamma_n}$ and let $\mathcal{D}_n^0$ be the group of divisors with support in $\mathrm{cusps}(\Gamma_n)^0$. We define a map

$$\delta_n^0 : \mathcal{E}_n(\overline{\mathbb{Q}}_p)^0 \to \mathcal{D}_n^0 \otimes \overline{\mathbb{Q}}_p$$

via

$$\delta_n^0(E) = \sum_{x \in \mathrm{cusps}(\Gamma_n)^0} r_{n,E}(x) \cdot (x). \tag{5.12}$$

By Lemma 4.4 this is an isomorphism.

Lemma 5.13. *Let $n \geq 1$ and suppose that $E \in \mathcal{E}_n(\overline{\mathbb{Q}}_p)^0$ satisfies $\delta_n^0(E) \in \mathcal{D}_n^0 \otimes \mathcal{O}$. Then $E \in \mathcal{E}_n(\mathcal{O})^0$.*

Proof. Fix $n \geq 1$ and E as above, and choose any $F \in \mathcal{E}_n(\mathcal{O})$ such that $\delta_n^0(F) = \delta_n^0(E)$. By Lemma 4.4, $\delta_n^0(eF) = \delta_n^0(E)$ so that $eF = E$. Now by Proposition 5.9, $eF \in \mathcal{E}_n(\mathcal{O})$.

We come now to the main result.

Theorem 5.14. *Let $N > 1$ be squarefree. Then $\beta \in \mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$ annihilates $\mathcal{C}_n^0$ for all $n \geq 1$.*

Proof. Fix $n \geq 1$ and for each $x \in \mathrm{cusps}(\Gamma_n)^0$ let I_x be the unique element of $\mathcal{E}_n(\mathbb{Z}_p)^0$ such that $\delta_n^0(I_x) = (x)$. To show that β annihilates $\mathcal{C}_n^0$ we must show that $I_x|\beta \in \mathcal{E}_n[\mathbb{Z}_p]$ for all $x \in \mathrm{cusps}(\Gamma_n)^0$. However, the set of ordinary cusps of divisor $p^n d$ for a given $d \mid N$ is generated by $\left[\begin{smallmatrix} 1 \\ p^n d \end{smallmatrix}\right]$ and N_n. Since $E \in \mathcal{E}_n[\mathbb{Z}_p]$ if and only if $E|\langle j \rangle \in \mathcal{E}_n[\mathbb{Z}_p]$, $j \in \mathbb{Z}(p^n N)^*$, we need only show that each $I_d|\beta \in \mathcal{E}_n[\mathbb{Z}_p]$ where $I_d = I_x$ for $x = \left[\begin{smallmatrix} 1 \\ p^n d \end{smallmatrix}\right]$.

Fix $d \mid N$. For each $\kappa \in \mathcal{X}_N^{\mathrm{arith},+}$ of signature $(\xi, 2)$ and level $m \leq n$ let η and χ be the unique primitive characters, of conductors dividing N/d and $p^n d$ respectively, such that $\xi = \overline{\eta}\chi$. Let $I_{\kappa,d,n}$ and $E_{\kappa,d,n}$ be as in Definition 4.7 and Proposition 4.8. From Proposition 4.8 we have the relation $E_{\kappa,d,n} = \beta(\xi) I_{\kappa,d,n}$; since $\mathbb{Z}_{p,N}^*$ acts on $I_{\kappa,d,n}$ by ξ we also have $E_{\kappa,d,n} = I_{\kappa,d,n}|\beta$. Now G_n and N_n act on $I_{\kappa,d,n}$ by η and $\eta\chi$ respectively; using (3.11) it follows that

$$\delta_n^0(I_{\kappa,d,n}) = \sum \eta(b)\overline{\chi}(a) \begin{bmatrix} a \\ bp^n d \end{bmatrix};$$

the sum over cusps of divisor $p^n d$. From this we obtain

$$I_d = \frac{2}{\phi(p^n N)} \sum_{\eta,\chi} I_{\kappa,d,n},$$

and

$$I_d|\beta = \frac{2}{\phi(p^n N)} \sum_{\eta,\chi} E_{\kappa,d,n}.$$

Let $E_d = I_d|\beta$. By the definitions of the $E_{\kappa,d,n}$ we have

$$E_d = \sum_{u|(N/d)} \sum_{v|d} \mu(uv)\phi_2\left(\frac{s^* v'}{p^n d/v}, \frac{r^* u'}{N/du}\right) \Big| \begin{pmatrix} p^n d & 0 \\ 0 & 1 \end{pmatrix},$$

where $s^* N/d \equiv 1 \pmod{p^n d}$ and $r^* p^n d \equiv 1 \pmod{N/d}$. In particular $\delta_n(E_d) \in \mathcal{D}_n \otimes \mathbb{Q}$. Since I_d is ordinary, and $\delta_n^0(I_d) = \left[\frac{1}{p^n d}\right]$, by Lemma 5.13, $I_d \in \mathcal{E}_n(\mathbb{Z}_p)^0$ which is preserved by $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^*]]$. As a consequence $\delta_n(E_d) \in \mathcal{D}_n \otimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p.

We now use Theorem 5.7. Let $\alpha \in C_{p^n N}$ have conductor $Q = q^r$. Fix some $E_{\kappa,d,n}$ as above. To calculate the special value $\Lambda(E_{\kappa,d,n}, \alpha, 1)$ we use the identities

$$L(1, \eta\alpha) = \frac{-\pi i}{\tau(\overline{\eta}\overline{\alpha})} \mathbf{B}_1(\overline{\eta}\overline{\alpha}); \qquad L(0, \alpha\chi_p) = -\mathbf{B}_1(\alpha\chi_p),$$

and for primitive characters ε_1 and ε_2 of relatively prime conductors M_1 and M_2,

$$\frac{\tau(\overline{\varepsilon}_1)\tau(\overline{\varepsilon}_2)}{\tau(\overline{\varepsilon}_1\overline{\varepsilon}_2)} = \varepsilon_2(M_1)\varepsilon_1(M_2).$$

Using Proposition 4.8(c) we obtain

$$\begin{aligned} L(E_{\kappa,d,n}, \alpha, s) = 2\tau(\overline{\eta})\overline{\eta}(p^n d)\chi(N/d) &\prod_{l|(N/N_1 d)} (\alpha(l) l^{1-s} - \overline{\eta}(l) l) \\ &\times \prod_{l|d} (1 - \alpha\chi(l) l^{2-s}) L(s, \eta\alpha) L(s-1, \alpha\chi_p), \end{aligned}$$

where $\mathrm{cond}(\eta) = N_1$, and so

$$\begin{aligned} \Lambda(E_{\kappa,d,n}, \alpha, 1) &= \overline{\eta}(p^n d)\chi(N/d)\eta(Q)\alpha(N_1) \\ &\quad \times \prod_{l|(N/N_1 d)} (\alpha(l) - \overline{\eta}(l) l) \prod_{l|d} (1 - \alpha\chi(l) l) \mathbf{B}_1(\overline{\eta}\overline{\alpha}) \mathbf{B}_1(\alpha\chi_p) \\ &= \Big[\overline{\eta}(p^n d)\eta(Q) \prod_{l|(N/d)} (1 - \overline{\eta}\overline{\alpha}(l) l) \mathbf{B}_1(\overline{\eta}\overline{\alpha})\Big] \\ &\quad \times \Big[\alpha\chi(N/d) \prod_{l|d} (1 - \alpha\chi(l) l) \mathbf{B}_1(\alpha\chi_p)\Big]. \end{aligned}$$

Let $A_1 \in \mathbb{Q}[\mathbb{Z}(NQ/d)^*]$ be defined by

$$A_1 = \sum_{u|(N/d)} \mu(u) \sum_{x \in \mathbb{Z}(NQ/d)^*} \mathbf{B}_1\left(\frac{xu'}{NQ/du}\right) \cdot [xr],$$

where $uu' \equiv 1 \pmod{NQ/du}$, and where $r \in \mathbb{Z}(NQ/d)^*$ satisfies $Qr \equiv p^n d \pmod{N/d}$ and $r \equiv 1 \pmod{Q}$. Similarly, if $d > 1$, we define $A_2 \in \mathbb{Q}[\mathbb{Z}(p^n dQ)^*]$ via

$$A_2 = \sum_{v|d} \mu(v) \sum_{x \in \mathbb{Z}(p^n dQ)^*} \mathbf{B}_1\left(\frac{xv'}{p^n dQ/v}\right) \cdot [xs],$$

where $vv' \equiv 1 \pmod{p^n dQ/v}$, and where $s \in \mathbb{Z}(p^n dQ)^*$ satisfies $s \equiv N/d \pmod{p^n dQ}$. Then $A_1 \in (NQ)^{-1}\mathbb{Z}[\mathbb{Z}(NQ/d)^*]$ and by Lemma 1.11 $A_2 \in (2NQ)^{-1}\mathbb{Z}[\mathbb{Z}(p^n dQ)^*]$. If $d = 1$ we define A_2 by

$$A_2 = \sum_{x \in \mathbb{Z}(p^n N)^*} \left[\mathbf{B}_1\left(\frac{x}{p^n Q}\right) - \mathbf{B}_1\left(\frac{xQ'}{p^n}\right)\right] \cdot [x],$$

where $QQ' \equiv 1 \pmod{p^n}$. It follows easily that $A_2 \in (2Q)^{-1}\mathbb{Z}[\mathbb{Z}(p^n Q)^*]$. Then by a straightforward modification of Lemma 1.6 and Corollary 1.7, defining integration in the obvious way,

$$\Lambda(E_{\kappa,d,n}, \alpha, 1) = \left(\int \overline{\eta}\,\overline{\alpha}\, dA_1\right)\left(\int \alpha\chi\, dA_2\right).$$

Thus $\Lambda(E_{\kappa,d,n}, \alpha, 1)$ is an element of $(2N^2)^{-1}\mathbb{Z}[\xi][\alpha, 1/Q]$. Writing $A_1 = \sum a_1(y)\cdot[y]$, let A_1' be defined by $A_1' = \sum a_1(y)\cdot[y']$, where $y \equiv y' \pmod{N/d}$ and $yy' \equiv 1 \pmod{Q}$. Then

$$\int \overline{\eta}\,\overline{\alpha}\, dA_1 = \int \overline{\eta}\alpha\, dA_1'.$$

Using the isomorphism $\mathbb{Z}(p^n N)^* \cong \mathbb{Z}(N/d)^* \times \mathbb{Z}(p^n d)^*$ we may identify $A = A_1' A_2$ as an element of $(2N^2Q^2)^{-1}\mathbb{Z}[\mathbb{Z}(p^n NQ)^*]$. Now let $\sigma : \mathbb{Z}(p^n NQ)^* \to \mathbb{Z}$ be the map given by (i) $\sigma(m) = 1$ if $m \equiv \pm 1 \pmod{p^n N}$ and (ii) $\sigma(m) = 0$ otherwise. Then

$$\Lambda(E_d, \alpha, 1) = \frac{2}{\phi(p^n N)} \sum_{\xi} \int \xi\alpha\, dA = \int \sigma\alpha\, dA \in (2N^2)^{-1}\mathbb{Z}[\alpha, 1/Q].$$

Let L be the product of the denominators of the $r_{n,E_d}(x)$ for $x \in \mathrm{cusps}(\Gamma_n)$, and let $K = (4N^2L)^{-1}\mathbb{Z}$. Then $\frac{1}{2}\Lambda(E_d, \alpha, 1) \in K[\alpha, 1/Q]$. Then by (5.6) and Theorem 5.7, $E \in \mathcal{E}_{\Gamma_n}[K] \subseteq \mathcal{E}_n[\mathbb{Z}_p]$, completing the proof.

Note that for $n \geq m$ we have inclusions $\mathcal{C}_m(\mathcal{O})^0 \to \mathcal{C}_n(\mathcal{O})^0$ and may define the direct limit $\mathcal{C}_{\mathcal{O}}^0 = \varinjlim \mathcal{C}_n(\mathcal{O})^0$ which is naturally an $\mathcal{O}[[\mathbb{Z}_{p,N}^*]]$-

module. Let $\mathcal{C}^*_{\mathcal{O}} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{C}^0_{\mathcal{O}}, \mathcal{O}/\mathcal{K})$, where $\mathcal{K}$ is the fraction field of $\mathcal{O}$. We equip $\mathcal{C}^*_{\mathcal{O}}$ with an action of $\mathcal{O}[[\mathbb{Z}^*_{p,N}]]$ by defining for $\sigma \in \mathcal{C}^*_{\mathcal{O}}$, $(\sigma|\lambda)(\Phi) = \sigma(\Phi|\lambda)$ for $\lambda \in \mathcal{O}[[\mathbb{Z}^*_{p,N}]]$ and $\Phi \in \mathcal{C}^0_{\mathcal{O}}$. If $\mathcal{O} = \mathbb{Z}_p$ we will simply write $\mathcal{C}^0 = \mathcal{C}^0_{\mathbb{Z}_p}$ and $\mathcal{C}^* = \mathcal{C}^*_{\mathbb{Z}_p}$.

Now let $\mathcal{O}$ be the ring formed by adjoining the values of even $\xi \in K(Np)$ to $\mathbb{Z}_p$. We have a canonical decomposition $\mathbb{Z}^*_{p,N} = (1 + p\mathbb{Z}_p) \times \Delta$, where $\Delta \cong \mathbb{Z}(pN)^*$. For even $\xi \in K(pN)$ we define the idempotent $\varepsilon_\xi \in \mathcal{O}[\Delta]$ by

$$\varepsilon_\xi = \frac{1}{\phi(pN)} \sum_{r \in \Delta} \overline{\xi}(r) \cdot [r].$$

Now suppose $p \nmid \phi(N)$. We define the ξ part of $\mathcal{C}^*_{\mathcal{O}}$ by $\mathcal{C}^*_{\mathcal{O},\xi} = \mathcal{C}^*_{\mathcal{O}}|\varepsilon_\xi$. Then $\mathcal{C}^*_{\mathcal{O}} = \bigoplus \mathcal{C}^*_{\mathcal{O},\xi}$; the sum over even $\xi \in K(Np)$. Given $\nu \in \mathcal{O}[[\mathbb{Z}^*_{p,N}]]$ we let $\nu_\xi \in \mathcal{O}[[1 + p\mathbb{Z}_p]]$ be defined via $\nu_\xi|\varepsilon_\xi = \nu|\varepsilon_\xi$.

THEOREM 5.15. *Suppose that $N > 1$ is squarefree and that $p \nmid \phi(N)$. Let $\xi \in K(Np)$ be even and suppose that $f_\xi \in \mathcal{O}[[1 + p\mathbb{Z}_p]]$ is a divisor of β_ξ which annihilates $\mathcal{C}^*_{\mathcal{O},\xi}$. Then $f_\xi = \iota\beta_\xi$ for some unit ι.*

Proof. Fix an even $\xi \in K(Np)$ and let $d \,|\, N$ and $n \geq 1$ be given. Let η and χ be the unique primitive Dirichlet characters of conductors dividing respectively N/d and $p^n d$ such that $\overline{\eta}\chi = \xi$. To prove that β_ξ annihilates $\mathcal{C}^*_{\mathcal{O},\xi}$ we must show that $E_{\kappa,d,n} = I_{\kappa,d,n}|\beta_\xi \in \mathcal{E}_n[\mathcal{O}]^0$, where $I_{\kappa,d,n} \in \mathcal{E}_n(\mathcal{O})^0$ satisfies

$$\delta^0_n(I_{\kappa,d,n}) = \sum \eta(b)\overline{\chi}(a) \begin{bmatrix} a \\ bp^n d \end{bmatrix};$$

the sum over cusps of divisor $p^n d$. This is essentially shown in the proof of Theorem 5.14 using Theorem 5.7.

Now suppose $f_\xi g_\xi = \beta_\xi$ for $f_\xi, g_\xi \in \mathcal{O}[[1 + p\mathbb{Z}_p]]$ and that $\mathcal{C}^*_{\mathcal{O},\xi}|f_\xi = 0$. Then f_ξ annihilates the cuspidal group $\mathcal{C}_1(\mathcal{O})^0$. To prove the theorem it suffices to find $E \in \mathcal{E}_1(\mathcal{O})^0$ such that $E|f_\xi \in \mathcal{E}_1[\mathcal{O}]^0$ only if f_ξ and β_ξ are associates. Given E, this will follow from Theorem 5.7.

Let E' be the unique element of $\mathcal{E}_1(\mathcal{O})^0$ such that

$$\delta^0_1(E') = \sum \overline{\xi}(a) \begin{bmatrix} a \\ pN \end{bmatrix}.$$

Let $\alpha \in \mathcal{C}^-_{pN}$ have conductor dividing q^r with q and $r = r_q$ as in Theorem 5.8. Then

$$\Lambda(E'|f_\xi, \alpha, 1) = g_\xi(0)^{-1}\Lambda(E'|\beta_\xi, \alpha, 1) = g_\xi(0)^{-1}B(\alpha\xi),$$

where

$$B(\alpha\xi) = \mathbf{B}_1(\overline{\alpha})\Big(\prod_{l|N}(1 - \alpha\xi(l)l)\Big)\mathbf{B}_1(\alpha\xi_p)$$

(see the proof of Theorem 5.14), and where $g_\xi(0) = \int d\beta$ is the constant term of g. Without loss of generality we may assume this lies in $\mathbb{Z}[\xi]$. The expression for $\Lambda(E'|f_\xi, \alpha\alpha_q, 1)$ is similar with $\alpha\alpha_q$ replacing α. Since α_q is odd this term vanishes.

Using the definition in Proposition 4.8, by inspection we have $E'|\beta_\xi$ satisfying $\delta_1(E'|\beta_\xi) \in \mathcal{D}_1 \otimes \mathbb{Q}[\xi]$. Since $E'|\beta_\xi \in \mathcal{E}_1(\mathcal{O})^0$, there exists $L \in \mathbb{Z}$ prime to p such that $LE'|\beta_\xi \in \mathcal{E}_1(\mathbb{Z}[\xi])$. Now let $E = 4LN^2E'$. Then $\frac{1}{2}\Lambda_\pm(E|f_\xi, \alpha, 1) = 2LN^2 g_\xi(0)^{-1}B(\alpha\xi)$. As shown in the proof of Theorem 5.14 we have $B(\alpha\xi) \in (2N^2)^{-1}\mathbb{Z}[\xi][\alpha, 1/Q]$. Therefore by Theorem 5.7, with $K = L\mathbb{Z}[\xi]$, $E|\beta_\xi \in \mathcal{E}_1[L\mathbb{Z}[\xi]]$. Given a prime $\mathbf{P}$ of $\overline{\mathbb{Q}}$ lying over p and not dividing $2LNq$, by Theorem 5.8 we may assume α is chosen so that $2LN^2B(\alpha\xi)$ is not divisible by $\mathbf{P}$. Then by Theorem 5.7, $g_\xi(0)$ is a unit of $\mathcal{O}$.

Now we consider the case for general N. If $N > 1$ then it is possible to design a canonical basis for each $\mathcal{E}_k(\Gamma_n; \overline{\mathbb{Q}}_p)^0$ with properties similar to those in the squarefree case. One should expect an analogue of β in $\mathbb{Z}_p[[A^*]]$ which annihilates each $\mathcal{C}_n^0$.

Let $N = 1$. The following analogue of Theorem 5.15 is an immediate consequence of Mazur and Wiles' study of the groups $\mathcal{C}_\chi^{(n)}$ ([12], Chapter 4).

Theorem 5.16 (Mazur–Wiles). *Let $N = 1$, and $\xi \in K(p)$ be even such that $\xi \neq 1$ or ω^{-2}. Then if $f_\xi \in \Lambda$ is a divisor of β_ξ which annihilates $\mathcal{C}_\xi^*$, $f_\xi = \iota\beta_\xi$ for some unit ι. If $\xi = 1$ or ω^{-2}, then $\mathcal{C}_\xi^*$ is trivial.*

Proof. Let c be a prime different from p. We will show that the measure $\beta_{2,c}$ annihilates $\mathcal{C}^*$. Suppose that M and M' are positive integers with $M|M'$. By a result of Fricke–Wohlfahrt $\Gamma(M)$ is generated by $\Gamma(M')$ and the parabolic elements of $\Gamma(M)$ ([24]; see the proof of Theorem 1.2 in Chapter 5 of [11]). This implies that $\Gamma_1(M)$ is generated by $\Gamma_1(M')$ and the parabolic elements of $\Gamma_1(M)$, as $\Gamma_1(M)$ is itself generated by its parabolic elements and $\Gamma(M)$. In particular this gives

$$\mathcal{E}_n(\mathbb{Z}_p)^0 \cap \mathcal{E}_{\Gamma_1(p^n c)}[\mathbb{Z}_p] = \mathcal{E}_n[\mathbb{Z}_p]^0;$$

therefore by Theorem 5.13 $\beta_{2,c}$ annihilates $\mathcal{C}_n^0$, and therefore $\mathcal{C}^*$.

Now let ξ be an even power of ω, and let $\chi \in K(p^\infty)$. Then χ may be regarded as a map from $\mathbb{Z}_p$ to $\overline{\mathbb{Q}}_p$; if χ factors through $1 + p\mathbb{Z}_p$ then χ is said to be a character of the second kind. For any such character we have $\int \chi\, d\beta_{2,c,\xi} = -\frac{1}{2}(1 - \chi\xi(c)c^2)\mathbf{B}_2(\chi\xi_p)$. Since this is also true for the distribution $(1 - \xi(c)c^2[\langle c\rangle])\beta_\xi$, these measures are equal. Choosing c so that $\omega(c)$ is a primitive $(p-1)$st root of unity, it follows that $(1 - \xi(c)c^2[\langle c\rangle])$ is a unit if $\xi \neq \omega^{-2}$, in which case $\beta_\xi \in \Lambda$ annihilates $\mathcal{C}_\xi^*$. The remainder of the proof follows as in the proof of Theorem 5.15; the last part derives from the fact that $\beta_{2,c,\xi}$ is a unit if $\xi = 1$ or ω^{-2}.

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