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On the coefficients of the Taylor expansion of the Dirichlet L-function at $s = 1$

by

Hideaki Ishikawa (Niigata)

1. Introduction and results. We consider the Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi(n)$ is a Dirichlet character modulo q, and denote its nth derivative at $s = 1$ by $L^{(n)}(1,\chi)$. These derivatives have been widely studied from a number theoretical point of view. Berger [1], Selberg and Chowla [7] and Deninger [2] obtained representations of $L'(1,\chi)$ by elementary functions. Kanemitsu [4] gave similar results for $L^{(n)}(1,\chi)$ for $n \geq 2$. Toyoizumi [8] obtained an upper bound for $L^{(n)}(1,\chi)$ for real non-principal χ. We can write $L^{(n)}(1,\chi)$ in the form

$$L^{(n)}(1,\chi) = (-1)^n \sum_{a=1}^{q} \chi(a)\gamma_n(a,q), \tag{1}$$

where the numbers $\gamma_n(a,q)$ are defined by

$$\gamma_n(a,q) = \lim_{N\to\infty} \Bigg(\sum_{0\leq m\equiv a \,(\mathrm{mod}\, q)}^{N} \frac{\log^n m}{m} - \frac{\log^{n+1} N}{q(n+1)} \Bigg) \tag{2}$$

and called generalized Euler constants for arithmetical progressions. Hence the study of $L^{(n)}(1,\chi)$ is closely related to that of $\gamma_n(a,q)$. Kanemitsu [4] proved that $\gamma_n(a,q)$ can be expressed in terms of classical functions. K. Dilcher [3] derived further properties of $\gamma_n(a,q)$, calculated $\gamma_n(a,q)$ explicitly in many cases ([3], p. 271), and computed many approximate values of $\gamma_n(a,q)$ ([3], p. 280).

In this paper we are interested in $L^{(n)}(1,\chi)$ as a function of n with fixed q and χ and study the asymptotic behavior of $L^{(n)}(1,\chi)$ as $n \to \infty$. As a

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byproduct, we derive a relation (Proposition 1) between $L^{(n)}(1,\chi)$ and the Gauss sum $\tau(\chi)=\sum_{a=1}^{q-1}\chi(a)e^{2\pi ia/q}$. We set

$$S_\mu^+(N)=\sharp\left\{n\le N:\left|\arg\frac{(-1)^nL^{(n)}(1,\chi)}{i^\alpha\tau(\chi)}\right|<\mu\right\},$$
$$S_\mu^-(N)=\sharp\left\{n\le N:\left|\arg\frac{(-1)^nL^{(n)}(1,\chi)}{-i^\alpha\tau(\chi)}\right|<\mu\right\}$$

where $\alpha=0$ or 1 according as $\chi(-1)=1$ or -1. Then we have

Theorem 1. *Given an arbitrarily small number $\mu>0$ and any number λ with $0<\lambda<1$, for sufficiently large N we have*

$$S_\mu^+(N)=\frac{1}{2}N+O\left(\frac{N}{\log^\lambda N}\right),\qquad S_\mu^-(N)=\frac{1}{2}N+O\left(\frac{N}{\log^\lambda N}\right).$$

Theorem 1 asserts that for sufficiently large n, almost all values of $L^{(n)}(1,\chi)$ are located near the line in the complex plane passing through the origin whose argument coincides with that of $i^\alpha\tau(\chi)$. This seems interesting since the value $L^{(n)}(1,\chi)=\lim_{s\to1+0}L^{(n)}(s,\chi)$ can be computed using only real-variable methods, e.g., by using (1), (2) and Euler–Maclaurin summation formula (see, e.g., [3], p. 280, where an error term is given), while $\tau(\chi)$ is an essential constant in the functional equation, i.e., an object of complex analysis.

The precise asymptotic behavior of $|L^{(n)}(1,\chi)|$ is given in the following theorems.

Theorem 2. *There exists an n_0 such that for all $n\ge n_0$*

$$|L^{(n)}(1,\chi)|\le q^{n/\log n-1/2}\exp\left(n\log\log n-\frac{n\log\log n}{\log n}\right).$$

By Cauchy's estimate for Taylor coefficients, for any fixed real number $r>0$ we have

$$|L^{(n)}(1,\chi)|\le n!\frac{M_r}{r^n} \tag{3}$$

where $M_r=\max_{|z-1|\le r}|L(z,\chi)|$. The right-hand side in (3) is $\ll e^{n\log n}$ as $n\to\infty$, while Theorem 2 implies the bound $\ll e^{n\log\log n}$. Hence the bound of Theorem 2 is much better than what can be obtained by Cauchy's estimate.

The next theorem shows that Theorem 2 is almost best possible.

Theorem 3. *There exist infinitely many n such that*

$$|L^{(n)}(1,\chi)|\ge q^{n/\log n-1/2}\exp\left(n\log\log n-\frac{n\log\log n}{\log n}-C_1\frac{n}{\log n}\right)$$

where C_1 is an absolute constant.

Toyoizumi [8] proved an upper bound for $L^{(n)}(1,\chi)$ for a real non-principal χ, which gives a sharp bound in terms of q:

Assume that q is cube-free. Then for $\varepsilon > 0$ we have

$$|L^{(n)}(1,\chi)| \le \left(\frac{1}{(k+1)4^{k+1}} \cdot \frac{L(1+\varepsilon,\chi)}{\zeta(1+\varepsilon)} + \varepsilon\right) \log^{n+1} q$$

if $q > q_0(\varepsilon)$.

At the first glance, this result seems to contradict with our Theorem 3. But the proof of this result requires that q_0 is larger than $\exp\left[\frac{1}{1+\varepsilon}n\right]$ to ensure that the function $(\log x)^n/x$ is decreasing in the required area of partial summation. Hence Toyoizumi's result is valid only when $\exp\left[\frac{1}{1+\varepsilon}n\right] \ll q_0$.

Our proof, whose essential idea is due to Matsuoka ([5] and [6]), is based on the functional equation for Dirichlet L-functions and the saddle point method. We first prove an asymptotic formula for $L^{(n)}(1,\chi)$.

Proposition 1. *Let $P(x) = \cos x$ or $\sin x$ according as $\chi(-1) = 1$ or -1 and χ be a primitive character modulo q. Then there exists an $n_0 > e^q$ such that for all $n > n_0$*

$$(4)\quad (-1)^n L^{(n)}(1,\chi) = i^{\alpha}\frac{\tau(\chi)}{q} q^{n/\log n} e^{n\log\log n + H_q(n)} \cdot [P(F_q(n)) + E_{q,\alpha}(n)]$$

where $H_q(n)$ and $F_q(n)$ are real valued functions satisfying

$$H_q(n) = -\frac{n\log\log n}{\log n} - \frac{n}{\log n}(\log 2\pi + 1) + O\left(\frac{n(\log\log n)^2}{\log^2 n}\right),$$
$$F_q(n) = -\frac{1}{2}\pi\frac{n}{\log n} + O\left(\frac{n\log\log n}{\log^2 n}\right)$$

and $E_{q,\alpha}(n)$ is a complex valued function satisfying $E_{q,\alpha}(n) = O(1/\log n)$. Each O-constant depends only on q.

Theorem 2 is a consequence of this proposition. Note that by the method of [5] one can show a more precise (but more complicated) asymptotic expansion, which, however, is not needed in this paper.

Taking the argument on both sides in (4), it follows that

$$\arg\frac{(-1)^n L^{(n)}(1,\chi)}{i^{\alpha}\tau(\chi)} = \arg[P(F_q(n)) + E_{q,\alpha}(n)].$$

The right side here must be treated carefully. When the oscillating function $P(F_q(n))$ is small, then $E_{q,\alpha}(n)$ is larger than the "main" term. Hence in Proposition 2, we show that the error terms $E_{q,\alpha}(n)$ are small in most cases.

Proposition 2. *Let c be a positive constant, and let m be a sufficiently large positive integer so that $m - c\log m > e^q$. Then for all n with $|n-m| <$*

$c\log m$, we have

$$
\begin{aligned}
&(-1)^n L^{(n)}(1,\chi)\\
&\quad = i^\alpha \frac{\tau(\chi)}{q} q^{n/\log n} e^{n\log\log n + H_q(n)} \cdot \left[P\left(F_q(m) - \frac{1}{2}\pi\frac{n-m}{\log m}\right) + E^*_{q,\alpha}(m)\right]
\end{aligned}
$$

where $E^*_{q,\alpha}(m) = O(\log\log m/(\log m))$. Here the O-constant depends on c and q.

Theorems 1 and 3 can be deduced from these propositions (see Section 5).

2. Lemmas for Proposition 1. To prove Proposition 1, we employ the saddle point method. The integrand to be investigated is $e^{\Phi_q(z)}$ with

$$\Phi_q(z) = z\log q - (n+1)\log z - z\log 2\pi i + \log\Gamma(z).$$

In this section, we prove some lemmas on the saddle point of the function $\Phi_q(z)$. We omit the details since they are similar to the lemmas in [5].

Lemma 1. *Let $z = x + yi$ and $n > \log^3 q$ be a sufficiently large positive integer. Then in the region $n^{1/2} < x < n, 0 < y < x$, the equation*

$$\frac{d}{dz}\Phi_q(z) = 0$$

has the unique solution $x + yi = a + bi$.

Proof. Let x be fixed and $h_q(y) = \Im(z\Phi'_q(z))$. Then it follows that $h_q(y) = 0$ has a unique solution in $0 < y < x$. Denote this solution y by y_x and put $z_x = x + y_x i$ and $u_q(x) = \Re(z_x\Phi'_q(z_x))$. Then

$$
\begin{aligned}
(5)\qquad u_q(x) &= x(\log q - \log 2\pi) - (n+1) + \frac{1}{2}\pi y_x + x\log\sqrt{x^2+y_x^2}\\
&\quad - y_x\arg(x+y_x) - \frac{1}{2} + \Re(z_x J'(z_x)),
\end{aligned}
$$

where $J(z)$ is the error term in Stirling's asymptotic formula for $\log\Gamma(z)$ ([9], p. 251), defined by

$$J(z) = 2\int_0^\infty \frac{\arctan(t/z)}{e^{2\pi t}-1}\,dt \ll |z|^{-1} \qquad \text{for } \Re z > 0.$$

We have $\frac{\partial}{\partial x}u_q(x) > 0$ for $n^{1/2} < x < n$. Using (5) we obtain $u_q(n^{1/2}) < 0$ and $u_q(n) > 0$.

Hence $u_q(x) = 0$ has the unique solution in $n^{1/2} < x < n$. The difference with respect to Matsuoka's Lemma 1 (see [5], p. 49) is that we must add a restriction $\log^3 q < n$ to ensure that $u_q(n^{1/2}) < 0$. ∎

Lemma 2. *If $n > e^q$, then*

$$\frac{n}{\log n} < a < \frac{n}{\log n} + \frac{2n\log\log n}{\log^2 n}, \qquad b = \frac{1}{2}\pi\frac{n}{\log^2 n} + O\left(\frac{n\log\log n}{\log^3 n}\right).$$

Proof. We easily see that

$$n = a\log aq - a\log 2\pi + O\left(\frac{a}{\log aq}\right) \tag{6}$$

since $a+bi$ is a solution of $\Phi_q'(z) = 0$. This gives the upper bound for a. Now assume $a \le n/\log n$. Then we have

$$n < a\log aq \le \frac{n}{\log n}\log\left(\frac{n}{\log n}q\right).$$

As $q < \log n$, we have

$$\frac{n}{\log n}\log\left(\frac{n}{\log n}q\right) < n,$$

which is a contradiction. Consequently, we have $a > n/\log n$. The estimate for b is proved similarly. ∎

Note that above estimations are independent of q, as we assumed $q < \log n$.

LEMMA 3. *Let $g_q(y) = \Re\Phi_q(a+yi)$, $f_q(y) = \Im\Phi_q(a+yi)$. Then*

$$g_q(y) \text{ is strictly increasing in } 0 \le y \le b, \tag{7}$$

$$g_q(y) \text{ is strictly decreasing in } b \le y \le a, \tag{8}$$

$$g_q''(b) = -\frac{\log aq}{a} + \frac{\log 2\pi - 1}{a} + O\left(\frac{1}{a\log aq}\right), \tag{9}$$

$$f_q''(b) = \frac{\pi}{a} + O\left(\frac{1}{a\log aq}\right), \tag{10}$$

$$g_q(b) - g_q(b+\Delta) > \frac{1}{3}(\log aq)^3, \tag{11}$$

$$g_q(b) - g_q(b-\Delta) > \frac{1}{3}(\log aq)^3 \tag{12}$$

where $\Delta = a^{1/2}\log aq$.

Proof. The proof follows the similar argument of Matsuoka's Lemma 3 (see [5], p. 52). ∎

3. Proof of Proposition 1. We expand $L(s,\chi)$ into the Taylor series at $s=1$:

$$L(s,\chi) = \sum_{n=0}^{\infty}\frac{L^{(n)}(1,\chi)}{n!}(s-1)^n.$$

Putting $s = 1-z$, we have

$$(-1)^n L^{(n)}(1,\chi) = \frac{n!}{2\pi i}\int_C \frac{1}{z^{n+1}}L(1-z,\chi)\,dz$$

where C is the counter-clockwise circular path with center $z=0$ and radius $\varrho>0$. Next we deform C into C', the rectangular path with corners $(c\pm Ri), (-R\pm Ri)$ where R and c are positive numbers to be chosen later. If $n-c+1/2>0$, the contribution of the horizontal segments and the left side of the rectangle tend to 0 as $R\to\infty$. As a result,

$$(13)\quad \begin{aligned}&(-1)^n L^{(n)}(1,\chi)\\ &\quad=\frac{n!}{2\pi i}\int\limits_{E_1}\frac{1}{z^{n+1}}L(1-z,\chi)\,dz+\frac{n!}{2\pi i}\int\limits_{E_2}\frac{1}{z^{n+1}}L(1-z,\chi)\,dz\\ &\quad=H_1+H_2\end{aligned}$$

where E_1 is a vertical path from $c+0i$ to $c+\infty i$ and E_2 is a path from $c-\infty i$ to $c-0i$. Now we use the functional equation. Suppose first that $\chi(-1)=1$. Then

$$H_1=\frac{n!}{2\pi i}\int\limits_{E_1}\frac{1}{z^{n+1}}\cdot\frac{\tau(\chi)}{q}\left(\frac{q}{2\pi}\right)^z 2\cos\frac{1}{2}\pi z\cdot\Gamma(z)L(z,\overline{\chi})\,dz.$$

Writing $\cos\frac{1}{2}\pi z=(e^{\frac{1}{2}\pi zi}+e^{-\frac{1}{2}\pi zi})/2$, we will see later that the contribution from the term $e^{\frac{1}{2}\pi zi}$ is an error term. Next we have $L(c+yi,\overline{\chi})=1+\sum_{k=2}^{\infty}\overline{\chi}(k)/k^{c+y}$. The contribution from $\sum_{k=2}^{\infty}\overline{\chi}(k)/k^{c+y}$ is small, since we will take the real part c of the path E_1 large. Hence we expect the main term to be

$$\frac{n!}{2\pi i}\cdot\frac{\tau(\chi)}{q}\int\limits_{E_1}\frac{1}{z^{n+1}}\left(\frac{q}{2\pi}\right)^z e^{-\frac{1}{2}\pi zi}\Gamma(z)\,dz.$$

We write the integrand as $e^{\Phi_q(z)}$ where

$$\Phi_q(z)=z\log q-(n+1)\log z-z\log 2\pi i+\log\Gamma(z).$$

The saddle point $a+bi$ of $e^{\Phi_q(z)}$, and of $\Phi_q(z)$, is estimated in Lemma 1. Henceforth we set $c=a$.

Treating H_2 similarly, we see that the main term in the estimate for H_2 is given by

$$\overline{\frac{n!}{2\pi}\int\limits_0^{\infty}e^{\Phi_q(a+yi)}\,dy}.$$

Hence it follows that

$$(14)\qquad (-1)^n L^{(n)}(1,\chi)=\frac{n!}{\pi}\cdot\frac{\tau(\chi)}{q}\left(\Re\int\limits_0^{\infty}e^{\Phi_q(a+yi)}\,dy+V_1\right)$$

where V_1 is an error term which we will estimate later.

Next we split the integral in (14) into

$$\Re\int_0^\infty e^{\Phi_q(a+yi)}dy = \Re\int_{b-\Delta}^{b+\Delta} e^{\Phi_q(a+yi)}\,dy + V_2$$

where V_2 is the integral along the remainder of the path. We take $\Delta = a^{1/2}\log aq$, which will ensure that V_2 is small. From now on, we denote by V_i $(i = 1, 2, \ldots)$ the expected error terms.

By using Taylor's formula, there exists η $(b-\Delta < \eta < b+\Delta)$ such that

$$\Phi_q(a+yi) = \Phi_q(a+bi) - \frac{W}{2}(y-b)^2 + O\biggl(\lim_{y\to\eta}\frac{d^3}{dy^3}\Phi_q(a+yi)\Delta^3\biggr),$$

where W is defined by

$$W = -\lim_{y\to b}\frac{d^2}{d^2y}\Phi_q(a+yi) = \lim_{z\to a+bi}\frac{d^2}{d^2z}\Phi_q(x+yi).$$

Here the second equality is justified since

$$\frac{d}{dz}\Phi_q(z) = -i\frac{\partial}{\partial y}\Re\Phi_q(x+yi) + \frac{\partial}{\partial y}\Im\Phi_q(x+yi)$$

by the Cauchy–Riemann equations. (We do not use a notation like $\Phi''(a+bi)$ to avoid confusion.) Then it follows that

$$\text{(15)}\qquad \exp\biggl[O\biggl(\lim_{y\to\eta}\frac{d^3}{dy^3}\Phi_q(a+yi)\Delta^3\biggr)\biggr] = 1 + O\biggl(\frac{\log aq}{a^2}\Delta^3\biggr) = 1 + O\biggl(\frac{\log^4 aq}{a^{1/2}}\biggr).$$

Hence we have

$$\text{(16)}\qquad \Re\int_{b-\Delta}^{b+\Delta} e^{\Phi_q(a+yi)}\,dy = \Re\biggl(e^{\Phi_q(a+bi)}\int_{b-\Delta}^{b+\Delta} e^{-(W/2)(y-b)^2}\,dy + V_3\biggr),$$

where V_3 is an error term to be treated later. Finally we write the integral in (16) as

$$\Re e^{\Phi_q(a+bi)}\biggl(\int_{-\infty}^{\infty} - \int_{b+\Delta}^{\infty} - \int_{-\infty}^{b-\Delta} e^{-(W/2)(y-b)^2}\,dy\biggr) = M + V_4 + V_5$$

where M is the desired main term

$$M = \Re\biggl(e^{\Phi_q(a+bi)}\sqrt{\frac{2}{W}}\pi^{1/2}\biggr) = \frac{\sqrt{2\pi}\,e^{g_q(b)}}{|W|^{1/2}}\cos\biggl(f_q(b) - \frac{1}{2}\arg W\biggr).$$

By (9), (10) and Lemma 2 we have $\arg W = O(\log^{-1} n)$. We denote $1/|W|^{1/2}$

by W_n. Then it is easily seen that

$$W_n = \frac{n^{1/2}}{\log n}\left[1 + O\left(\left(\frac{\log\log n}{\log n}\right)^{1/4}\right)\right].$$

We write $F_q(n)$ instead of $f_q(b)$ to indicate that $f_q(b)$ is a function of n. We then have

$$M = \sqrt{2\pi}\, e^{g_q(b)} W_n \cdot [\cos F_q(n) + O(\log^{-1} n)]. \tag{17}$$

It remains to estimate the error terms V_i. The term V_1 can be split into $V_1 = \sum_{j=1}^{2} \frac{1}{2}(I_j + I_j')$ where

$$I_1 = \int_0^\infty \frac{1}{(a+yi)^{n+1}}\left(\frac{q}{2\pi}\right)^{a+yi} e^{\frac{1}{2}\pi(a+yi)i}\Gamma(a+yi)L(a+yi,\overline{\chi})\,dy,$$

$$I_2 = \int_0^\infty \frac{1}{(a+yi)^{n+1}}\left(\frac{q}{2\pi}\right)^{a+yi} e^{-\frac{1}{2}\pi(a+yi)i}\Gamma(a+yi)\sum_{k=2}^{\infty}\overline{\chi}(k)/k^{a+yi}\,dy,$$

$$I_1' = \int_{-\infty}^0 \frac{1}{(a+yi)^{n+1}}\left(\frac{q}{2\pi}\right)^{a+yi} e^{-\frac{1}{2}\pi(a+yi)i}\Gamma(a+yi)L(a+yi,\overline{\chi})\,dy,$$

$$I_2' = \int_{-\infty}^0 \frac{1}{(a+yi)^{n+1}}\left(\frac{q}{2\pi}\right)^{a+yi} e^{\frac{1}{2}\pi(a+yi)i}\Gamma(a+yi)\sum_{k=2}^{\infty}\overline{\chi}(k)/k^{a+yi}\,dy.$$

For I_1 we have

$$I_1 \ll \int_0^\infty \frac{1}{(a^2+y^2)^{(1/2)(n-a+3/2)}}\left(\frac{q}{2\pi}\right)^{a} e^{-\frac{1}{2}\pi y - y\arg(a+yi)}\,dy$$

by using Stirling's formula for $\log\Gamma(z)$ ([9], p. 251). The right-hand side is

$$\ll q^a(2\pi e)^{-a}a^{-(n-a+1/2)} \ll \exp[g_q(0) + \log a],$$

since $g_q(0) = a\log q - (n-a+3/2)\log a - a(\log 2\pi + 1) + O(1)$. Thus we have

$$I_1 \ll e^{g_q(b)}\exp\left[-\frac{(\log aq)^3}{3} + \log a\right] \ll e^{g_q(b)}\left(\frac{\log n}{n}\right)^{1/3}$$

by using Lemma 3.

For I_2 it follows that

$$I_2 \ll e^{g_q(b)}\left(\frac{\log n}{n}\right)^{1/3} e^{-\frac{1}{10}\cdot\frac{n}{\log n}}.$$

We have the same estimates for I_1' and I_2'. Summing up, we have

$$V_1 \ll e^{g_q(b)}\left(\frac{\log n}{n}\right)^{1/3}.$$

For V_3, (15) and Lemma 3 give the bound

$$V_3 \ll e^{g_q(b)} \int_{b-\Delta}^{b+\Delta} e^{\frac{1}{2}g_q''(b)(y-b)^2} \frac{\log^4 aq}{a^{1/2}}\,dy \ll e^{g_q(b)} \Delta \frac{\log^4 aq}{a^{1/2}} = e^{g_q(b)} \log^5 aq.$$

Hence it follows that

$$V_3 \ll e^{g_q(b)} \log^5 n.$$

The terms V_2, V_4 and V_5 are $\ll e^{g_q(b)}((\log n)/n)^{1/3}$. We omit the details, since the proofs are straightforward. Hence we see that

$$\sum_{i=1}^{5} V_i \ll e^{g_q(b)} \log^5 n. \tag{18}$$

Combining (17) and (18), we obtain

$$\begin{aligned}(-1)^n L^{(n)}(1,\chi) &= \frac{n!}{\pi} \cdot \frac{\tau(\chi)}{q}\Big(M + \sum_{i=1}^{5} V_i\Big)\\ &= \frac{n!}{\pi} \cdot \frac{\tau(\chi)}{q}\Big(\sqrt{2\pi}\, e^{g_q(b)} W_n \cdot \Big[\cos F_q(n) + O\Big(\frac{1}{\log n}\Big)\Big] + \sum_{i=1}^{5} V_i\Big)\\ &= \sqrt{\frac{2}{\pi}} \cdot \frac{\tau(\chi)}{q} n! e^{g_q(b)} W_n \cdot [\cos F_q(n) + E_q(n)]\end{aligned}$$

where $E_q(n) \ll \log^{-1} n$. Proposition 1 for $\chi(-1) = 1$ easily follows from this formula by using Stirling's formula for $n!$.

In the case $\chi(-1) = -1$ the result follows by a similar argument. ■

4. Proof of Proposition 2. Proposition 2 is equivalent to Matsuoka's Theorem 1 in [6], p. 281, and its proof is similar. The proof depends on the following lemma:

LEMMA 4. *Let c be a positive constant, and let m be a sufficiently large positive integer so that $m - c\log m > e^q$. Then for all n with $|n-m| < c\log m$, we have*

$$F_q(n) = F_q(m) - \frac{1}{2}\pi\frac{n-m}{\log m} + O\Big(\frac{\log\log m}{\log m}\Big)$$

where the O-constant depends on c and q.

Proof. This can be proved as in [6], p. 281, since we may regard the conductor q as a constant. ■

5. Proof of theorems. We first show the following:

LEMMA 5. *For arbitrary $\mu > 0$ and any number λ satisfying $0 < \lambda < 1$ there exists an $m_0(\mu, \lambda)$ such that for all $m > m_0$,*

$$S_\mu^+(m + 4\log m) - S_\mu^+(m) = 2\log m + O(\log^{1-\lambda} m), \tag{19}$$

$$S_\mu^-(m + 4\log m) - S_\mu^-(m) = 2\log m + O(\log^{1-\lambda} m). \tag{20}$$

Proof. Denote by $[x]$ the greatest integer not exceeding x and $\{x\} = x - [x]$. Using Proposition 2 and setting $\{F_q(m)/(2\pi)\} = \theta_m$, we have

$$\begin{aligned}&(-1)^n L^{(n)}(1,\chi)\\ &\quad = i^\alpha \frac{\tau(\chi)}{q} q^{n/\log n} e^{n\log\log n + H_q(n)} \cdot \left[P\left(2\pi\left(\theta_m - \frac{1}{4}\cdot\frac{n-m}{\log m}\right)\right) + E_{q,\alpha}^*(m)\right]\end{aligned}$$

for all n in the interval $(m, m + 4\log m]$. Setting $n - m = k$, we have

$$\theta_m - \frac{1}{4}\cdot\frac{n-m}{\log m} = \theta_m - \frac{1}{4}\cdot\frac{1}{\log m}k \quad (k = 1, 2, \ldots, [4\log m]).$$

Suppose first $\chi(-1) = 1$. Then

$$P\left(2\pi\left(\theta_m - \frac{1}{4}\cdot\frac{n-m}{\log m}\right)\right) = \cos\left(2\pi\left\{\theta_m - \frac{1}{4}\cdot\frac{1}{\log m}k\right\}\right).$$

The right-hand side is greater than $\sin(\pi\varepsilon/2)$, provided

$$\begin{gathered} 0 \le \left\{\theta_m - \frac{1}{4}\cdot\frac{1}{\log m}k\right\} < \frac{1}{4} - \frac{1}{4}\varepsilon \quad \text{or} \\ \frac{3}{4} + \frac{1}{4}\varepsilon < \left\{\theta_m - \frac{1}{4}\cdot\frac{1}{\log m}k\right\} < 1. \end{gathered} \tag{21}$$

If we take $\varepsilon = \log^{-\lambda} m$ where λ is fixed number satisfying $0 < \lambda < 1$, then $\sin\frac{1}{2}\pi\varepsilon > \log^{-\lambda} m$.

The number of integers $k = 1, 2, \ldots, [4\log m]$ satisfying

$$\cos\left(2\pi\left(\theta_m - \frac{1}{4}\cdot\frac{k}{\log m}\right)\right) > \frac{1}{\log^\lambda m}$$

is $2\log m - 2\log^{1-\lambda} m + O(1)$. Thus

$$\left|\arctan\left(\frac{\Im E_{q,\alpha}^*(m)}{\cos\left(2\pi\left\{\theta_m - \frac{k}{4\log m}\right\}\right) + \Re E_{q,\alpha}^*(m)}\right)\right| \le A\frac{\log\log m}{\log^{1-\lambda} m}$$

for k satisfying (21), where A is a constant depending only on q. Hence we have

$$S_\mu^+(m + 4\log m) - S_\mu^+(m) \ge 2\log m - 2\log^{1-\lambda} m + O(1) \tag{22}$$

if we choose m large enough such that

$$A\frac{\log\log m}{\log^{1-\lambda} m} < \mu.$$

Analogously, we obtain

$$S_\mu^-(m+4\log m) - S_\mu^-(m) \ge 2\log m - 2\log^{1-\lambda} m + O(1). \tag{23}$$

Noting that

$$\begin{aligned}[4\log m] - (S_\mu^{\mp}(m+4\log m) - S_\mu^{\mp}(m)) \\ \ge S_\mu^{\pm}(m+4\log m) - S_\mu^{\pm}(m),\end{aligned} \tag{24}$$

we see that (19) and (20) follow from (22)–(24). ∎

Proof of Theorem 1. Set $N_0 = N$, $N_1 + 4\log N_1 = N_0, \ldots, N_i + 4\log N_i = N_{i-1}$. Then it follows from Lemma 5 that

$$S_\mu^{\pm}(N_{i-1}) - S_\mu^{\pm}(N_i) = 2\log N_i + O(\log^{1-\lambda} N_i)$$

provided N_i is sufficiently large. For sufficiently large l, we have

$$N_l = N^{1/2} + A(N)\log N$$

where $A(N)$ is a function of N satisfying $0 \le A(N) \le 1$ and therefore

$$\begin{aligned}&\sum_{i=1}^{l}(S_\mu^{\pm}(N_{i-1}) - S_\mu^{\pm}(N_i)) \\ &\qquad = \frac{1}{2}(N - N^{1/2} - A(N)\log N) + \sum_{i=1}^{l} O(\log^{1-\lambda} N_i).\end{aligned}$$

We see that

$$\sum_{i=1}^{l} O(\log^{1-\lambda} N_i) \ll \log^{-\lambda} N_l \sum_{i=1}^{l} \log N_i \le A_1\frac{N}{\log^{\lambda} N}$$

where A_1 is an absolute constant since N_l is chosen sufficiently large, which depends on λ and μ. Hence Theorem 1 for the case $\chi(-1) = 1$ follows immediately.

The case $\chi(-1) = -1$ is proved likewise. ∎

Proof of Theorems 2 and 3. Theorem 2 follows from Proposition 1. To show Theorem 3, we take $n = m + [4\theta_m \log m] - \alpha\log m$. Then by Proposition 2,

$$\begin{aligned}&(-1)^n L^{(n)}(1,\chi) \\ &\qquad = i^\alpha q^{n/\log n}\frac{\tau(\chi)}{q}e^{n\log\log n + H_q(n)} \cdot \left[1 + O\left(\frac{1}{\log^2 m}\right) + E_{q,\alpha}^*(m)\right].\end{aligned}$$

The expression in brackets is close to 1 for infinitely many m. Hence, Theorem 3 follows. ∎

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Graduate School of Science and Technology
Niigata University
Niigata, 950-2181, Japan
E-mail: isikawah@qed.sc.niigata-u.ac.jp

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