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On the Andrianov-type identity for power series attached to Jacobi forms and its application

by

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1. Introduction. The theory of Jacobi forms (that is, automorphic forms on the Jacobi group and its generalizations to higher degree) has been studied by several authors (cf. [7, 29, 20, 21, 11]). In particular, Shintani introduced the standard L-function attached to a Jacobi form of arbitrary degree, and afterward Murase derived in a series of papers [20, 21] its meromorphic continuation and functional equation by making use of its integral expression. Moreover, Murase and Sugano derived in [22] an expression of the standard L-function attached to a Jacobi form in terms of a power series generated by eigenvalues of Hecke operators. In this paper, we derive a local expression of the standard L-function attached to a Jacobi form in terms of a power series related to its Fourier coefficients. This can be regarded as an analogue of Andrianov's identity in [1] for Siegel modular forms. As an application, we shall also prove a rationality theorem for a formal power series related to a polynomial appearing in the theory of local densities of quadratic forms, which is very similar to the result obtained in [6] by Böcherer and Sato.

Let us describe our main results precisely. Let p be an arbitrary rational prime. For any non-zero element a of the field $\mathbb{Q}_p$ of p-adic numbers, we put

$$\chi_p(a) = \begin{cases} 1 & \text{if } \mathbb{Q}_p(a^{1/2}) = \mathbb{Q}_p, \\ -1 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is unramified}, \\ 0 & \text{if } \mathbb{Q}_p(a^{1/2})/\mathbb{Q}_p \text{ is ramified}. \end{cases}$$

Let n be a positive even integer. For each non-degenerate half-integral symmetric matrix B' of degree n over the ring $\mathbb{Z}_p$ of p-adic integers, we define

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the local Siegel series with complex parameter s by

$$b_p(B';s) := \sum_{R\in \mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(-B'R))\,\mu_p(R)^{-s},$$

where $\mu_p(R) = [\mathbb{Z}_p^n R + \mathbb{Z}_p^n : \mathbb{Z}_p^n]$, and $\mathbf{e}_p$ is the standard additive character of $\mathbb{Q}_p$. It is well-known that such singular series appear naturally in the study of Fourier coefficients of Siegel Eisenstein series of degree n and there exists a unique polynomial $F_p(B';X)$ in one variable X such that

$$b_p(B';s) = \frac{(1-p^{-s})\prod_{i=1}^{n/2}(1-p^{2i-2s})}{1-\xi_p(B')p^{n/2-s}} F_p(B';p^{-s}),$$

where $\xi_p(B') = \chi_p((-1)^{n/2}\det(2B'))$ (cf. [18]). Let B be a non-degenerate symmetric matrix of degree $n-1$ over a subring R of $\mathbb{Z}_p$ satisfying the condition

(1.1) $(B + {}^t r_B r_B)/4$ is a half-integral symmetric matrix over R for some $r_B \in R^{n-1}$.

Then we can associate B with a non-degenerate half-integral symmetric matrix

$$B^{(1)} = \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B + {}^t r_B r_B)/4 \end{pmatrix}$$

of degree n over R. Here we easily see that the vector r_B is uniquely determined by B modulo $2R^{n-1}$, and therefore $B^{(1)}$ is uniquely determined by B up to $\mathrm{GL}_n(R)$-equivalence. For such a B over $\mathbb{Z}_p$, we define a polynomial $F_p^{(1)}(B;X)$ in X by

$$F_p^{(1)}(B;X) := F_p(B^{(1)};X)$$

and put

$$G_p^{(1)}(B;X) = \sum_{D\in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathrm{M}_{n-1}(\mathbb{Z}_p)\cap \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D) F_p^{(1)}(B[D^{-1}];X)(p^n X^2)^{\mathrm{ord}_p(\det D)},$$

where $\pi_p(D)$ denotes the generalized local Möbius function, that is, $\pi_p(D) = (-1)^i p^{i(i-1)/2}$ or 0 according as $D \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\left(\begin{array}{c|c} \mathbf{1}_{n-1-i} & \\ \hline & p\mathbf{1}_i \end{array}\right)\mathrm{GL}_{n-1}(\mathbb{Z}_p)$ for some $0 \le i \le n-1$ or not. We note that these polynomials do not depend on the choice of r_B. In addition, we also define a polynomial $\mathbf{B}_p^{(1)}(B;t)$ in one variable t by

$$\mathbf{B}_p^{(1)}(B;t) := \frac{(1-\xi_p(B^{(1)})p^{-(n-1)/2}t)\prod_{i=1}^{n/2-1}(1-p^{-2i+1}t^2)}{G_p^{(1)}(B;p^{-n+1/2}t)}.$$

On the other hand, for any positive even integers k and n, let ϕ be a Jacobi form of weight k and of index 1 with respect to the Jacobi modular group Γ^J_{n-1} of degree $n-1$, and $\sigma(\phi)$ a Siegel modular form of weight $k-1/2$ with respect to the congruence subgroup $\Gamma_0^{(n-1)}(4)$ of the Siegel modular group of degree $n-1$ corresponding to ϕ under the Eichler–Zagier–Ibukiyama correspondence σ (cf. §§2.3 and 2.4 below). Let $\mathbf{D}_p^{(n-1)}(\mathbb{Z})$ be the set of all $(n-1)\times(n-1)$ matrices with entries in $\mathbb{Z}$ whose determinant is a power of p. For each positive definite half-integral symmetric matrix B of degree $n-1$ over $\mathbb{Z}$, we define a power series $\widetilde{G}_{\phi,p}(B;t)$ in t by

$$\widetilde{G}_{\phi,p}(B;t) := \sum_{D\in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z})} \pi_p(D)C_{\sigma(\phi)}(B[D^{-1}])(p^k t)^{\mathrm{ord}_p(\det D)},$$

where $C_{\sigma(\phi)}(B)$ denotes the Bth Fourier coefficient of $\sigma(\phi)$. Then our first main result is the following:

Theorem 1.1 (cf. Theorem 3.1 below). *Suppose that ϕ is a Hecke eigenform, that is, a common eigenfunction of all Hecke operators, whose Satake p-parameter is of the form $(\chi_\phi^{(1)}(p),\dots,\chi_\phi^{(n-1)}(p))$ up to the action of the Weyl group. Then, for each positive definite half-integral symmetric matrix B of degree $n-1$ over $\mathbb{Z}$ satisfying the condition* (1.1), *we have*

$$\frac{\mathbf{B}_p^{(1)}(B;p^{n-1/2}t)\widetilde{G}_{\phi,p}(B;t)}{\prod_{i=1}^{n-1}(1-\chi_\phi^{(i)}(p)p^{n-1/2}t)(1-\chi_\phi^{(i)}(p)^{-1}p^{n-1/2}t)} = \sum_{W\in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z})} C_{\sigma(\phi)}(B[W])p^{-(k-n-1)\,\mathrm{ord}_p(\det W)}t^{\mathrm{ord}_p(\det W)}.$$

This can be regarded as an analogue of the so-called Andrianov identity, which was obtained in the study of standard L-functions attached to Siegel modular forms of integral weight (cf. [1], see also [5]). We also note that the above identity for $p\neq 2$ can be derived from a similar result for Siegel modular forms of half-integral weight due to Shimura and Zhuravlev (cf. Corollary 5.2 in [25], see also Theorem 1.1 in [28]). However, we cannot use their results to prove the above identity for $p=2$.

Next, we explain an application of the above result to the rationality of a certain formal power series related to the polynomial $F_p^{(1)}(B;X)$. For each non-degenerate half-integral symmetric matrix B of degree $n-1$ over $\mathbb{Z}_p$ satisfying the condition (1.1), we define a Laurent polynomial $\widetilde{F}_p^{(1)}(B;X)$ in X by

$$\widetilde{F}_p^{(1)}(B;X) := X^{-\mathrm{ord}_p((-1)^{n/2}\det(2B^{(1)})\mathfrak{d}(B^{(1)})^{-1})/2}F_p^{(1)}(B;p^{-(n+1)/2}X),$$

and put

$$\widetilde{G}_p^{(1)}(B;X,t) = \sum_{D\in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathrm{M}_{n-1}(\mathbb{Z}_p)\cap \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \pi_p(D)\widetilde{F}_p^{(1)}(B[D^{-1}];X)t^{\mathrm{ord}_p(\det D)},$$

where $\mathfrak{d}(B^{(1)})$ is the discriminant of the field extension

$$\mathbb{Q}_p(\sqrt{(-1)^{n/2}\det(2B^{(1)})})/\mathbb{Q}_p.$$

We note that the functional equation $\widetilde{F}_p^{(1)}(B;X) = \widetilde{F}_p^{(1)}(B;X^{-1})$ holds (cf. [12]). Thus $\widetilde{F}_p^{(1)}(B;X)$ is a polynomial in $X+X^{-1}$, and therefore $\widetilde{G}_p^{(1)}(B;X,t)$ is a polynomial in $X+X^{-1}$ and t. Now we put

$$R_p^{(1)}(B;X,t) = \sum_{W\in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathrm{M}_{n-1}(\mathbb{Z}_p)\cap \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \widetilde{F}_p^{(1)}(B[W];X)t^{\mathrm{ord}_p(\det W)}.$$

By applying Theorem 1.1 to the Jacobi Eisenstein series, we obtain the following:

THEOREM 1.2 (cf. Theorem 3.4 below). *Let n be a positive even integer. If B is a non-degenerate half-integral symmetric matrix of degree $n-1$ over $\mathbb{Z}_p$ satisfying the condition* (1.1), *then*

$$R_p^{(1)}(B;X,t) = \frac{\mathbf{B}_p^{(1)}(B;p^{n/2-1}t)\widetilde{G}_p^{(1)}(B;X,t)}{\prod_{j=1}^{n-1}(1-p^{j-1}Xt)(1-p^{j-1}X^{-1}t)}.$$

We note that Böcherer and Sato ([6]) obtained a similar identity for a half-integral symmetric matrix of degree n. The above identity will play an important role in proving a conjecture on the period of the Ikeda lift proposed in [13] by Ikeda (cf. [16, 17]).

Notation. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}\,x)$ for any $x \in \mathbb{C}$. For each rational prime p, let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ be the field of p-adic rational numbers and the ring of p-adic integers, respectively. We denote by ord_p the valuation of $\mathbb{Q}_p$ normalized as $\mathrm{ord}_p(p) = 1$, and by $\mathbf{e}_p$ the continuous additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for any $x \in \mathbb{Q}$, which will be called the *standard additive character* of $\mathbb{Q}_p$.

Let R be a commutative ring. We denote by $R^\times$ the unit group of R, and by $\mathrm{M}_{m,n}(R)$ the set of $m\times n$ matrices with entries in R. In particular, we write $\mathrm{M}_n(R) = \mathrm{M}_{n,n}(R)$ and $R^n = \mathrm{M}_{1,n}(R)$. We denote by $\mathbf{1}_n, \mathbf{0}_n \in \mathrm{M}_n(R)$ the unit matrix and the zero matrix of degree n, respectively. We put $\mathrm{GL}_n(R) = \{U \in \mathrm{M}_n(R) \mid \det U \in R^\times\}$, where $\det U$ is the determinant of U. For $X \in \mathrm{M}_{m,n}(R)$ and $A \in \mathrm{M}_m(R)$, we write $A[X] = {}^tXAX \in$

$\mathrm{M}_n(R)$, where ${}^tX$ denotes the transpose of X. Let $\mathrm{Sym}_n(R)$ be the set of symmetric matrices of degree n with entries in R. If R is an integral domain of characteristic different from 2, let $\mathrm{Sym}_n^*(R)$ be the set of all half-integral symmetric matrices of degree n over R, that is,

$$\mathrm{Sym}_n^*(R) := \left\{ T = (t_{ij}) \in \mathrm{Sym}_n(\mathrm{Frac}(R)) \,\middle|\, \begin{matrix} t_{ii} \in R & (1 \leq i \leq n), \\ 2t_{ij} \in R & (1 \leq i \neq j \leq n) \end{matrix} \right\},$$

where $\mathrm{Frac}(R)$ is the field of fractions of R. In addition, for any subset $\mathcal{S}$ of $\mathrm{Sym}_n(R)$, we denote by $\mathcal{S}^\times$ the subset of $\mathcal{S}$ consisting of all non-degenerate elements in $\mathcal{S}$. In particular, if R is a subring of $\mathbb{R}$, we denote by $\mathcal{S}_{>0}$ (resp. $\mathcal{S}_{\geq 0}$) the subset of $\mathcal{S}$ consisting of all positive definite (resp. semi-positive definite) matrices. For any commutative ring R, the group $\mathrm{GL}_n(R)$ acts on $\mathrm{Sym}_n(R)$ in the following way:

$$\mathrm{GL}_n(R) \times \mathrm{Sym}_n(R) \ni (U, A) \mapsto A[U] \in \mathrm{Sym}_n(R).$$

For a subgroup G of $\mathrm{GL}_n(R)$, and a subset $\mathcal{S}$ of $\mathrm{Sym}_n(R)$ stable under the action of G, we denote by $\mathcal{S}/G$ the set of G-orbits in $\mathcal{S}$. For a subring R' of R we define an equivalence relation on $\mathrm{Sym}_n(R)$ as follows: for any $A_1, A_2 \in \mathrm{Sym}_n(R)$,

$$A_1 \sim_{R'} A_2 \overset{\text{def}}{\Longleftrightarrow} A_2 = A_1[U] \text{ for some } U \in \mathrm{GL}_n(R'). \tag{1.2}$$

For square matrices $X \in \mathrm{M}_m(R)$ and $Y \in \mathrm{M}_n(R)$, we write $X \perp Y = \begin{pmatrix} X & \\ & Y \end{pmatrix}$. In particular, we often write $x \perp Y$ instead of $(x) \perp Y$ for any $x \in R$. We can simply write the diagonal matrix with entries $x_1, \ldots, x_n$ in R by $x_1 \perp \cdots \perp x_n$.

2. Preliminaries

2.1. Siegel modular forms of integral weight. Let $G_n(\mathbb{R})$ be the real symplectic group of degree n, that is,

$$G_n(\mathbb{R}) := \mathrm{Sp}_n(\mathbb{R}) = \{ M \in \mathrm{GL}_{2n}(\mathbb{R}) \mid {}^tMJ_nM = J_n \},$$

where $J_n = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}$. For any $S \in \mathrm{Sym}_n(\mathbb{R})$ and $A \in \mathrm{GL}_n(\mathbb{R})$, we put $\mathbf{n}_n(S) = \begin{pmatrix} \mathbf{1}_n & S \\ \mathbf{0}_n & \mathbf{1}_n \end{pmatrix}$ and $\mathbf{d}_n(A) = \begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix}$, respectively. We easily see that the elements $\mathbf{n}_n(S), \mathbf{d}_n(A)$ and J_n generate $G_n(\mathbb{R})$. The discrete subgroup $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z}) = G_n(\mathbb{R}) \cap \mathrm{M}_{2n}(\mathbb{Z})$ of $G_n(\mathbb{R})$ is called the *Siegel modular group* of degree n. For any $N \in \mathbb{Z}_{>0}$, we denote by $\Gamma_0^{(n)}(N)$ the congruence subgroup of Γ_n defined by

$$\Gamma_0^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv \mathbf{0}_n \pmod{N} \right\}.$$

We denote the Siegel upper half-space of degree n by $\mathfrak{H}_n$, that is,

$$\mathfrak{H}_n := \{ Z = X + \sqrt{-1}\,Y \in \mathrm{Sym}_n(\mathbb{C}) \mid Y > 0 \text{ (positive definite)} \}.$$

For any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n(\mathbb{R})$ and $Z \in \mathfrak{H}_n$, we easily see that $j(M, Z) := CZ + D \in \mathrm{GL}_n(\mathbb{C})$ and we put $M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}$. As is well-known, this defines a transitive action of $G_n(\mathbb{R})$ on $\mathfrak{H}_n$.

For any $k \in \mathbb{Z}$, a $\mathbb{C}$-valued holomorphic function $F(Z)$ on $\mathfrak{H}_n$ is called a (*holomorphic*) *Siegel modular form* of degree n and weight k if it satisfies the following two conditions:

(i) $F(M\langle Z\rangle) = \det(j(M, Z))^k F(Z)$ for any $M \in \Gamma_n$;

(ii) F possesses a Fourier expansion of the form

$$F(Z) = \sum_{B \in \mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} A_F(B)\mathbf{e}(\mathrm{tr}(BZ)),$$

where $\mathrm{tr}(*)$ denotes the trace of a matrix.

In particular, a Siegel modular form F is called a *cusp form* if it satisfies the stronger condition $A_F(B) = 0$ unless $B > 0$ (positive definite). We denote by $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ the $\mathbb{C}$-vector spaces consisting of all Siegel modular forms and Siegel cusp forms of degree n and weight k, respectively. For further details on Siegel modular forms of integral weight, see [1] or [8].

2.2. Review of the theory of Jacobi forms of higher degree. In this subsection, we introduce some basic facts on Jacobi forms of integral weight whose index is a scalar. For further details on Jacobi forms, see [7, 20, 21, 29].

2.2.1. *Jacobi group and complex analytic Jacobi forms.* Let $G_n = \mathrm{Sp}_n(\mathbb{Q}) = \{M \in \mathrm{GL}_{2n}(\mathbb{Q}) \mid {}^t M J_n M = J_n\}$; we naturally identify G_n with its image under the natural inclusion

$$G_n \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto [M] := \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{array}\right) \in G_{n+1}.$$

We denote by H_n the Heisenberg group consisting of all elements of the form

$$[(\lambda, \mu), \kappa] := \left(\begin{array}{cc|cc} 1 & 0 & \kappa & \mu \\ 0 & \mathbf{1}_n & {}^t\mu & \mathbf{0}_n \\ \hline & & 1 & 0 \\ & & 0 & \mathbf{1}_n \end{array}\right) \left(\begin{array}{cc|cc} 1 & \lambda & & \\ 0 & \mathbf{1}_n & & \\ \hline & & 1 & 0 \\ & & -{}^t\lambda & \mathbf{1}_n \end{array}\right)$$

for some $(\lambda, \mu) \in \mathbb{Q}^n \oplus \mathbb{Q}^n$ and $\kappa \in \mathbb{Q}$. Then

$$G_n^J := \{[(\lambda, \mu), \kappa] \cdot [M] \in G_{n+1} \mid [(\lambda, \mu), \kappa] \in H_n,\ M \in G_n\}$$

is a $\mathbb{Q}$-algebraic subgroup of G_{n+1}; it is called the *Jacobi group* of degree n. We note that the Jacobi group G_n^J is a semi-direct product $G_n \ltimes H_n$, and forms a connected non-reductive $\mathbb{Q}$-algebraic group with the center

$$Z_n^J = \{[(0,0),\kappa] \mid \kappa \in \mathbb{Q}\}.$$

It is easy to see the following:

LEMMA 2.1. *For each* $[(\lambda,\mu),\kappa], [(\lambda',\mu'),\kappa'] \in H_n$, *and* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n$, *we have*

$$(2.1) \quad [(\lambda,\mu),\kappa]\cdot[(\lambda',\mu'),\kappa'] = [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+2\lambda\,{}^t\mu'],$$

$$(2.2) \quad [(\lambda,\mu),\kappa]\cdot[M] = [M]\cdot[(\lambda A+\mu C,\lambda B+\mu D),\kappa+(\lambda A+\mu C)\,{}^t(\lambda B+\mu D)-\lambda\,{}^t\mu].$$

Proof. Since it is an easy calculation, we omit the proof. ∎

According to the action of $G_{n+1}(\mathbb{R}) = \mathrm{Sp}_{n+1}(\mathbb{R})$ on the Siegel upper half-space $\mathfrak{H}_{n+1}$, the group $G_n^J(\mathbb{R})$ of real points of G_n^J naturally acts on the space $\mathfrak{H}_n \times \mathbb{C}^n$ as follows. For each $g = [(\lambda,\mu),\kappa]\cdot[M] \in G_n^J(\mathbb{R})$ with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n(\mathbb{R})$ and $(\tau,z) \in \mathfrak{H}_n \times \mathbb{C}^n$, we put

$$g\langle\tau,z\rangle := (M\langle\tau\rangle, z(C\tau+D)^{-1}+\lambda M\langle\tau\rangle+\mu).$$

We easily see that this action is transitive and the stabilizer of the point $(\sqrt{-1}\,\mathbf{1}_n,0) \in \mathfrak{H}_n \times \mathbb{C}^n$ in $G_n^J(\mathbb{R})$ coincides with $Z_n^J(\mathbb{R})\cdot K_\infty$, where K_∞ is the stabilizer of $\sqrt{-1}\,\mathbf{1}_n \in \mathfrak{H}_n$ in $G_n(\mathbb{R})$, that is,

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G_n(\mathbb{R}) \,\middle|\, A+\sqrt{-1}\,B \text{ is unitary} \right\}.$$

The map $g \mapsto g\langle\sqrt{-1}\,\mathbf{1}_n,0\rangle$ induces a diffeomorphism of the quotient $G_n^J(\mathbb{R})/(Z_n^J(\mathbb{R})\cdot K_\infty)$ onto $\mathfrak{H}_n \times \mathbb{C}^n$.

Let l and m be non-negative integers. For any $\mathbb{C}$-valued function $\phi(\tau,z)$ on $\mathfrak{H}_n \times \mathbb{C}^n$, we define the action of $g \in G_n^J(\mathbb{R})$ on ϕ by

$$(\phi|_{l,m}g)(\tau,z) := J_{l,m}(g,(\tau,z))^{-1}\phi(g\langle\tau,z\rangle),$$

where for $g = [(\lambda,\mu),\kappa]\cdot[M]$, we put

$$J_{l,m}(g,(\tau,z)) := \det(C\tau+D)^l$$
$$\times\,\mathbf{e}(-m\kappa - m\tau[{}^t\lambda] - 2m\lambda\,{}^tz - m\lambda\,{}^t\mu + m\{(C\tau+D)^{-1}C\}[{}^t(z+\lambda\tau+\mu)]).$$

It is easy to see that for any $g_i \in G_n^J(\mathbb{R})$ $(i=1,2)$,

$$(\phi|_{l,m}g_1)|_{l,m}g_2 = \phi|_{l,m}(g_1g_2).$$

In particular, it follows from Lemma 2.1 that for any $M, M' \in G_n(\mathbb{R})$ and $[(\lambda,\mu),\kappa], [(\lambda',\mu'),\kappa'] \in H_n(\mathbb{R})$, we have

$$\begin{cases} \phi|_{l,m}[M]|_{l,m}[M'] = \phi|_{l,m}[MM'], \\ \phi|_{l,m}[(\lambda,\mu),\kappa]|_{l,m}[(\lambda',\mu'),\kappa'] = \phi|_{l,m}[(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+2\lambda\,{}^t\mu'], \\ \phi|_{l,m}[M]|_{l,m}\left[(\lambda,\mu)M, \kappa + (\lambda,\mu)M\begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{pmatrix}{}^tM\,{}^t(\lambda,\mu) - \lambda\,{}^t\mu\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \phi|_{l,m}[(\lambda,\mu),\kappa]\cdot[M]. \end{cases}$$

We also define a subgroup of $G_n^J(\mathbb{R})$ by $\Gamma_n^J := \Gamma_n \ltimes H_n(\mathbb{Z})$, where $H_n(\mathbb{Z})$ is a subgroup of $H_n(\mathbb{R})$ consisting of all elements with integral entries.

Let l and m be positive integers. A holomorphic function $\phi(\tau,z)$ on $\mathfrak{H}_n \times \mathbb{C}^n$ is called a (*holomorphic*) *Jacobi form* of degree n, weight l and index m if it satisfies the following two conditions:

(i) $\phi|_{l,m}\gamma = \phi$ for any $\gamma \in \Gamma_n^J$.

(ii) ϕ possesses a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{T\in \mathrm{Sym}_n^*(\mathbb{Z}),\, r\in\mathbb{Z}^n} c_\phi(T,r)\mathbf{e}(\mathrm{tr}(T\tau) + r\,{}^tz)$$

with $c_\phi(T,r) = 0$ unless $4mT - {}^trr \geq 0$.

In particular, a Jacobi form ϕ is called *cuspidal* if it satisfies the stronger condition $c_\phi(T,r) = 0$ unless $4mT - {}^trr > 0$. We denote by $J_{l,m}(\Gamma_n^J)$ and $J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$ the $\mathbb{C}$-vector spaces consisting of all Jacobi forms and cuspidal Jacobi forms of degree n, weight l and index m, respectively.

As an important example of Jacobi form, we consider Fourier–Jacobi coefficients of Siegel modular forms of arbitrary degree $n > 1$. For any $k \in \mathbb{Z}$, let $F \in M_k(\Gamma_n)$ possess a Fourier expansion

$$F(Z) = \sum_{B'\in \mathrm{Sym}_n^*(\mathbb{Z})_{\geq 0}} A_F(B')\mathbf{e}(\mathrm{tr}(B'Z)) \quad (Z \in \mathfrak{H}_n),$$

and we put

$$Z = \begin{pmatrix} \tau' & z \\ {}^tz & \tau \end{pmatrix} \quad \text{with } \tau \in \mathfrak{H}_{n-1},\ z \in \mathbb{C}^{n-1} \text{ and } \tau' \in \mathfrak{H}_1.$$

Then we have the so-called Fourier–Jacobi expansion

$$F\left(\begin{pmatrix} \tau' & z \\ {}^tz & \tau \end{pmatrix}\right) = \sum_{m=0}^{\infty} \phi_m(\tau,z)\mathbf{e}(m\tau'),$$

where

$$\phi_m(\tau,z) = \sum_{\substack{T\in \mathrm{Sym}_{n-1}^*(\mathbb{Z}),\, r\in\mathbb{Z}^{n-1} \\ 4mT - {}^trr \geq 0}} A_F\left(\begin{pmatrix} m & r/2 \\ {}^tr/2 & T \end{pmatrix}\right)\mathbf{e}(\mathrm{tr}(T\tau) + r\,{}^tz). \tag{2.3}$$

We easily see that $\phi_m \in J_{k,m}(\Gamma_{n-1}^J)$ for each $m \in \mathbb{Z}_{>0}$. In particular, if $F \in S_k(\Gamma_n)$, then $\phi_m \in J_{k,m}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$.

As another example, if k is an even integer such that $k > n+1$, for each $m \in \mathbb{Z}_{>0}$, we define the *Jacobi Eisenstein series* of degree $n-1$, weight k and index m by

$$\mathfrak{E}_{k,m}^{(n-1)}(\tau, z) := \sum_{\gamma \in P_{n-1}^J \cap \Gamma_{n-1}^J \backslash \Gamma_{n-1}^J} J_{k,m}(\gamma, (\tau, z))^{-1} \quad (\tau \in \mathfrak{H}_{n-1},\, z \in \mathbb{C}^{n-1}),$$

where

$$P_{n-1}^J := \bigl\{[(\lambda, \mu), \kappa] \cdot \bigl[\bigl(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\bigr)\bigr] \in G_{n-1}^J \,\big|\, C = \mathbf{0}_{n-1},\, \lambda = 0\bigr\}.$$

We easily see that the right-hand side of the above definition is absolutely convergent and $\mathfrak{E}_{k,m}^{(n-1)} \in J_{k,m}(\Gamma_{n-1}^J)$. Moreover, Böcherer ([4]) showed that for any $m \in \mathbb{Z}_{>0}$, there exists a certain relation between $\mathfrak{E}_{k,m}^{(n-1)}$ and the mth coefficient $e_{k,m}^{(n-1)}$ of the above Fourier–Jacobi expansion of the Siegel Eisenstein series $E_k^{(n)} \in M_k(\Gamma_n)$. In particular, when $m = 1$, we have $\mathfrak{E}_{k,1}^{(n-1)} = e_{k,1}^{(n-1)}$.

For later use, we give an explicit formula for the Fourier coefficients of $e_{k,1}^{(n-1)}$ in case n is even. Let k be a positive even integer such that $k > n+1$. The *Siegel Eisenstein series* $E_k^{(n)}$ of weight k with respect to Γ_n is defined by

$$E_k^{(n)}(Z) = \sum_{(C,D)} \det(CZ + D)^{-k} \quad (Z \in \mathfrak{H}_n)$$

where (C, D) runs through a complete set of representatives of the equivalence classes of coprime symmetric pairs of size n. For each positive definite half-integral symmetric matrix B' of degree n, we denote by $\mathfrak{d}(B')$ the discriminant of the field extension $\mathbb{Q}(\sqrt{(-1)^{n/2}\det(2B')})/\mathbb{Q}$ and put $\mathfrak{f}(B') = \sqrt{(-1)^{n/2}\det(2B')/\mathfrak{d}(B')}$. It is well-known that $\mathfrak{f}(B') \in \mathbb{Z}_{>0}$. Furthermore, we denote by $\chi_{B'}$ the Kronecker character corresponding to the above field extension. For each $B' \in \mathrm{Sym}_n^*(\mathbb{Z})_{>0}$, the B'th Fourier coefficient $A_k^{(n)}(B')$ of $E_k^{(n)}$ is described as

$$\begin{aligned} A_k^{(n)}(B') = {} & \xi(n,k) L(1 - k/2 + n/2, \chi_{B'}) \mathfrak{f}(B')^{k-(n+1)/2} \\ & \times \prod_{p \mid \mathfrak{f}(B')} \widetilde{F}_p(B'; p^{k-(n+1)/2}), \end{aligned} \tag{2.4}$$

where

$$\xi(n,k) = 2^{n/2} \zeta(1-k)^{-1} \prod_{i=1}^{n/2} \zeta(1 + 2i - 2k)^{-1},$$

$L(s,\chi_{B'})$ denotes the Dirichlet L-function associated with $\chi_{B'}$, and

$$\widetilde{F}_p(B';X) = X^{-\mathrm{ord}_p(\mathfrak{f}(B'))} F_p(B';p^{-(n+1)/2}X).$$

We note that if $B \in \mathrm{Sym}^*_{n-1}(\mathbb{Z})_{>0}$ satisfies condition (1.1), then $\widetilde{F}^{(1)}_p(B;X) = \widetilde{F}_p(B^{(1)};X)$. Thus we have

Proposition 2.1. *Under the same assumption as above, let $e^{(n-1)}_{k,1}$ possess a Fourier expansion*

$$e^{(n-1)}_{k,1}(\tau,z) = \sum_{T\in\mathrm{Sym}^*_{n-1}(\mathbb{Z}),\, r\in\mathbb{Z}^{n-1}} c^{(n-1)}_{k,1}(T,r)\mathbf{e}(\mathrm{tr}(T\tau) + r\,{}^t z).$$

Then, for each $T \in \mathrm{Sym}^*_{n-1}(\mathbb{Z})$ such that $B_T = 4T - {}^t rr > 0$ with $r \in \mathbb{Z}^{n-1}$, we have

$$\begin{aligned} &c^{(n-1)}_{k,1}(T,r) \\ &\quad = \xi(n,k)L(1-k+n/2,\chi_{B^{(1)}_T})\mathfrak{f}(B^{(1)}_T)^{k-(n+1)/2} \prod_{p\mid \mathfrak{f}(B^{(1)}_T)} \widetilde{F}^{(1)}_p(B_T;p^{k-(n+1)/2}),\end{aligned}$$

where

$$B^{(1)}_T = \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & (B_T + {}^t rr)/4 \end{pmatrix} = \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix} \in \mathrm{Sym}^*_n(\mathbb{Z})_{>0}.$$

Proof. Since

$$c^{(n-1)}_{k,1}(T,r) = A^{(n)}_k(B^{(1)}_T),$$

the assertion immediately follows from (2.4). ∎

Returning to the general theory of Jacobi forms, we now consider the action of Hecke operators on Jacobi forms. Let $M \in \mathrm{Sp}_n(\mathbb{Q})$ and decompose the double coset $\Gamma^J_n M \Gamma^J_n$ into disjoint right cosets:

$$\Gamma^J_n M \Gamma^J_n = \bigsqcup_{i=1}^{d} \Gamma^J_n\, g_i,$$

where d is the number of right cosets, that is, $d = [\Gamma^J_n M \Gamma^J_n : \Gamma^J_n]$. Then, for any $\phi \in J_{l,m}(\Gamma^J_n)$, we define the action of the double coset $\Gamma^J_n M \Gamma^J_n$ on ϕ by

$$\phi|_{l,m}\Gamma^J_n M \Gamma^J_n := \sum_{i=1}^{d} \phi|_{l,m} g_i,$$

where the summation on the right-hand side is well-defined. We easily see that for any $\gamma \in \Gamma^J_n$,

$$(\phi|_{l,m}\Gamma^J_n M \Gamma^J_n)|_{l,m}\gamma = \phi|_{l,m}\Gamma^J_n M \Gamma^J_n,$$

that is, $\phi|_{l,m}\Gamma_n^J M\Gamma_n^J \in J_{l,m}(\Gamma_n^J)$. Moreover, if $\phi \in J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$, we have $\phi|_{l,m}\Gamma_n^J M\Gamma_n^J \in J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$. Here we note that each double coset $\Gamma_n^J M\Gamma_n^J$ with $M \in G_n(\mathbb{Q})$ contains a unique representative of the form

$$\mathbf{d}_n(\delta_1 \perp \cdots \perp \delta_n) = (\delta_1 \perp \cdots \perp \delta_n) \perp (\delta_1^{-1} \perp \cdots \perp \delta_n^{-1})$$

with $0 < \delta_1|\cdots|\delta_n$. Moreover, let $D = \delta_1 \perp \cdots \perp \delta_n$ and $D' = \delta_1' \perp \cdots \perp \delta_n'$ be two diagonal matrices with $0 < \delta_1|\cdots|\delta_n$, $0 < \delta_1'|\cdots|\delta_n'$. We easily see that if $(\delta_n, \delta_n') = 1$, then for any $\phi \in J_{l,m}(\Gamma_n^J)$,

$$\phi|_{l,m}\Gamma_n^J\mathbf{d}_n(DD')\Gamma_n^J = \phi|_{l,m}\Gamma_n^J\mathbf{d}_n(D)\Gamma_n^J|_{l,m}\Gamma_n^J\mathbf{d}_n(D')\Gamma_n^J.$$

A Jacobi form $\phi \in J_{l,1}(\Gamma_n^J)$ is called a *Hecke eigenform* if it is a common eigenfunction of all actions of double cosets $\Gamma_n^J M\Gamma_n^J$ with $M \in G_n(\mathbb{Q})$, that is, for any $M \in G_n(\mathbb{Q})$, the equation

$$\phi|_{l,m}\Gamma_n^J M\Gamma_n^J = \lambda_\phi(M)\phi$$

holds with some $\lambda_\phi(M) \in \mathbb{C}$. We easily see from the above argument that ϕ is a Hecke eigenform if and only if it satisfies for any rational prime p and $D = p^{\alpha_1} \perp \cdots \perp p^{\alpha_n} \in \mathbf{D}_p^{(n)}(\mathbb{Z})$ with $0 \le \alpha_1 \le \cdots \le \alpha_n$,

$$\phi|_{l,m}\Gamma_n^J\mathbf{d}_n(D)\Gamma_n^J = \lambda_\phi(D)\phi$$

with $\lambda_\phi(D) \in \mathbb{C}$.

2.2.2. *Jacobi forms on the adele group.* Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and let $\Psi_\mathbb{A}$ be the character of $\mathbb{Q}\setminus\mathbb{A}$ such that $\Psi_\mathbb{A}(x_\infty) = \mathbf{e}(x_\infty)$ for any $x_\infty \in \mathbb{R}$. In addition, for each $m \in \mathbb{Z}$, we put $\Psi_\mathbb{A}^m(\kappa) = \Psi_\mathbb{A}(m\kappa)$ for any $\kappa \in \mathbb{A}$. We denote by $G_n^J(\mathbb{A})$ the adele group of the Jacobi group G_n^J defined in the previous subsection. It follows from the strong approximation theorem for G_n^J that

$$G_n^J(\mathbb{A}) = G_n^J(\mathbb{Q})G_n^J(\mathbb{R})K_{\mathrm{fin}}^J,$$

where $K_{\mathrm{fin}}^J := \prod_{p<\infty} G_n^J(\mathbb{Z}_p)$.

Let l and m be positive integers. A $\mathbb{C}$-valued function f on $G_n^J(\mathbb{A})$ is called a *Jacobi form* of weight l and index m if it satisfies the following two conditions:

(i) The transformation formula

$$f([(0,0),\kappa]\gamma g k_\infty k_{\mathrm{fin}}) = \det(j(k_\infty, \sqrt{-1}\,\mathbf{1}_n))^{-l}\Psi_\mathbb{A}^m(\kappa)f(g)$$

holds for any $\kappa \in \mathbb{A}$, $\gamma \in G_n^J(\mathbb{Q})$, $g \in G_n^J(\mathbb{A})$, $k_\infty \in K_\infty$ and $k_{\mathrm{fin}} \in K_{\mathrm{fin}}^J$.

(ii) For any $(\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^n$, we fix an element $g_\infty \in G_n^J(\mathbb{R})$ such that $g_\infty\langle\sqrt{-1}\,\mathbf{1}_n, 0\rangle = (\tau, z)$ and put

$$\Phi_f(\tau, z) := J_{l,m}(g_\infty, (\sqrt{-1}\,\mathbf{1}_n, 0))f(g_\infty), \tag{2.5}$$

with the factor of automorphy $J_{l,m} : G_n^J(\mathbb{R}) \times (\mathfrak{H}_n \times \mathbb{C}^n) \to \mathbb{C}$ defined in §2.2.1. Here we easily see that the value Φ_f does not depend on the choice of g_∞. Then the function Φ_f is holomorphic on $\mathfrak{H}_n \times \mathbb{C}^n$.

In particular, a Jacobi form f is called *cuspidal* if it satisfies the further condition that

$$|\det(\mathrm{Im}(\tau))^{l/2} \exp(-2m\pi \operatorname{tr}(\mathrm{Im}(\tau)^{-1}[{}^t\mathrm{Im}(z)]))\Phi_f(\tau, z)|$$

is bounded on $\mathfrak{H}_n \times \mathbb{C}^n$. We denote by $J_{l,m}(G_n^J(\mathbb{A}))$ and $J_{l,m}^{\mathrm{cusp}}(G_n^J(\mathbb{A}))$ the $\mathbb{C}$-vector spaces of the Jacobi forms and cuspidal Jacobi forms of weight l and index m on the group $G_n^J(\mathbb{A})$, respectively.

It is easy to see that for each $f \in J_{l,m}(G_n^J(\mathbb{A}))$, the associated function Φ_f is an element of $J_{l,m}(\Gamma_n^J)$. In particular, if $f \in J_{l,m}^{\mathrm{cusp}}(G_n^J(\mathbb{A}))$, then $\Phi_f \in J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$. Furthermore we have

LEMMA 2.2. *The map $J_{l,m}(G_n^J(\mathbb{A})) \ni f \mapsto \Phi_f \in J_{l,m}(\Gamma_n^J)$ induces $\mathbb{C}$-linear isomorphisms $J_{l,m}(G_n^J(\mathbb{A})) \cong J_{l,m}(\Gamma_n^J)$ and $J_{l,m}^{\mathrm{cusp}}(G_n^J(\mathbb{A})) \cong J_{l,m}^{\mathrm{cusp}}(\Gamma_n^J)$.*

Proof. Since it is straightforward, we omit the proof. ∎

2.3. Standard L-functions attached to Jacobi forms. In this subsection we study Shintani's standard L-functions attached to Jacobi forms. In particular, we derive an explicit formula for the standard L-function attached to the Jacobi Eisenstein series of index 1. It might be given in a classical way, but here we treat it adelically.

Let p be an arbitrary rational prime. For simplicity, we write G_p^J, G_p, K_p^J, K_p and Z_p^J instead of $G_n^J(\mathbb{Q}_p)$, $G_n(\mathbb{Q}_p)$, $G_n^J(\mathbb{Z}_p)$, $G_n(\mathbb{Z}_p)$ and $Z_n^J(\mathbb{Q}_p)$, respectively. We denote by Ψ_p and $|*|_p$ the restriction of $\Psi_{\mathbb{A}}$ to $\mathbb{Q}_p$ and the p-adic valuation of $\mathbb{Q}_p$ normalized as $|p|_p = p^{-1}$, respectively. Let $\mathscr{H}_p = \mathscr{H}(G_p^J, K_p^J; \Psi_p)$ be the $\mathbb{C}$-algebra consisting of $\mathbb{C}$-valued functions φ on G_p^J satisfying the following two conditions:

(i) The equation
$$\varphi([(0,0),\kappa]kgk') = \Psi_p(\kappa)\varphi(g)$$
holds for any $\kappa \in \mathbb{Q}_p$, $k, k' \in K_p^J$ and $g \in G_p^J$.

(ii) The function φ is compactly supported modulo Z_p^J.

We note that $\mathscr{H}_p$ forms a $\mathbb{C}$-algebra with the convolution product

$$(\varphi_1 * \varphi_2)(g) := \int_{Z_p^J \backslash G_p^J} \varphi_1(gx^{-1})\varphi_2(x)\,dx \quad (\varphi_1, \varphi_2 \in \mathscr{H}_p),$$

where dx is a Haar measure on $Z_p^J \setminus G_p^J$ normalized by $\int_{Z_p^J \backslash Z_p^J K_p^J} dx = 1$. The algebra $\mathscr{H}_p$ is called the *Hecke algebra* of (G_p^J, K_p^J) with respect to the additive character Ψ_p.

We put

$$N_p^J := \{[(0,\mu),0]\mathbf{d}_n(A)\mathbf{n}_n(S) \in G_p^J \mid \mu \in \mathbb{Q}_p^n,\ A \in U_{n,p},\ S \in \mathrm{Sym}_n(\mathbb{Q}_p)\},$$
$$T_p = T(\mathbb{Q}_p) := \{\mathbf{d}_n(t_1 \perp \cdots \perp t_n) \in G_p \mid t_i \in \mathbb{Q}_p^\times\}$$

and $T(\mathbb{Z}_p) := T_p \cap K_p$, where $U_{n,p} \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is the group of upper unipotent matrices. We fix Haar measures $d\mathbf{n}$ and dt on N_p^J and T_p respectively normalized by

$$\int\limits_{N_p^J \cap K_p^J} d\mathbf{n} = 1 \quad \text{and} \quad \int\limits_{T(\mathbb{Z}_p)} dt = 1.$$

We define the module $\delta_{N_p^J}(t)$ of $t \in T_p$ to be the ratio $d(t\mathbf{n}t^{-1})/d\mathbf{n}$. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we put

$$\pi_\alpha = p^{\alpha_1} \perp \cdots \perp p^{\alpha_n} \in \mathrm{GL}_n(\mathbb{Q}_p).$$

We easily see that

$$\delta_{N_p^J}(\pi_\alpha) = p^{-\sum_{i=1}^n (2n+3-2i)\alpha_i}.$$

Let $X_0(T_p)$ be the group of unramified characters of T_p, that is,

$$X_0(T_p) := \{\chi \in \mathrm{Hom}(T_p, \mathbb{C}^\times) \mid \chi \text{ is trivial on } T(\mathbb{Z}_p)\}.$$

In particular, if $n = 1$, then $X_0(T_p)$ coincides with the group $X_0(\mathbb{Q}_p^\times)$ consisting of all unramified characters of $\mathbb{Q}_p^\times$. For any $\chi \in X_0(T_p)$ and $\varphi \in \mathscr{H}_p$, we define the *zonal spherical function* $\widehat{\omega}_\chi(\varphi)$ by

$$\widehat{\omega}_\chi(\varphi) := \sum_{\alpha \in \mathbb{Z}^n} \chi^{-1}(\mathbf{d}_n(\pi_\alpha))\widetilde{\varphi}(\mathbf{d}_n(\pi_\alpha)),$$

where

$$\widetilde{\varphi}(t) := \delta_{N,p}^J(t)^{-1/2} \int\limits_{N_p^J} \varphi(\mathbf{n}t)\, d\mathbf{n} \qquad (t \in T_p).$$

It was shown by Murase that the map $\varphi \mapsto \widehat{\omega}_\chi(\varphi)$ gives a $\mathbb{C}$-algebra homomorphism of $\mathscr{H}_p$ to $\mathbb{C}$ and that every $\mathbb{C}$-algebra homomorphism of $\mathscr{H}_p$ to $\mathbb{C}$ is given by $\varphi \mapsto \widehat{\omega}_\chi(\varphi)$ for some $\chi \in X_0(T_p)$ (cf. Proposition 4.10 and Theorem 4.15 in [20]).

On the other hand, for any $\chi \in X_0(T_p)$, let ϕ_χ be the $\mathbb{C}$-valued function on G_p^J defined by

$$\phi_\chi([(0,0),\kappa]\mathbf{n}t[(\lambda,0),0]k) = \Psi_p(\kappa)(\chi\delta_{N_p^J}^{-1/2})(t)\,\mathrm{char}_{\mathbb{Z}_p^n}(\lambda)$$

for any $\kappa \in \mathbb{Q}_p$, $\mathbf{n} \in N_p^J$, $t \in T_p$, $\lambda \in \mathbb{Q}_p^n$ and $k \in K_p^J$, where we denote by $\mathrm{char}_{\mathbb{Z}_p^n}$ the characteristic function of $\mathbb{Z}_p^n$. Here we note that each $\chi \in X_0(T_p)$ can be written in the form

$$\chi(\mathbf{d}_n(t_1 \perp \cdots \perp t_n)) = \chi^{(1)}(t_1) \cdots \chi^{(n)}(t_n),$$

with uniquely determined n unramified characters $\chi^{(1)},\dots,\chi^{(n)} \in X_0(\mathbb{Q}_p^\times)$. In that case, we simply write $\chi=(\chi^{(1)},\dots,\chi^{(n)})$. For each $\chi=(\chi^{(1)},\dots,\chi^{(n)}) \in X_0(T_p)$, we easily see that

$$\phi_\chi([(0,0),\kappa]\mathbf{n}t[(\lambda,0),0]k) = \Psi_p(\kappa)\prod_{i=1}^n \chi^{(i)}(t_i)|t_i|_p^{(2n+3-2i)/2}\,\mathrm{char}_{\mathbb{Z}_p^n}(\lambda) \tag{2.6}$$

for any $\kappa \in \mathbb{Q}_p$, $\mathbf{n} \in N_p^J$, $t = \mathbf{d}_n(t_1 \perp \cdots \perp t_n) \in T_p$, $\lambda \in \mathbb{Q}_p^n$ and $k \in K_p^J$.

For each rational prime p, we define the action of the Hecke algebra $\mathscr{H}_p$ on the space $J_{l,1}(G_n^J(\mathbb{A}))$ as follows: for any $f \in J_{l,1}(G_n^J(\mathbb{A}))$ and $\varphi \in \mathscr{H}_p$,

$$(f*\varphi)(g) := \int_{Z_p^J\backslash G_p^J} f(gx^{-1})\varphi(x)\,dx \qquad (g \in G_n^J(\mathbb{A})).$$

A Jacobi form $f \in J_{l,1}(G_n^J(\mathbb{A}))$ is called a *Hecke eigenform* if it is a common eigenfunction of all elements of $\bigotimes_p \mathscr{H}_p$, that is, for any rational prime p and $\varphi \in \mathscr{H}_p$, the equation

$$f*\varphi = \lambda_f(\varphi)f$$

holds with some $\lambda_f(\varphi) \in \mathbb{C}$. Since, for each p, the map $\lambda_f : \mathscr{H}_p \to \mathbb{C}$ gives a $\mathbb{C}$-algebra homomorphism of $\mathscr{H}_p$ to $\mathbb{C}$, it determines a $\chi_f \in X_0(T_p)$ such that

$$\lambda_f(\varphi) = \widehat{\omega}_{\chi_f}(\varphi)$$

for any $\varphi \in \mathscr{H}_p$. Then the *Satake p-parameter* of f is defined to be the orbit of $\chi_f = (\chi_f^{(1)},\dots,\chi_f^{(n)})$ in $X_0(T_p)$ under the action of the Weyl group W_n of type C_n isomorphic to the semi-direct product of S_n and $\{\pm1\}^n$. We also call the vector $(\chi_f^{(1)}(p),\dots,\chi_f^{(n)}(p)) \in (\mathbb{C}^\times)^n/W_n$ the Satake p-parameter of f. Then, for a Hecke eigenform $f \in J_{l,1}(G_n^J(\mathbb{A}))$, we define the *standard L-function attached to ϕ* by

$$L(s,f,\mathrm{St}) := \prod_p \prod_{i=1}^n \{(1-\chi_f^{(i)}(p)p^{-s})(1-\chi_f^{(i)}(p)^{-1}p^{-s})\}^{-1},$$

which was introduced by Shintani in his unpublished paper, and afterwards was studied by Murase (cf. [20, 21]).

By Lemma 2.2, for each $f \in J_{l,1}(G_n^J(\mathbb{A}))$, we obtain the associated element $\Phi_f \in J_{l,1}^{\mathrm{cusp}}(\Gamma_n^J)$. Then we easily have the following relation between the action of the Hecke algebra $\mathscr{H}_p$ on f and the operation $\Phi_f|_{l,1}\Gamma_n^J M \Gamma_n^J$ for some $M \in G_n(\mathbb{Z}[p^{-1}])$:

Lemma 2.3. *Let $f \in J_{l,1}(G_n^J(\mathbb{A}))$. For any $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n$ with $0 \le \alpha_1 \le \cdots \le \alpha_n$, we have*

$$\Phi_{f*\varphi_\alpha} = \Phi_f|_{l,1}\Gamma_n^J \mathbf{d}_n(\pi_\alpha)\Gamma_n^J.$$

Here φ_α is the element of $\mathscr{H}_p$ defined by

$$\varphi_\alpha(g) = \begin{cases} \Psi_p(\kappa) & \text{if } g \in Z_p^J K_p^J \mathbf{d}_n(\pi_\alpha) K_p^J \text{ and } g = [(0,0),\kappa] k \mathbf{d}_n(\pi_\alpha) k', \\ 0 & \text{otherwise,} \end{cases}$$

where $\kappa \in \mathbb{Q}_p$ and $k, k' \in K_p^J$. In particular, if f is a Hecke eigenform, then Φ_f is also a Hecke eigenform in the sense of §2.2.1.

Let $\phi \in J_{l,1}(\Gamma_n^J)$ be the Hecke eigenform corresponding to a Hecke eigenform $f \in J_{l,1}(G_n^J(\mathbb{A}))$ via the mapping defined in (2.5), that is, $\phi = \Phi_f$. By Lemma 2.3, we naturally define the standard L-function attached to ϕ as $L(s,\phi,\mathrm{St}) := L(s,f,\mathrm{St})$, that is,

$$L(s,\phi,\mathrm{St}) := \prod_{p<\infty} \prod_{i=1}^{n} \{(1-\chi_\phi^{(i)}(p)p^{-s})(1-\chi_\phi^{(i)}(p)^{-1}p^{-s})\}^{-1},$$

where we put $\chi_\phi^{(i)}(p) = \chi_f^{(i)}(p)$ for $i = 1,\dots,n$.

If ϕ is a cuspidal Hecke eigenform, the following analytic properties of $L(s,\phi,\mathrm{St})$ have been shown by Murase ([21]):

Lemma 2.4 (cf. [21]). *If $\phi \in J_{l,1}^{\mathrm{cusp}}(\Gamma_n^J)$ is a Hecke eigenform, then the standard L-function $L(s,\phi,\mathrm{St})$ has a meromorphic continuation to the entire complex plane $\mathbb{C}$. More precisely, the function*

$$L^*(s,\phi,\mathrm{St}) = \prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+l-1/2-i) L(s,\phi,\mathrm{St}),$$

with $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$, is meromorphic on $\mathbb{C}$ and satisfies the functional equation

$$L^*(1-s,\phi,\mathrm{St}) = \varepsilon_n L^*(s,\phi,\mathrm{St}),$$

where

$$\varepsilon_n = \begin{cases} -1 & \text{if } n \equiv 1,2 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Remark. Murase derived similar properties for the standard L-functions attached to more general cuspidal Jacobi forms whose index is a matrix.

In the rest of this subsection we consider the standard L-function attached to the Jacobi Eisenstein series $\mathfrak{E}_{l,1}^{(n)} \in J_{l,1}(G_n^J(\mathbb{A}))$.

For any quasi-character $\xi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$, we define a $\mathbb{C}$-valued function $\widetilde{\phi_\xi}$ on $G_n^J(\mathbb{A})$ by

$$\widetilde{\phi_\xi}([(0,\mu),\kappa] g [(\lambda,0),0] k_\infty k_{\mathrm{fin}}) = \xi(\det(A)) \varphi_0(\lambda) j(k_\infty, \sqrt{-1}\,\mathbf{1}_n)^{-l}$$

for any $\kappa \in \mathbb{A}$, $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_n^J(\mathbb{A})$, $k_\infty \in K_\infty$ and $k_{\mathrm{fin}} \in K_{\mathrm{fin}}^J$, where $\varphi_0 = \prod_v \varphi_{0,v}$,

$$\varphi_{0,v}(\lambda) = \begin{cases} \mathrm{char}_{\mathbb{Z}_p^n}(\lambda) & \text{if } v = p < \infty, \\ \exp(-2\pi\lambda\,{}^t\lambda) & \text{if } v = \infty. \end{cases}$$

Then we define the Eisenstein series E_ξ on $G_n^J(\mathbb{A})$ associated with ξ by

$$E_\xi(g) := \sum_{\gamma \in P_n^J(\mathbb{Q})\backslash G_n^J(\mathbb{Q})} \widetilde{\phi_\xi}(\gamma g) \quad (g \in G_n^J(\mathbb{A})).$$

In particular, we denote by $\mathcal{E}_{l,1}^{(n)}$ the Eisenstein series on $G_n^J(\mathbb{A})$ associated with a special character $\xi_l(x) = |x|_{\mathbb{A}}^l$ $(x \in \mathbb{A}^\times)$. We easily see that $\mathcal{E}_{l,1}^{(n)}$ is an element of $J_{l,1}(G_n^J(\mathbb{A}))$ and corresponds to the Jacobi Eisenstein series $\mathfrak{E}_{l,1}^{(n)} \in J_{l,1}(\Gamma_n^J)$ in the same manner as in Lemma 2.2. Hence we also call $\mathcal{E}_{l,1}^{(n)}$ the Jacobi Eisenstein series of weight l and index 1. Then we have

Proposition 2.2. *The Jacobi Eisenstein series $\mathcal{E}_{l,1}^{(n)} \in J_{l,1}(G_n^J(\mathbb{A}))$ is a Hecke eigenform, that is, for any $\varphi \in \bigotimes_p \mathscr{H}_p$,*

$$\mathcal{E}_{l,1}^{(n)} * \varphi = \lambda_{\mathcal{E}}(\varphi)\mathcal{E}_{l,1}^{(n)}$$

with $\lambda_{\mathcal{E}}(\varphi) \in \mathbb{C}^\times$. Moreover, the Satake p-parameter of $\mathcal{E}_{l,1}^{(n)}$ is taken to be of the form

$$(p^{l-(n+1)+i-1/2})_{1\le i\le n}$$

up to the action of the Weyl group W_n.

Proof. For any quasi-character ξ of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$, we take a $\chi = (\chi^{(1)}, \dots, \chi^{(n)}) \in X_0(T_p)$ such that

$$\chi^{(i)}(t_i) = \xi(t_i)|t_i|_p^{-(2n+3-2i)/2} \quad (t_i \in \mathbb{Q}_p^\times) \tag{2.7}$$

for each $1 \le i \le n$. Then, by (2.6) and the definition of $\widetilde{\phi_\xi}$, we have $\widetilde{\phi_\xi} = \phi_\chi$. Therefore it suffices to prove that for any $\varphi \in \mathscr{H}_p$ and $\lambda \in \mathbb{Q}_p^n$,

$$(\phi_\chi * \varphi)([(\lambda, 0), 0]) = c \cdot \mathrm{char}_{\mathbb{Z}_p^n}(\lambda) \tag{2.8}$$

with some $c \in \mathbb{C}^\times$. Indeed, if $\lambda \notin \mathbb{Z}_p^n$, there exists $0 \ne \mu \in \mathbb{Z}_p^n$ such that $\Psi_p(\lambda\,{}^t\mu) \ne 1$. Thus we have

$$\begin{aligned}(\phi_\chi * \varphi)([(\lambda, 0), 0]) &= (\phi_\chi * \varphi)([(\lambda, 0), 0] \cdot [(0, \mu), 0]) \\ &= (\phi_\chi * \varphi)([(\lambda, \mu), \lambda\,{}^t\mu]) \\ &= (\phi_\chi * \varphi)([(0, \mu), \lambda\,{}^t\mu] \cdot [(\lambda, 0), 0]) \\ &= \Psi_p(\lambda\,{}^t\mu)\,(\phi_\chi * \varphi)([(\lambda, 0), 0]),\end{aligned}$$

and $(\phi_\chi * \varphi)([(\lambda, 0), 0]) = 0$. Now we have proved that the Eisenstein series E_ξ is a Hecke eigenform. Moreover, it follows from (2.8) that

$$c = (\phi_\chi * \varphi)(1_{G_p^J}) = \int_{Z_p^J\backslash G_p^J} \phi_\chi(g)\varphi(g^{-1})\,dg = \widehat{\omega}_\chi(\varphi)$$

and hence the eigenvalue $\lambda_{\mathcal{E}}(\varphi)$ coincides with the zonal spherical function $\widehat{\omega}_\chi(\varphi)$. Therefore it follows from (2.7) that

$$\chi^{(i)}(t_i) = \xi_l(t_i)|t_i|_p^{-(2n+3-2i)/2} = |t_i|_p^{l-(2n+3-2i)/2}$$

for each i. By substituting $t_i = p$, we obtain $\chi^{(i)}(p) = p^{-l+(2n+3-2i)/2}$ and complete the proof. ∎

By Proposition 2.2, we obtain the following conclusion:

Corollary. *Let l be a positive even integer such that $l > n+2$. Then*

$$L(s, \mathcal{E}_{l,1}^{(n)}, \mathrm{St}) = L(s, \mathfrak{E}_{l,1}^{(n)}, \mathrm{St}) = \prod_{i=1}^{n} \zeta(s - l + 1/2 + i)\zeta(s + l - 1/2 - i).$$

In particular, $L(s, \mathcal{E}_{l,1}^{(n)}, \mathrm{St})$ and $L(s, \mathfrak{E}_{l,1}^{(n)}, \mathrm{St})$ converge absolutely for $\mathrm{Re}(s) > l-n-1/2$. *In addition, they have meromorphic continuations to the entire complex plane $\mathbb{C}$ and satisfy functional equations under $s \mapsto 1-s$.*

Remark. Let k and n be positive even integers such that $k > n+1$. As mentioned in §2.1, $\mathfrak{E}_{k,1}^{(n-1)}$ coincides with the first Fourier–Jacobi coefficient $e_{k,1}^{(n-1)}$ of the Siegel Eisenstein series $E_k^{(n)} \in M_k(\Gamma_n)$ of degree n and weight k. Thus it follows from the Corollary to Proposition 2.2 that

$$\begin{aligned} L(s, e_{l,1}^{(n)}, \mathrm{St}) & \\ &= \prod_p \prod_{i=1}^{n-1} \{(1 - p^{k-(n+1)/2} p^{-s+i-n/2})(1 - (p^{k-(n+1)/2})^{-1} p^{-s+i-n/2})\}^{-1} \\ &= \prod_{i=1}^{n-1} L(s + k - 1/2 - i, E_{2k-n}^{(1)}), \end{aligned}$$

where $E_{2k-n}^{(1)} \in M_{2k-n}(\Gamma_1)$. Moreover, replacing $e_{k,1}^{(n-1)}$ by the first Fourier–Jacobi coefficient $\phi_1 \in J_{k,1}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$ of a Siegel cusp form $F \in S_k(\Gamma_n)$ which is connected to a normalized Hecke eigenform $f \in S_{2k-n}(\Gamma_1)$ via a lifting procedure due to Ikeda (cf. [12]), we also obtain a similar explicit formula for the standard L-function attached to ϕ_1 (cf. [10]).

2.4. Eichler–Zagier–Ibukiyama correspondence between Jacobi forms and Siegel modular forms of half-integral weight. For later use, we recall that there exists a natural $\mathbb{C}$-linear correspondence from the space of Jacobi forms of even integral weight and of index 1 into that of Siegel modular forms of half-integral weight.

For any $(\tau, z) \in \mathfrak{H}_n \times \mathbb{C}^n$ and $(r_1, r_2) \in \mathbb{Q}^n \oplus \mathbb{Q}^n$, we define the *theta series* of characteristic (r_1, r_2) by

$$\theta_{(r_1,r_2)}(\tau, z) = \theta^{(n)}_{(r_1,r_2)}(\tau, z) := \sum_{\lambda \in \mathbb{Z}^n} \mathbf{e}((\tau/2)[{}^t(\lambda + r_1)] + (\lambda + r_1)\,{}^t(z + r_2)).$$

In particular, for any $r \in \mathbb{Z}^n$, we put $\theta_r(\tau, z) = \theta^{(n)}_r(\tau, z) := \theta^{(n)}_{(r/2,0)}(2\tau, 2z)$. We note that the function $\theta_r(\tau, z)$ depends only on $r \bmod 2\mathbb{Z}^n$. For a fixed $\tau \in \mathfrak{H}_n$, it is known that $(\theta_r(\tau, z))_{r \in \mathbb{Z}^n/2\mathbb{Z}^n}$ forms a basis of the $\mathbb{C}$-vector space $\Theta^{(n)}_\tau$ consisting of all $\mathbb{C}$-valued holomorphic functions $\theta(z)$ on $\mathbb{C}^n$ which satisfy

$$\theta(z + \lambda\tau + \mu) = \mathbf{e}(-\operatorname{tr}(\tau[{}^t\lambda] + 2\,{}^t\lambda z))\theta(z)$$

for any $\lambda, \mu \in \mathbb{Z}^n$.

For any $\tau \in \mathfrak{H}_n$, we put

$$\theta(\tau) = \theta^{(n)}(\tau) := \theta^{(n)}_{(0,0)}(2\tau, 0) = \sum_{\lambda \in \mathbb{Z}^n} \mathbf{e}(\tau[{}^t\lambda]).$$

Then, for any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(n)}_0(4)$, we define *Shimura's factor of automorphy* by

$$J(M, \tau) = J^{(n)}(M, \tau) := \frac{\theta^{(n)}(M\langle\tau\rangle)}{\theta^{(n)}(\tau)}.$$

As is well-known,

$$J(M, \tau)^2 = (-1)^{(\det D - 1)/2} \det(C\tau + D).$$

For any $l \in \mathbb{Z}$, a holomorphic function $F(\tau)$ on $\mathfrak{H}_n$ is called a *Siegel modular form* of degree n and weight $l - 1/2$ if it satisfies the following two conditions:

(i) $F(M\langle\tau\rangle) = J(M, \tau)^{2l-1}F(\tau)$ for any $M \in \Gamma^{(n)}_0(4)$.

(ii) For any $M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_n$, the function $\det(C\tau + D)^{-l+1/2}F(M\langle\tau\rangle)$ possesses a Fourier expansion of the form

$$\det(C\tau + D)^{-l+1/2}F(M\langle\tau\rangle) = \sum_{B \in \mathrm{Sym}^*_n(\mathbb{Z})_{\geq 0}} C_{F,M}(B)\mathbf{e}(\operatorname{tr}(B\tau)/4),$$

where $\det(C\tau + D)^{-l+1/2}$ is an appropriately defined single-valued function of τ. We note that such a F possesses a usual Fourier expansion

$$F(\tau) = \sum_{B \in \mathrm{Sym}^*_n(\mathbb{Z})_{\geq 0}} C_F(B)\mathbf{e}(\operatorname{tr}(B\tau)).$$

In particular, a Siegel modular form F is called a *cusp form* if it satisfies the stronger condition $C_{F,M}(B) = 0$ unless $B > 0$ (positive definite). We denote by $M_{l-1/2}(\Gamma^{(n)}_0(4))$ and $S_{l-1/2}(\Gamma^{(n)}_0(4))$ the $\mathbb{C}$-vector spaces of

Siegel modular forms and Siegel cusp forms of degree n and weight $l-1/2$, respectively.

We now define the *generalized Kohnen plus space* $M^+_{l-1/2}(\Gamma_0^{(n)}(4))$ to consist of all elements $F \in M_{l-1/2}(\Gamma_0^{(n)}(4))$ whose Fourier coefficients $C_F(B)$ satisfy the condition

$$C_F(B) = 0 \text{ unless } B \equiv (-1)^{l+1}\, {}^t r_B r_B \bmod 4\mathrm{Sym}^*_n(\mathbb{Z}) \text{ for some } r_B \in \mathbb{Z}^{n-1},$$

and put $S^+_{l-1/2}(\Gamma_0^{(n)}(4)) := M^+_{l-1/2}(\Gamma_0^{(n)}(4)) \cap S_{l-1/2}(\Gamma_0^{(n)}(4))$. These spaces were introduced by Kohnen ([19]) for $n = 1$, and by Ibukiyama ([11]) for general n.

Now, we recall an important fact that if l is even, then there exists a $\mathbb{C}$-linear isomorphism between the space $J_{l,1}(\Gamma_n^J)$ of Jacobi forms of index 1 and the generalized Kohnen plus space $M^+_{l-1/2}(\Gamma_0^{(n)}(4))$ defined as follows. Let $\phi \in J_{l,1}(\Gamma_n^J)$ possess a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{T \in \mathrm{Sym}^*_n(\mathbb{Z}),\, r \in \mathbb{Z}^n \\ 4T - {}^t rr \geq 0}} c_\phi(T, r)\mathbf{e}(\mathrm{tr}(T\tau) + r\, {}^t z).$$

Since, for each $\tau \in \mathfrak{H}_n$, $\phi(\tau, z)$ belongs to the space $\Theta_\tau^{(n)}$ generated by $(\theta_r(\tau, z))_{r \in \mathbb{Z}^n/2\mathbb{Z}^n}$, ϕ can be expressed as

$$\phi(\tau, z) = \sum_{r \in \mathbb{Z}^n/2\mathbb{Z}^n} h_r(\tau)\theta_r(\tau, z)$$

with some 2^n holomorphic functions $(h_r(\tau))_{r \in \mathbb{Z}^n/2\mathbb{Z}^n}$ on $\mathfrak{H}_n$ whose Fourier expansion is of the form

$$h_r(\tau) = \sum_{\substack{T \in \mathrm{Sym}^*_n(\mathbb{Z}) \\ 4T - {}^t rr \geq 0}} c_\phi(T, r)\mathbf{e}(\mathrm{tr}((T - {}^t rr/4)\tau)).$$

Then we put

$$\sigma(\phi)(\tau) = \sum_{r \in \mathbb{Z}^n/2\mathbb{Z}^n} h_r(4\tau).$$

The following statement was shown by Eichler and Zagier ([7]) in the case $n = 1$ and by Ibukiyama for general n:

Proposition 2.3 (cf. Theorems 1, 2 in [11]). *If l is even, then the map $\phi \mapsto \sigma(\phi)$ gives a $\mathbb{C}$-linear isomorphism*

$$J_{l,1}(\Gamma_n^J) \cong M^+_{l-1/2}(\Gamma_0^{(n)}(4)),$$

which is compatible with the actions of Hecke operators. Furthermore, its *restriction to the space* $J_{l,1}^{\mathrm{cusp}}(\varGamma_n^J)$ *also induces a* $\mathbb{C}$*-linear isomorphism*

$$J_{l,1}^{\mathrm{cusp}}(\varGamma_n^J) \cong S_{l-1/2}^{+}(\varGamma_0^{(n)}(4)).$$

We call it the Eichler–Zagier–Ibukiyama correspondence.

Remark. When l is odd, the space $J_{l,1}(\varGamma_n^J)$ is not isomorphic to the Kohnen plus space $M_{l-1/2}^{+}(\varGamma_0^{(n)}(4))$. However, we note that a similar claim is also valid for the space $J_{l,1}^{\mathrm{skew}}(\varGamma_n^J)$ of *skew holomorphic Jacobi forms*, which was shown by Skoruppa ([26, 27]) in the case $n = 1$ and by Arakawa ([2]) and Hayashida ([9]) for general n.

We easily see by the definition that the Fourier expansion of $\sigma(\phi)$ can be expressed in terms of Fourier coefficients of ϕ as

$$\sigma(\phi)(\tau) = \sum_{B \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} c_\phi((B + {}^t r_B r_B)/4, r_B)\mathbf{e}(\mathrm{tr}(B\tau)),$$

where r_B denotes an element of $\mathbb{Z}^n$ such that $B + {}^t r_B r_B \in 4\mathrm{Sym}_n^*(\mathbb{Z})$. We note that r_B is uniquely determined by B modulo $2\mathbb{Z}^n$, and furthermore $c_\phi((B + {}^t r_B r_B)/4, r_B)$ does not depend on the choice of the representative of $r_B \bmod 2\mathbb{Z}^n$. Moreover, if ϕ coincides with the first Fourier–Jacobi coefficient of a Siegel modular form $F \in M_l(\varGamma_{n+1})$, we have

$$\sigma(\phi)(\tau) = \sum_{B \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} A_F(B^{(1)})\mathbf{e}(\mathrm{tr}(B\tau)),$$

where $B^{(1)} \in \mathrm{Sym}_{n+1}^*(\mathbb{Z})$ denotes the matrix defined in §1, and $A_F(B^{(1)})$ is the $B^{(1)}$th Fourier coefficient of F. In particular, let n and k be positive even integers such that $k > n+1$ and take $\phi = e_{k,1}^{(n-1)} \in J_{k,1}(\varGamma_{n-1}^J)$. Then we have the following explicit formula for the Fourier coefficients of the associated form $\sigma(e_{k,1}^{(n-1)}) \in M_{k-1/2}^{+}(\varGamma_0^{(n-1)}(4))$:

Proposition 2.4. *Under the same assumption as in Proposition* 2.1, *let* $\sigma(e_{k,1}^{(n-1)})$ *possess a Fourier expansion*

$$\sigma(e_{k,1}^{(n-1)})(\tau) = \sum_{B \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} C_{k-1/2}^{(n-1)}(B)\mathbf{e}(\mathrm{tr}(B\tau)).$$

Then, for each $B \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})_{>0}$ *satisfying the condition* (1.1), *we have*

$$C_{k-1/2}^{(n-1)}(B) = \xi(n,k)L(1-k+n/2, \chi_{B^{(1)}})\mathfrak{f}(B^{(1)})^{k-(n+1)/2} \prod_{p \mid \mathfrak{f}(B^{(1)})} \widetilde{F}_p^{(1)}(B; p^{k-(n+1)/2}).$$

Proof. If $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}^*(\mathbb{Z})$ and $r \in \mathbb{Z}^{n-1}$, we have $C_{k-1/2}^{(n-1)}(B) = c_{k,1}^{(n-1)}(T, r)$. Thus the assertion follows from Proposition 2.1. ∎

3. Andrianov-type identity for power series attached to Jacobi forms. Throughout this section, let n and k be positive even integers such that $k > n+1$, and fix a rational prime p. For a subring R of $\mathbb{Z}_p$, we denote by $\mathrm{Sym}_{n-1}(R)^{(1)}$ the subset of $\mathrm{Sym}_{n-1}(R)^{\times}$ consisting of all elements which satisfy the condition (1.1) in §1:

$$\mathrm{Sym}_{n-1}(R)^{(1)} \\ = \{B \in \mathrm{Sym}_{n-1}(R)^{\times} \mid B + {}^t r_B r_B \in 4\mathrm{Sym}^*_{n-1}(R) \text{ for some } r_B \in R^{n-1}\}.$$

As mentioned in §1, with each $B \in \mathrm{Sym}_{n-1}(R)^{(1)}$ we can associate an element

$$B^{(1)} = \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B + {}^t r_B r_B)/4 \end{pmatrix} \in \mathrm{Sym}^*_n(R)^{\times}.$$

For $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we define a modified local Siegel series $b_p^{(1)}(B;t)$ as follows. For each $R \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}])$ and $r \in \mathbb{Z}_p^{n-1}$, if $R \in p^{-l}\mathrm{Sym}_{n-1}(\mathbb{Z}_p)$ with $l \geq 0$, we put

$$\omega(R;r) = p^{-(n-1)l}\mu_p(R)^{1/2} \sum_{x \in \mathbb{Z}_p^{n-1}/p^l\mathbb{Z}_p^{n-1}} \mathbf{e}_p(-R[{}^t x] + rR\,{}^t x/2 + xR\,{}^t r/2),$$

where $\mu_p(R) = [\mathbb{Z}_p^{n-1}R + \mathbb{Z}_p^{n-1} : \mathbb{Z}_p^{n-1}]$, and the right-hand side does not depend on the choice of l. Suppose that $B \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ is of the form $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. We put

$$b_p^{(1)}(B;t) = \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z}_p)} \omega(R;r)\mathbf{e}_p(-\mathrm{tr}(TR))t^{\mathrm{ord}_p(\mu_p(R))}.$$

We note that this series coincides with $\alpha_1(B,t)$ of [23] if $p \neq 2$ and $r = 0$. As will be shown later, the above definition does not depend on the choice of T and r (cf. Proposition 3.1 below).

On the other hand, if $m > 1$, for each $S \in \mathrm{Sym}^*_{m-1}(\mathbb{Z}_p)$, $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$, $r \in \mathbb{Z}_p^{n-1}$ and $e \in \mathbb{Z}_{>0}$, we put

$$\mathcal{A}_e(S,T,r) \\ := \left\{ X \in \mathrm{M}_{m,n-1}(\mathbb{Z}_p)/p^e\mathrm{M}_{m,n-1}(\mathbb{Z}_p) \;\middle|\; \begin{array}{l} (-1 \perp S)[X] + {}^t r\boldsymbol{x}_1/2 \\ \qquad + {}^t\boldsymbol{x}_1 r/2 - T \in p^e\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p) \end{array} \right\},$$

where $\boldsymbol{x}_1 \in \mathbb{Z}_p^{n-1}$ denotes the first row of X. We easily check that it is well-defined. Furthermore, if both S and $\left(\begin{smallmatrix} 1 & r/2 \\ {}^t r/2 & T \end{smallmatrix}\right)$ are non-degenerate, then $p^{e(-m(n-1)+n(n-1)/2)}\#\mathcal{A}_e(S,T,r)$ has the same value for each

$$e \geq \mathrm{ord}_p\left(\det\begin{pmatrix} 1 & r/2 \\ {}^t r/2 & T \end{pmatrix}\right);$$

this value will be denoted by $\alpha_p^{(1)}(S,T,r)$. We note that $\alpha_p^{(1)}(S,T,r)$ coincides with the usual local density $\alpha_p(-1 \perp S, T)$ if $r=0$. Then we obtain the following lemmas:

LEMMA 3.1. *Suppose that $B \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)^\times$ is of the form $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then*

$$b_p^{(1)}(B; p^{-k+1/2}) = \alpha_p(H_{k-1}, T, r),$$

where

$$H_{k-1} = \underbrace{H \perp \cdots \perp H}_{k-1} \quad \text{with} \quad H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \in \mathrm{Sym}_2^*(\mathbb{Z}_p).$$

In particular, $b_p^{(1)}(B;t) = 0$ unless $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$.

Proof. By Lemma 3.4 of [24], we have

$$\begin{aligned}
&b_p^{(1)}(B; p^{-k+1/2}) \\
&= \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z}_p)} \ \sum_{x \in \mathbb{Z}_p^{n-1}/p^l \mathbb{Z}_p^{n-1}} \mathbf{e}_p(-R[{}^t x] + rR\,{}^t x/2 + xR\,{}^t r/2) \\
&\qquad\qquad \times p^{-(k-1)\,\mathrm{ord}_p(\mu_p(R))} p^{-(n-1)l} \mathbf{e}_p(-\mathrm{tr}(TR)) \\
&= \sum_R \sum_x \mathbf{e}_p(-R[{}^t x] + rR\,{}^t x/2 + xR\,{}^t r/2) p^{-(n-1)l} \mathbf{e}_p(-\mathrm{tr}(TR)) p^{-2l(k-1)n} \\
&\qquad\qquad \times \sum_{Y \in \mathrm{M}_{2k-2,n-1}(\mathbb{Z}_p)/p^l \mathrm{M}_{2k-2,n-1}(\mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(H_{k-1}[Y]R)) \\
&= \sum_R \sum_x \sum_Y \mathbf{e}_p(\mathrm{tr}((-{}^t xx + H_{k-1}[Y] + {}^t rx/2 + {}^t xr/2 - T)R)) p^{-l(2k-1)(n-1)} \\
&= \#\mathcal{A}_l(H_{k-1}, T, r) p^{-l((2k-1)(n-1) - n(n-1)/2)}.
\end{aligned}$$

Thus the assertion holds. ∎

LEMMA 3.2. *Suppose that $B \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)^\times$ is of the form $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$. Then*

$$\alpha_p(H_k, B^{(1)}) = (1 - p^{-k}) \alpha_p(H_{k-1}, T, r).$$

Proof. The proof is similar to that of Proposition 2.4 in [14]; we give a sketch. For each $\xi = (\xi_i) \in \mathbb{Z}_p^{2k}$, we put

$$\mathcal{A}_e(H_k, B^{(1)}) = \{X \in \mathrm{M}_{2k,n}(\mathbb{Z}_p)/p^e \mathrm{M}_{2k,n}(\mathbb{Z}_p) \mid H_k[X] - B^{(1)} \in p^e \mathrm{Sym}_n^*(\mathbb{Z}_p)\}$$

and

$$\begin{aligned}
&\mathcal{A}_e(H_k, B^{(1)}; \xi) \\
&\quad = \{X = (x_{ij}) \in \mathcal{A}_e(H_k, B^{(1)}) \mid x_{i1} \equiv \xi_i \pmod{p^e} \text{ for } 1 \le i \le 2k\}.
\end{aligned}$$

We easily see that $\mathcal{A}_e(H_k, B^{(1)};\xi) \neq \emptyset$ only if $\xi \in \mathcal{A}_e(H_k, 1)$. Fix such a ξ. Then $\xi \not\equiv 0 \pmod{p\mathbb{Z}_p^{2k}}$. Thus by Lemma 2.3 in [14], we can take $U \in \mathrm{GL}_{2k}(\mathbb{Z}_p)$ and $K \in \mathrm{Sym}^*_{2k-2}(\mathbb{Z}_p)$ such that

$$\text{(i)}\ \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \perp K = H_k[U]; \qquad \text{(ii)}\ K \sim_{\mathbb{Z}_p} H_{k-1}; \qquad \text{(iii)}\ U^{-1}\xi = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

For each $X \in \mathcal{A}_e(H_k, B^{(1)};\xi)$, we write $X = ({}^t\xi \mid Y)$ with $Y \in \mathrm{M}_{2k,n-1}(\mathbb{Z}_p)$, and

$$Y = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ Y_3 \end{pmatrix} \qquad \text{with } \boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{Z}_p^{n-1} \text{ and } Y_3 \in \mathrm{M}_{2k-2,n-1}(\mathbb{Z}_p).$$

Then, by an easy calculation, we have

$$\boldsymbol{y}_1 + \boldsymbol{y}_2/2 - r/2 \in p^e\mathbb{Z}_p^{n-1}$$

and

$$-{}^t\boldsymbol{y}_1\boldsymbol{y}_1 + K[Y_3] + {}^t\boldsymbol{y}_1\boldsymbol{y}_2/2 + {}^t\boldsymbol{y}_2\boldsymbol{y}_1/2 - T \in p^e\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p).$$

Thus we have

$$-{}^t\boldsymbol{y}_1\boldsymbol{y}_1 + K[Y_3] + {}^tr\boldsymbol{y}_1/2 + {}^t\boldsymbol{y}_1 r/2 - T \in p^e\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p),$$

that is, $\left(\begin{smallmatrix} \boldsymbol{y}_1 \\ Y_3 \end{smallmatrix}\right) \in \mathcal{A}_e(H_{k-1}, T, r)$. Moreover, we easily see that $Y \mapsto \left(\begin{smallmatrix} \boldsymbol{y}_1 \\ Y_3 \end{smallmatrix}\right)$ induces a bijection between $\mathcal{A}_e(H_k, B^{(1)};\xi)$ and $\mathcal{A}_e(H_{k-1}, T, r)$. Thus

$$\begin{aligned} &p^{e(-2kn+n(n+1)/2)}\#\mathcal{A}_e(H_k, B^{(1)}) \\ &\quad = p^{e(-2k+1)}\#\mathcal{A}_e(H_k, 1)p^{e(-(2k-1)(n-1)+n(n-1)/2)}\#\mathcal{A}_e(H_{k-1}, T, r) \\ &\quad = \alpha_p(H_k, 1)\alpha_p(H_{k-1}, T, r) = (1 - p^{-k})\alpha_p(H_{k-1}, T, r). \end{aligned}$$

Hence the assertion holds. ∎

Now, by combining Lemmas 3.1 and 3.2, we obtain the following:

PROPOSITION 3.1. *For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$ and $s \in \mathbb{C}$, we have*

$$b_p^{(1)}(B; p^{-s+1/2}) = (1 - p^{-s})^{-1}\, b_p(B^{(1)}; s).$$

Proof. It is well-known that for each $B' \in \mathrm{Sym}^*_n(\mathbb{Z}_p)^\times$ with $n < 2k$, the Siegel series $b_p(B'; s)$ in §1 satisfies the equation

$$b_p(B'; k) = \alpha_p(H_k, B').$$

Hence, by Lemmas 3.1 and 3.2, we have

$$b_p^{(1)}(B; p^{-k+1/2}) = (1 - p^{-k})^{-1} b_p(B^{(1)}; k)$$

for infinitely many k, and hence the assertion follows. ∎

Remark. The definition of the series $b_p^{(1)}(B;t)$ for $B = 4T - {}^t rr$ with $T \in \mathrm{Sym}_{n-1}(\mathbb{Q}_p)$ and $r \in \mathbb{Z}_p^{n-1}$ does not depend on the choice of T and r. Indeed, if $T \in \mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)$, the vector r is uniquely determined by B modulo $2\mathbb{Z}_p^{n-1}$, and the matrix $\left(\begin{smallmatrix} 1 & r/2 \\ {}^t r/2 & T \end{smallmatrix}\right)$ is uniquely determined by B up to $\mathrm{GL}_n(\mathbb{Z}_p)$-equivalence. Thus, by Proposition 3.1, $b_p^{(1)}(B;t)$ is uniquely determined by B. If $T \notin \mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)$, we have $b_p^{(1)}(B;t) = 0$. Furthermore, if $B = 4T' - {}^t r' r'$ is another expression, then T' does not belong to $\mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)$ either. This proves that $b_p^{(1)}(B;t)$ is well-defined.

Now we put

$$\widetilde{b}_p^{(1)}(B;t) := \sum_{D \in \mathrm{GL}_{n-1}(\mathbb{Z}_p) \backslash \mathbf{D}_p^{(n-1)}(\mathbb{Z}_p)} \pi_p(D)\, b_p^{(1)}(B[D^{-1}];t)\, (p^{n-1}\, t^2)^{\mathrm{ord}_p(\det D)}.$$

Then, by Proposition 3.1, we obtain the following rationality theorem for the polynomial $\mathbf{B}_p^{(1)}(B;t)$ defined in §1:

Proposition 3.2. *For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have*

$$\mathbf{B}_p^{(1)}(B;p^{n-1/2}t)\widetilde{b}_p^{(1)}(B;p^{1/2}t) = \prod_{i=1}^{n-1}(1 - p^{2i}t^2).$$

Next, we study the standard L-function attached to a Hecke eigenform and some power series related to it. For a Hecke eigenform $\phi \in J_{k,1}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$, and $D \in \mathbf{D}_p^{(n-1)}(\mathbb{Z})$, let

$$\phi|_{k,1}\Gamma_{n-1}^J \mathbf{d}_{n-1}(D)\Gamma_{n-1}^J = \lambda_\phi(D)\phi$$

with $\lambda_\phi(D) \in \mathbb{C}$. Then we define a power series $Z_p(t,\phi)$ by

$$Z_p(t,\phi) := \sum_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \lambda_\phi(D) t^{\mathrm{ord}_p(\det D)},$$

where $\mathbf{ED}_p^{(n-1)}(\mathbb{Z})$ denotes the set of all elementary divisors of the form $p^{\alpha_1} \perp \cdots \perp p^{\alpha_{n-1}}$ with $0 \le \alpha_1 \le \cdots \le \alpha_{n-1}$. The following statement is shown by Murase and Sugano:

Proposition 3.3 (cf. Lemma 6.5 in [22], see also Theorem 5.5 in [3]). *Let $\phi \in J_{k,1}(\Gamma_{n-1}^J)$ be a Hecke eigenform whose Satake p-parameter is of the form $(\chi_\phi^{(1)}(p), \ldots, \chi_\phi^{(n-1)}(p)) \in (\mathbb{C}^\times)^{n-1}/W_{n-1}$. Then*

$$Z_p(t,\phi) = \prod_{i=1}^{n-1} \frac{1 - p^{2i}t^2}{(1 - \chi_\phi^{(i)}(p)p^{n-1/2}t)(1 - \chi_\phi^{(i)}(p)^{-1}p^{n-1/2}t)}.$$

Let

$$\mathscr{Z}_p^{(n-1)} := \left\{ \begin{pmatrix} V \\ W \end{pmatrix} \in \mathrm{M}_{2n-2,n-1}(\mathbb{Z}) \,\middle|\, V, W \in \mathbf{D}_p^{(n-1)}(\mathbb{Z}),\ \gcd(V,W)=1 \right\},$$

where $\gcd(V,W)=1$ means that V and W are coprime to each other. For each $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right) \in \mathscr{Z}_p^{(n-1)}$, $R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])$ and $(\lambda_1,\lambda_2) \in \mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}$, we put

$$M_{V,W,R} := \left(\begin{array}{c|c} {}^tW^{-1}\,{}^tV & {}^tW^{-1}RV^{-1} \\ \hline \mathbf{0}_{n-1} & WV^{-1} \end{array} \right) \in G_{n-1}(\mathbb{Z}[p^{-1}])$$

and

$$[\lambda_1,\lambda_2] := [(\lambda_1,\lambda_2), \lambda_1\,{}^t\lambda_2] = \left(\begin{array}{cc|cc} 1 & \lambda_1 & 0 & \lambda_2 \\ 0 & \mathbf{1}_{n-1} & {}^t\lambda_2 & \mathbf{0}_{n-1} \\ \hline 0 & 0 & 1 & 0 \\ 0 & \mathbf{0}_{n-1} & -\,{}^t\lambda_1 & \mathbf{1}_{n-1} \end{array} \right) \in H_{n-1}(\mathbb{Z}).$$

By combining Lemma 2.1 and some easy calculation (cf. [5]), we obtain the following:

Lemma 3.3. *We have*

$$\begin{aligned} \Gamma_{n-1}^J G_{n-1}(\mathbb{Z}[p^{-1}])\Gamma_{n-1}^J &= \bigcup_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \Gamma_{n-1}^J \mathbf{d}_{n-1}(D)\Gamma_{n-1}^J \\ &= \bigsqcup_{\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)} \bigsqcup_{R} \bigsqcup_{(\lambda_1,\lambda_2)} \Gamma_{n-1}^J [M_{V,W,R}] \cdot [\lambda_1,\lambda_2], \end{aligned}$$

where $\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)$, R *and* (λ_1,λ_2) *run respectively over*

- $(\mathbf{1}_{n-1} \perp \mathrm{GL}_{n-1}(\mathbb{Z})) \setminus \mathscr{Z}_p^{(n-1)}/\mathrm{GL}_{n-1}(\mathbb{Z})$,
- $\mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/{}^tW\mathrm{Sym}_{n-1}(\mathbb{Z})W$, *and*
- $(\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1}) + (\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1})M_{V,W,R}/(\mathbb{Z}^{n-1} \oplus \mathbb{Z}^{n-1})M_{V,W,R}$.

Furthermore, if $M_{V,W,R} \in \Gamma_{n-1}^J \mathbf{d}_{n-1}(D)\Gamma_{n-1}^J$ *with* $D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})$, *we have* $\mathrm{ord}_p(\det D) = \mathrm{ord}_p(\det V \det W\, \mu_p(R))$.

Therefore, we get the following explicit formula for the actions of Hecke operators:

Corollary. *For each* $\phi \in J_{k,1}(\Gamma_{n-1}^J)$, *we have*

$$\sum_{D \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})} (\phi\,|_{k,1} \Gamma_{n-1}^J \mathbf{d}_{n-1}(D)\Gamma_{n-1}^J)(\tau,z) = \sum_{\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)} \sum_{R} p^{(-2n+3)\delta_{V,W,R}} \det V^{k-1}$$

$$\times \det W^{-k} \sum_{(\lambda_1,\lambda_2)\in(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})/p^{\delta_{V,W,R}}(\mathbb{Z}^{n-1\,t}V\oplus\mathbb{Z}^{n-1})} \mathbf{e}(\tau[{}^t\lambda_1]+2\lambda_1{}^tz)$$
$$\times \phi(\tau[VW^{-1}]+R[W^{-1}],(z+\lambda_1\tau+\lambda_2)VW^{-1}),$$

where $\binom{V}{W}$ *and* R *run over the sets stated in Lemma* 3.3, *and* $\delta_{V,W,R} = \mathrm{ord}_p(\det V \det W \mu_p(R))$.

Proof. For each $\binom{V}{W} \in \mathscr{Z}_p^{(n-1)}$ and $R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])$, we have

$$\Gamma_{n-1}^J M_{V,W,R} \Gamma_{n-1}^J = \Gamma_{n-1}^J \mathbf{d}_{n-1}(D)\Gamma_{n-1}^J$$

for some $D = p^{\alpha_1}\perp\cdots\perp p^{\alpha_{n-1}} \in \mathbf{ED}_p^{(n-1)}(\mathbb{Z})$. Then we have

$$\begin{aligned}(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1}) &+ (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}/(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})M_{V,W,R}\\ &\simeq (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1}) + (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})\mathbf{d}_{n-1}(D)/(\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})\mathbf{d}_{n-1}(D)\\ &\simeq \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}D.\end{aligned}$$

It follows from Lemma 3.3 that $\#(\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}D) = p^{\delta_{V,W,R}}$ and $\alpha_1,\dots,\alpha_{n-1} \le \delta_{V,W,R}$. Thus we have a natural surjection

$$\pi : (\mathbb{Z}^{n-1}\oplus\mathbb{Z}^{n-1})/p^{\delta_{V,W,R}}(\mathbb{Z}^{n-1\,t}V\oplus\mathbb{Z}^{n-1}) \to \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}D,$$

and $\#\ker(\pi) = p^{(2n-3)\delta_{V,W,R}} \det V$. Thus the assertion holds. ∎

By the above corollary, we obtain the following conclusion:

Proposition 3.4. *Suppose that* $\phi\in J_{k,1}(\Gamma_{n-1}^J)$ *is a Hecke eigenform and the associated form* $\sigma(\phi) \in M_{k-1/2}^+(\Gamma_0^{(n-1)}(4))$ *under the Eichler–Zagier–Ibukiyama correspondence possesses a Fourier expansion*

$$\sigma(\phi)(\tau) = \sum_{B\in\mathrm{Sym}_{n-1}^*(\mathbb{Z}_p)_{\ge 0}} C_{\sigma(\phi)}(B)\mathbf{e}(\mathrm{tr}(B\tau)).$$

Then, for each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z})_{>0}^{(1)}$, *we have*

$$\begin{aligned}\prod_{i=1}^{n-1} &\frac{1-p^{2i}t^2}{(1-\chi_\phi^{(i)}(p)p^{n-1/2}t)(1-\chi_\phi^{(i)}(p)^{-1}p^{n-1/2}t)} C_{\sigma(\phi)}(B)\\ &= \sum_{\binom{V}{W}} b_p^{(1)}(B[{}^tV^{-1}];t)\, C_{\sigma(\phi)}(B[{}^tV^{-1}][W])\\ &\qquad \times p^{-(k-n-1)\,\mathrm{ord}_p(\det W)} p^{k\,\mathrm{ord}_p(\det V)} t^{\mathrm{ord}_p(\det V\det W)},\end{aligned}$$

where $\binom{V}{W}$ *runs over the set stated in Lemma* 3.3.

Proof. We put

$$\Lambda_p(t) = \sum_{D\in\mathbf{ED}_p^{(n-1)}(\mathbb{Z})} \Gamma_{n-1}^J \mathbf{d}_{n-1}(D)\Gamma_{n-1}^J\, t^{\mathrm{ord}_p(\det D)}.$$

By the Corollary to Lemma 3.3, we have

$$
\begin{aligned}
&(\phi|_{k,1}\Lambda_p(t))(\tau,z) = \sum_T \sum_r c_\phi(T,r) \\
&\times \sum_{\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right) \in (\mathbf{1}_{n-1}\perp \mathrm{GL}_{n-1}(\mathbb{Z}))\backslash \mathscr{X}_p^{(n-1)}/\mathrm{GL}_{n-1}(\mathbb{Z})} p^{(k-1)\operatorname{ord}_p(\det V) - k \operatorname{ord}_p(\det W)} t^{\operatorname{ord}_p(\det V \det W)} \\
&\times \mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}\,{}^tV]\tau + {}^t(r\,{}^tW^{-1}\,{}^tV)z)) \\
&\times \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/{}^tW \mathrm{Sym}_{n-1}(\mathbb{Z})W} \mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}]R))\, t^{\operatorname{ord}_p(\mu_p(R))} \\
&\times \sum_{\lambda_1 \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}.{}^tV} p^{-(2n-3)\delta_{V,W,R}} \mathbf{e}(\mathrm{tr}(2\,{}^t\lambda_1 z + {}^t(r\,{}^tW^{-1}\,{}^tV + \lambda_1)\lambda_1\tau)) \\
&\times \sum_{\lambda_2 \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} \mathbf{e}(\mathrm{tr}({}^t(r\,{}^tW^{-1}\,{}^tV + \lambda_1)\lambda_2)).
\end{aligned}
$$

Since

$$
\sum_{\lambda_2 \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} \mathbf{e}(\mathrm{tr}({}^t(r\,{}^tW^{-1}\,{}^tV + \lambda_1)\lambda_2))
= \begin{cases} p^{(n-1)\delta_{V,W,R}} & \text{if } r\,{}^tW^{-1} \in \mathbb{Z}^{n-1}, \\ 0 & \text{otherwise,} \end{cases}
$$

and

$$
\begin{aligned}
&\sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/{}^tW \mathrm{Sym}_{n-1}(\mathbb{Z})W} \mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}]R)) t^{\operatorname{ord}_p(\mu_p(R))} \\
&= \begin{cases} (\det W)^n \displaystyle\sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\mathrm{tr}(T[{}^tW^{-1}]R)) t^{\operatorname{ord}_p(\mu_p(R))} \\ \qquad\qquad \text{if } T[{}^tW^{-1}] \in \mathrm{Sym}^*_{n-1}(\mathbb{Z}), \\ 0 \qquad\qquad \text{otherwise,} \end{cases}
\end{aligned}
$$

we have

$$
\begin{aligned}
&(\phi|_{k,1}\Lambda_p(t))(\tau,z) \\
&\quad = \sum_T \sum_r \sum_{\left(\begin{smallmatrix} V \\ W \end{smallmatrix}\right)} p^{k \operatorname{ord}_p(\det V) + (-k+n+1)\operatorname{ord}_p(\det W)} t^{\operatorname{ord}_p(\det V \det W)} \\
&\quad\quad \times \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\mathrm{tr}(TR))(pt)^{\operatorname{ord}_p(\mu_p(R))}
\end{aligned}
$$

$$\times \sum_{\lambda_1 \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}\,{}^tV} p^{-(n-1)\delta_{V,W,R}} c_\phi(T[{}^tW], r\,{}^tW)$$
$$\times \mathbf{e}(\mathrm{tr}({}^t(r\,{}^tV + 2\lambda_1)z))\,\mathbf{e}(\mathrm{tr}((T[{}^tV] + {}^t(r\,{}^tV + \lambda_1)\lambda_1)\tau)).$$

For a fixed $r_0 \in \mathbb{Z}^{n-1}$, we put

$$\mathcal{S}_1(r_0) = \{\lambda_1 \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}\,{}^tV \mid 2\lambda_1 \equiv r_0 \bmod \mathbb{Z}^{n-1}\,{}^tV\},$$

and

$$\mathcal{S}_2(r_0) = \{r \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1} \mid r\,{}^tV \equiv r_0 \bmod 2\mathbb{Z}^{n-1}\}.$$

For each $\lambda_1 \in \mathcal{S}_1(r_0)$, the map $\lambda_1 \mapsto (2\lambda_1 - r_0)\,{}^tV^{-1}$ induces a bijection between $\mathcal{S}_1(r_0)$ and $\mathcal{S}_2(r_0)$. Thus we have

$$(\phi|_{k,1}\Lambda_p(t))(\tau,z) = \sum_T \sum_{r_0} \sum_{\binom{V}{W}} p^{k\,\mathrm{ord}_p(\det V)-(k-n-1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det V \det W)}$$
$$\times \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\mathrm{tr}(TR))(pt)^{\mathrm{ord}_p(\mu_p(R))} p^{-(n-1)\delta_{V,W,R}}$$
$$\times \sum_{r \in \mathcal{S}_2(r_0)} c_\phi(T[{}^tW], r\,{}^tW)\mathbf{e}(\mathrm{tr}({}^tr_0 z))\mathbf{e}(\mathrm{tr}((T[{}^tV]+({}^tr_0r_0 - {}^t(r\,{}^tV)(r\,{}^tV))/4)\tau))$$
$$= \sum_{T_0} \sum_{r_0} \mathbf{e}(\mathrm{tr}(T_0\tau + {}^tr_0 z))$$
$$\times \sum_{\binom{V}{W}} \sum_{r \in \mathcal{S}_2(r_0)} p^{k\,\mathrm{ord}_p(\det V)-(k-n-1)\,\mathrm{ord}_p(\det W)} p^{-(n-1)\delta_{V,W,R}}$$
$$\times c_\phi((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}][{}^tW] + ({}^trr/4)[{}^tW], r\,{}^tW)$$
$$\times \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} \mathbf{e}(\mathrm{tr}(((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}] + {}^trr/4)R))(pt)^{\mathrm{ord}_p(\mu_p(R))}.$$

Then, for a fixed $r \in \mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1}$, the map

$$(r + 2\mathbb{Z}^{n-1}) + 2p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}/2p^{\delta_{V,W,R}}\mathbb{Z}^{n-1} \ni r + 2u \mapsto u \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}$$

is a bijection, and we have

$$c_\phi((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}][{}^tW] + ({}^t(r+2u)(r+2u)/4)[{}^tW], (r+2u)\,{}^tW)$$
$$= c_\phi((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}][{}^tW] + ({}^trr/4)[{}^tW], r\,{}^tW).$$

Thus we have

$$(\phi|_{k,1}\Lambda_p(t))(\tau,z) = \sum_{T_0} \sum_{r_0} \mathbf{e}(\mathrm{tr}(T_0\tau + {}^tr_0 z))$$
$$\times \sum_{\binom{V}{W}} p^{k\,\mathrm{ord}_p(\det V)-(k-n-1)\,\mathrm{ord}_p(\det W)} t^{\mathrm{ord}_p(\det V \det W)}$$

$$\times \sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} (pt)^{\mathrm{ord}_p(\mu_p(R))} p^{-(n-1)\delta_{V,W,R}}$$
$$\times \sum_{\substack{r \in \mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1} \\ r\,{}^tV \equiv r_0 \bmod 2\mathbb{Z}^{n-1}}} c_\phi((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}][{}^tW] + ({}^trr/4)[{}^tW], r\,{}^tW)$$
$$\times \sum_{u \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} \mathbf{e}(\mathrm{tr}(((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}] + {}^trr/4 + {}^tuu + {}^tur/2 + {}^tru/2)R)).$$

We easily see that for an element $r \in \mathbb{Z}^{n-1}/2\mathbb{Z}^{n-1}$, the sum

$$\sum_{R \in \mathrm{Sym}_{n-1}(\mathbb{Z}[p^{-1}])/\mathrm{Sym}_{n-1}(\mathbb{Z})} (pt)^{\mathrm{ord}_p(\mu_p(R))} p^{-(n-1)\delta_{V,W,R}}$$
$$\times \sum_{u \in \mathbb{Z}^{n-1}/p^{\delta_{V,W,R}}\mathbb{Z}^{n-1}} \mathbf{e}(\mathrm{tr}(((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}] + {}^trr/4 + {}^tuu + {}^tur/2 + {}^tru/2)R))$$

equals $b_p^{(1)}((4T_0 - {}^tr_0r_0)[{}^tV^{-1}]; t)$ or 0 according as $(T_0 - {}^tr_0r_0/4)[{}^tV^{-1}] + {}^trr/4 \in \mathrm{Sym}^*_{n-1}(\mathbb{Z})$ (that is, $(4T_0 - {}^tr_0r_0)[{}^tV^{-1}] \in \mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}$) or not. In the former case, such a vector r is uniquely determined by T_0, r_0, and V, which will be denoted by $r_1 = r_1(T_0, r_0, V)$. Furthermore, we have

$$\begin{aligned}((4T_0 - {}^tr_0r_0)[{}^tV^{-1}] + {}^tr_1r_1)[{}^tV] \\ = (4T_0 - {}^tr_0r_0) + {}^t(r_1\,{}^tV)r_1\,{}^tV \in 4\mathrm{Sym}^*_{n-1}(\mathbb{Z}_p),\end{aligned}$$

and $r_1{}^tV \equiv r_0 \bmod 2\mathbb{Z}^{n-1}$ in that case. Thus

$$\begin{aligned}(\phi|_{k,1}\varLambda_p(t))(\tau, z) \\ = \sum_{T_0}\sum_{r_0} \mathbf{e}(\mathrm{tr}(T_0\tau + {}^tr_0z)) \sum_{\binom{V}{W}} p^{k\,\mathrm{ord}_p(\det V) - (k-n-1)\,\mathrm{ord}_p(\det W)} \\ \times t^{\mathrm{ord}_p(\det V \det W)} b_p^{(1)}((4T_0 - {}^tr_0r_0)[{}^tV^{-1}]; t) \\ \times c_\phi((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}][{}^tW] + ({}^tr_1r_1/4)[{}^tW], r_1{}^tW).\end{aligned}$$

Now we take an element $B \in \mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}$ so that $B = 4T_0 - {}^tr_0r_0$ with $T_0 \in \mathrm{Sym}^*_{n-1}(\mathbb{Z})$ and $r_0 \in \mathbb{Z}^{n-1}$. Then we have

$c_\phi(T_0, r_0) = C_{\sigma(\phi)}(B)$,
$c_\phi((T_0 - {}^tr_0r_0/4)[{}^tV^{-1}][{}^tW] + ({}^tr_1r_1/4)[{}^tW], r_1{}^tW) = C_{\sigma(\phi)}(B[{}^tV^{-1}][{}^tW])$,

and

$$b_p^{(1)}((4T_0 - {}^tr_0r_0)[{}^tV^{-1}]; t) = b_p^{(1)}(B[{}^tV^{-1}]; t).$$

Since $\phi|_{k,1}\varLambda_p(t) = Z_p(t, \phi)\phi$, the assertion follows immediately from Proposition 3.3. ∎

For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}_{>0}$, let $\widetilde{G}_{\phi,p}(B;t)$ be the polynomial in t defined in §1. Then, by making use of the same argument as in [5] combined with Propositions 3.2 and 3.4, we obtain the following:

Theorem 3.1. *Let n and k be positive even integers such that $k > n+1$. Suppose that $\phi \in J_{k,1}(\Gamma^J_{n-1})$ is a Hecke eigenform whose Satake p-parameter is of the form $(\chi^{(1)}_\phi(p), \dots, \chi^{(n-1)}_\phi(p)) \in (\mathbb{C}^\times)^{n-1}/W_{n-1}$. Then, for each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}_{>0}$, we have*

$$\frac{\mathbf{B}^{(1)}_p(B;p^{n-1/2}t)\widetilde{G}_{\phi,p}(B;t)}{\prod_{i=1}^{n-1}(1-\chi^{(i)}_\phi(p)p^{n-1/2}t)(1-\chi^{(i)}_\phi(p)^{-1}p^{n-1/2}t)} = \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathbf{D}^{(n-1)}_p(\mathbb{Z})} C_{\sigma(\phi)}(B[W])p^{-(k-n-1)\,\mathrm{ord}_p(\det W)}t^{\mathrm{ord}_p(\det W)}.$$

For each $D \in \mathrm{M}_{n-1}(\mathbb{Z}) \cap \mathrm{GL}_{n-1}(\mathbb{Q})$, we define the *generalized global Möbius function* $\pi(D)$ as $\prod_p \pi_p(D)$, where π_p is the local Möbius function defined in §1. We easily see that this is a finite product. For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}_{>0}$, we put

$$\widetilde{H}_\phi(B;s) = \sum_{D \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathrm{M}_{n-1}(\mathbb{Z})\cap \mathrm{GL}_{n-1}(\mathbb{Q})} \pi(D)C_{\sigma(\phi)}(B[D^{-1}])\det D^{-s+k} \quad (s \in \mathbb{C}),$$

which is a finite sum, and $\widetilde{H}_\phi(B;s) = \prod_p \widetilde{G}_{\phi,p}(B;p^{-s})$. In addition, we also put $\mathbf{B}^{(1)}(B;s) = \prod_p \mathbf{B}^{(1)}_p(B;p^{-s})$. Then Theorem 3.1 can be restated globally as follows:

Theorem 3.2. *Under the same situation as above, we have*

$$\mathbf{B}^{(1)}(B;s)L(s,\phi,\mathrm{St})\widetilde{H}_\phi(B;s+n-1/2) = \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z})\backslash \mathrm{M}_{n-1}(\mathbb{Z})\cap \mathrm{GL}_{n-1}(\mathbb{Q})} C_{\sigma(\phi)}(B[W])(\det W)^{-s-k+3/2}.$$

Moreover, by applying Theorem 3.1 to the Jacobi Eisenstein series $\mathfrak{E}^{(n-1)}_{k,1} = e^{(n-1)}_{k,1} \in J_{k,1}(\Gamma^J_{n-1})$, we obtain the following conclusion:

Theorem 3.3. *Let n and k be as above. For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$,*

$$\frac{\mathbf{B}^{(1)}_p(B;p^{n-1/2}t)\widetilde{G}^{(1)}_p(B;p^{k-(n+1)/2},p^{(n+1)/2}t)}{\prod_{i=1}^{n-1}(1-p^{j-1}p^{k-(n+1)/2}p^{(n+1)/2}t)(1-p^{j-1}p^{-k+(n+1)/2}p^{(n+1)/2}t)} = \sum_{W \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)\backslash \mathbf{D}^{(n-1)}_p(\mathbb{Z}_p)} \widetilde{F}^{(1)}_p(B[W];p^{k-(n+1)/2})(p^{(n+1)/2}t)^{\mathrm{ord}_p(\det W)},$$

where $\widetilde{F}^{(1)}_p(B;X)$ and $\widetilde{G}^{(1)}_p(B;X,t)$ are polynomials defined in §1.

Proof. Suppose that $B \in \mathrm{Sym}_{n-1}(\mathbb{Z})^{(1)}_{>0}$. The Bth Fourier coefficient of $\sigma(e^{(n-1)}_{k,1}) \in M^+_{k-1/2}(\Gamma^{(n-1)}_0(4))$ is expressed as

$$\xi(n,k)L(1-k/2+n/2,\chi_{B^{(1)}})\mathfrak{f}(B^{(1)})^{k-(n+1)/2} \prod_{p\mid \mathfrak{f}(B^{(1)})} \widetilde{F}^{(1)}_p(B;p^{k-(n+1)/2})$$

(cf. Proposition 2.4). Thus the assertion follows from Theorem 3.1 and the Corollary to Proposition 2.2. Moreover, we easily see that it can be extended to any $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$. ∎

For each $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, let $R^{(1)}_p(B;X,t)$ be the formal power series in $X+X^{-1}$ and t defined in §1. Eventually, we obtain the rationality for $R^{(1)}_p(B;X,t)$ as follows:

THEOREM 3.4. *Let n be a positive even integer. For $B \in \mathrm{Sym}_{n-1}(\mathbb{Z}_p)^{(1)}$, we have*

$$R^{(1)}_p(B;X,t) = \frac{\mathbf{B}^{(1)}_p(B;p^{n/2-1}t)\widetilde{G}^{(1)}_p(B;X,t)}{\prod_{j=1}^{n-1}(1-p^{j-1}Xt)(1-p^{j-1}X^{-1}t)}.$$

Proof. We write both sides of the above equation as power series in t as

$$R^{(1)}_p(B;X,t) = \sum_{i=1}^{\infty} A_i(X)t^i,$$

and

$$\frac{\mathbf{B}^{(1)}_p(B;p^{n/2-1}t)\widetilde{G}^{(1)}_p(B;X,t)}{\prod_{j=1}^{n-1}(1-p^{j-1}Xt)(1-p^{j-1}X^{-1}t)} = \sum_{i=1}^{\infty} B_i(X)t^i,$$

where for each i, $A_i(X)$ and $B_i(X)$ are polynomials in $X+X^{-1}$. Then, by Theorem 3.3,

$$A_i(p^{k-(n+1)/2}) = B_i(p^{k-(n+1)/2})$$

for infinitely many k. Thus $A_i(X) = B_i(X)$ for each i, completing the proof. ∎

REMARK. For a given pair of positive even integers n and k as in Theorem 3.1, let $f \in S_{2k-n}(\Gamma_1)$ be a Hecke eigenform, which possesses a Fourier expansion

$$f(z) = \sum_{N=1}^{\infty} a_f(N)\mathbf{e}(Nz) \quad (z \in \mathfrak{H}_1)$$

normalized by $a_f(1)=1$. For each rational prime p, we denote by α_p the Satake p-parameter of f, that is, an algebraic number determined by the condition $\alpha_p + \alpha_p^{-1} = a_f(p)\,p^{-k+(n+1)/2}$ uniquely up to inversion. By substituting $X=\alpha_p$ in the main identity of Theorem 3.4, we can also derive a result similar to Theorem 3.3 for a power series related to the first Fourier–Jacobi coefficient of a Siegel cusp form $F \in S_k(\Gamma_n)$ which is connected to f under Ikeda's lifting procedure (cf. [12]). We note that it will play an important role in a proof of Ikeda's conjecture on the period of such an F, which was proposed in [13] (cf. [16, 17]).

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References

[1] A. N. Andrianov, *Quadratic Forms and Hecke Operators*, Grundlehren Math. Wiss. 286, Springer, Berlin, 1987.

[2] T. Arakawa, *Siegel's formula for Jacobi forms*, Int. J. Math. 4 (1993), 689–719.

[3] —, *Jacobi Eisenstein series and a basis problem for Jacobi forms*, Comment. Math. Univ. St. Pauli 43 (1994), 181–216.

[4] S. Böcherer, *Über die Fourier–Jacobi-Entwicklung Siegelscher Eisensteinreihen*, Math. Z. 183 (1983), 21–46.

[5] —, *Eine Rationalitätsatz für formale Heckereihen zur Siegelschen Modulgruppe*, Abh. Math. Sem. Univ. Hamburg 56 (1986), 35–47.

[6] S. Böcherer and F. Sato, *Rationality of certain formal power series related to local densities*, Comment. Math. Univ. St. Pauli 36 (1987), 53–86.

[7] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progr. Math. 55, Birkhäuser Boston, Boston, MA, 1985.

[8] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren Math. Wiss. 254, Springer, Berlin, 1983.

[9] S. Hayashida, *Skew-holomorphic Jacobi forms of index 1 and Siegel modular forms of half-integral weight*, J. Number Theory 106 (2004), 200–218.

[10] S. Hayashida, H. Kawamura and N.-P. Skoruppa, *Linear correspondence between Jacobi forms of integral weight and Siegel modular forms of half-integral weight*, preprint, 2008.

[11] T. Ibukiyama, *On Jacobi forms and Siegel modular forms of half integral weights*, Comment. Math. Univ. St. Pauli 41 (1992), 109–124.

[12] T. Ikeda, *On the lifting of elliptic modular forms to Siegel cusp forms of degree 2n*, Ann. of Math. 154 (2001), 641–681.

[13] —, *Pullback of the lifting of elliptic cusp forms and Miyawaki's conjecture*, Duke Math. J. 131 (2006), 469–497.

[14] H. Katsurada, *An explicit formula for Siegel series*, Amer. J. Math. 121 (1999), 415–452.

[15] H. Katsurada and H. Kawamura, *A certain Dirichlet series of Rankin–Selberg type associated with the Ikeda lifting*, J. Number Theory 128 (2008), 2025–2052.

[16] —, —, *On Ikeda's conjecture on the period of the Ikeda lift and its application*, RIMS Kôukyûroku Bessatsu, to appear.

[17] —, —, *Ikeda's conjecture on the period of the Ikeda lift*, preprint, 2008.

[18] Y. Kitaoka, *Dirichlet series in the theory of Siegel modular forms*, Nagoya Math. J. 95 (1984), 73–84.

[19] W. Kohnen, *Modular forms of half-integral weight on $\Gamma_0(4)$*, Math. Ann. 248 (1980), 249–266.

[20] A. Murase, *L-functions attached to Jacobi forms of degree n. Part I. The basic identity*, J. Reine Angew. Math. 401 (1989), 122–156.

[21] —, *L-functions attached to Jacobi forms of degree n. Part II. Functional equation*, Math. Ann. 290 (1991), 247–276.

[22] A. Murase and T. Sugano, *Whittaker–Shintani functions on the symplectic group of Fourier–Jacobi type*, Compos. Math. 79 (1991), 321–349.

[23] G. Shimura, *On Eisenstein series*, Duke Math. J. 50 (1983), 417–476.

[24] —, *Euler products and Fourier coefficients of automorphic forms on symplectic groups*, Invent. Math. 116 (1994), 531–576.

[25] —, *Zeta functions and Eisenstein series on mataplectic groups*, ibid. 121 (1995), 21–60.

[26] N.-P. Skoruppa, *Explicit formulas for the Fourier coefficients of Jacobi and elliptic forms*, ibid. 102 (1990), 501–520.

[27] —, *Developments in the theory of Jacobi forms*, in: Automorphic Functions and their Applications (Khabarovsk, 1988), Acad. Sci. USSR, Inst. Appl. Math., Khabarovsk, 1990, 167–185.

[28] V. G. Zhuravlëv, *Euler expansions of theta-transformations of Siegel modular forms of half integer weight and their analytic properties*, Math. USSR Sb. 51 (1985), 169–190.

[29] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg 59 (1989), 191–224.

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