ACTA ARITHMETICA
145.3 (2010)

On certain Diophantine systems with infinitely many parametric solutions and applications

by

Maciej Ulas (Kraków and Warszawa)

1. Introduction. Let $f(x,y) = y^2 - x^n$, where n is an odd integer. In [4], we proved that for any quadruple a, b, c, d of distinct integers the set of rational parametric solutions of the system

$$\frac{f(x_1,y_1)}{a} = \frac{f(x_2,y_2)}{b} = \frac{f(x_3,y_3)}{c} = \frac{f(x_4,y_4)}{d}$$

is infinite. In the cited paper, this result was used to show that if $C_i : y^2 = x^n + a_i$, where $a_i \in \mathbb{Z}\backslash\{0\}$ are pairwise distinct, then there exists a polynomial $D \in \mathbb{Z}[t]$ such that the $\mathbb{Q}(t)$-rank of the Jacobian variety $\mathrm{Jac}(C_{i,D})$ is positive, where $C_{i,D} : y^2 = x^n + a_i D(t)$ for $i = 1,2,3,4$. Similar results were proved in [8, 9] and [3], where instead of $f(x,y)$, we considered $g(x,y) = (y^2 - x^3)/x$ and $g(x,y) = (y^2 - x^5)/x$ respectively. In the light of these results, it is natural to ask what can be said about a general system of the form

$$\frac{h(x_1,y_1)}{a_1} = \frac{h(x_2,y_2)}{a_2} = \cdots = \frac{h(x_k,y_k)}{a_k}, \tag{1}$$

where $h \in \mathbb{Z}[x,y]$ and k is a fixed positive integer. In general, this is a difficult question. The most interesting but difficult case is that of a homogeneous form h. It seems that the only pertinent results available concern the case where all a_i are equal and $\deg h = 2, 3$. In the case of $\deg h = 2$, the problem is related to the construction of a rational number A such that the curve $C : h(x,y) = A$ is rational over $\mathbb{Q}$. In the case of $\deg h = 3$, the problem is related to the construction of a rational number A such that the curve $C : h(x,y) = A$ (i.e. h is an elliptic curve with nonzero discriminant) has infinitely many solutions in rationals. In this connection, we mention the work of Choudhry and Wróblewski [1], who showed that if $h(x,y) = x^4 - y^4$

2010 *Mathematics Subject Classification*: 11D25, 11D41, 11G05, 11G30.
Key words and phrases: higher twists of superelliptic curves, rank of superelliptic jacobian, rational points.

DOI: 10.4064/aa145-3-7

then the system of equations

$$h(x_1, y_1) = h(x_2, y_2) = h(x_3, y_3)$$

has infinitely many nontrivial solutions in integers (a *nontrivial* solution is a solution that cannot be obtained from another by multiplication by a nonzero integer).

In this paper, we investigate the question of solvability of the system (1) with the additional condition $\min\{\deg_x h, \deg_y h\} \geq 3$ and h of the form $y^n - x^m$.

2. First result. In this section, we are interested in constructing rational parametric solutions of the system of Diophantine equations

$$\frac{y_1^3 - x_1^m}{a} = \frac{y_2^3 - x_2^m}{b} = \frac{y_3^3 - x_3^m}{c}, \tag{2}$$

where a, b, c are pairwise distinct integers and $m \in \mathbb{N}_+$ such that $(3, m) = 1$.

Before we state the main result of this section, let us recall what the torsion part of the curve $E : y^2 = x^3 + q$ looks like for a fixed $q \in \mathbb{Z}$ [7, p. 323]. If $q = 1$, then $\operatorname{Tors} E(\mathbb{Q}) \cong \mathbb{Z}/6\mathbb{Z}$. If $q \neq 1$ and q is a square in $\mathbb{Z}$, then $\operatorname{Tors} E(\mathbb{Q}) = \{\mathcal{O}, (0, \sqrt{q}), (0, -\sqrt{q})\}$. If $q = -432$, then $\operatorname{Tors} E(\mathbb{Q}) = \{\mathcal{O}, (12, 36), (12, -36)\}$. If $q \neq 1$ and q is a cube in $\mathbb{Z}$, then $\operatorname{Tors} E(\mathbb{Q}) = \{\mathcal{O}, (-\sqrt[3]{q}, 0)\}$. In the remaining cases, $\operatorname{Tors} E(\mathbb{Q}) = \{\mathcal{O}\}$. Therefore, if $\mathcal{E} : y^2 = x^3 + Q$, where $Q \in \mathbb{Z}[u, v, w] \setminus \mathbb{Z}$, is an elliptic curve defined over the field $\mathbb{Q}(u, v, w)$ and if on $\mathcal{E}$ there exists a $\mathbb{Q}(u, v, w)$-rational point $P = (x, y)$ with $xy \neq 0$, then the order of P in the group $\mathcal{E}(\mathbb{Q}(u, v, w))$ is not finite provided that $\mathcal{E}$ is not isomorphic to an elliptic curve defined over $\mathbb{Q}$. Thus, in that case the curve $\mathcal{E}$ over $\mathbb{Q}(u, v, w)$ has a positive rank.

Now we are ready to prove the following theorem.

THEOREM 2.1. *Let $a, b, c \in \mathbb{Z} \setminus \{0\}$ and suppose that $(3, m) = 1$. Then the system (2) has infinitely many rational three-parameter solutions.*

Proof. We consider the variety $\mathcal{U}$ defined by (2). Since we are looking for parametric solutions, we are interested in nontrivial points on $\mathcal{U}$, i.e. points which satisfy $y_i^3 \neq x_i^m$ and $x_i y_i \neq 0$, for $i = 1, 2, 3$. Let $\alpha, \beta \in \mathbb{Z}$ be such that $m\beta - 3\alpha = 1$. Note that this is the only place where we need the condition $(3, m) = 1$. Put

$$\begin{aligned} x_1 &= u^3 T^\beta, & x_2 &= v^3 T^\beta, & x_3 &= w^3 T^\beta, \\ y_1 &= p T^\alpha, & y_2 &= q T^\alpha, & y_3 &= r T^\alpha, \end{aligned} \tag{3}$$

where u, v, w are rational parameters and p, q, r, T are variables. Now, if $T = (bp^3 - aq^3)/(bu^{3m} - av^{3m})$, then the first equation defining the variety $\mathcal{U}$ is satisfied. On the other hand, if $T = (cq^3 - br^3)/(cv^{3m} - bw^{3m})$, then the second equation defining $\mathcal{U}$ is satisfied. From the above, after some necessary simplifications, we can see that in order to find $\mathbb{Q}(u,v,w)$-rational points on our variety we must show that the Diophantine equation

$$Ap^3+Bq^3+Cr^3=0, \tag{4}$$

where

$$A=bw^{3m}-cv^{3m},\qquad B=cu^{3m}-aw^{3m},\qquad C=av^{3m}-bu^{3m}, \tag{5}$$

has infinitely many nontrivial $\mathbb{Q}(u,v,w)$-rational solutions. From a geometric viewpoint, equation (4) defines a cubic curve $\mathcal{C}$ over the field $\mathbb{Q}(u,v,w)$. This is an elliptic curve with the $\mathbb{Q}(u,v,w)$-rational point $P=[u^m:v^m:w^m]$. Doubling the point P on the curve $\mathcal{C}$, we find that $2P=[p',q',r']$, where

$$\begin{aligned} p'&=-u^m(cu^{3m}v^{3m}+bu^{3m}w^{3m}-2av^{3m}w^{3m}),\\ q'&=-v^m(cu^{3m}v^{3m}-2bu^{3m}w^{3m}+av^{3m}w^{3m}),\\ r'&=w^m(2cu^{3m}v^{3m}-bu^{3m}w^{3m}-av^{3m}w^{3m}). \end{aligned} \tag{6}$$

In this case, the value of $T=T(u,v,w)$ is given by $T=-G(u^m,v^m,w^m)$, where

$$\begin{aligned} G(u,v,w)=&\,c^3u^9v^9+3bc^2u^9v^6w^3+3ac^2u^6v^9w^3+3b^2cu^9v^3w^6\\ &-21abcu^6v^6w^6+3a^2cu^3v^9w^6+b^3u^9w^9+3ab^2u^6v^3w^9\\ &+3a^2bu^3v^6w^9+a^3v^9w^9. \end{aligned}$$

Using these quantities, we find that a solution of the system (2) has the form

$$\begin{aligned} x_1&=u^3T(u,v,w)^\beta, & y_1&=p'(u,v,w)T(u,v,w)^\alpha,\\ x_2&=v^3T(u,v,w)^\beta, & y_2&=q'(u,v,w)T(u,v,w)^\alpha,\\ x_3&=w^3T(u,v,w)^\beta, & y_3&=r'(u,v,w)T(u,v,w)^\alpha. \end{aligned}$$

By a standard argument, we find that $\mathcal{C}$ is birationally equivalent to the elliptic curve with Weierstrass equation

$$\mathcal{E}: ZY^2=X^3+16A^2B^2C^2Z^3.$$

The mapping $\psi:\mathcal{E}\to\mathcal{C}$ is given by

$$\psi(X,Y,Z)=(-4ABCXYZ,-4ABC(BY^3-CZ^3),AX^3),$$

where A,B,C are given by (5).

Now we will see that the curve $\mathcal{E}$ has positive rank over the field $\mathbb{Q}(u,v,w)$. Note that on $\mathcal{E}$, we have the point $S=(X,Y,1)$, where

$$X=-\frac{4BCq'r'}{p'^2},\qquad Y=-\frac{4BC(Bq'^3-Cr'^3)}{p'^3}.$$

Here p',q',r' are defined by (6). Note also that $XY\neq 0$ in $\mathbb{Q}(u,v,w)$ and $16A^2B^2C^2$ is not of the form $a'F(u,v,w)^6$ for $a'\in\mathbb{Z}, F\in\mathbb{Z}[u,v,w]$. By the remark at the beginning of this section, the point S is of infinite order in the group $\mathcal{E}(\mathbb{Q}(u,v,w))$. This shows that the set of rational three-parameter solution of the system (2) is infinite. ∎

REMARK 2.2 (see also [2]). If C is a curve of genus $g \geq 2$ and if there exists a morphism from C to an elliptic curve E, then $\mathrm{Jac}(C)$, the Jacobian variety of C, is isogenous to $A \times E$, where A is an Abelian variety of dimension $g-1$. In particular, if the rank of E is positive then so is the rank of $\mathrm{Jac}(C)$.

As an application of our result, we prove the following theorem.

THEOREM 2.3. *Let $a_i \in \mathbb{Z} \setminus \{0\}$, $i = 1,2,3$. Suppose that $a_i \neq a_j$ for $i \neq j$, and let $m \in \mathbb{N}_+$ be such that $(3,m) = 1$. Consider the superelliptic curves defined by*

$$C_i : y^3 = x^{2m} + a_i, \qquad i = 1,2,3.$$

Then there exists a polynomial $D \in \mathbb{Z}[u,v,w]$ such that the Jacobian variety associated with the curve $C_{i,D} : y^3 = x^{2m} + a_i D(u,v,w)$ has a positive rank over $\mathbb{Q}(u,v,w)$ for $i = 1,2,3$.

Proof. First, note that from Theorem 2.1 we can deduce the existence of a polynomial $D \in \mathbb{Z}[u,v,w]$ such that the set of $\mathbb{Q}(u,v,w)$-rational points on the curve $C_{i,D}$ is nonempty for $i = 1,2,3$. (In fact, infinitely many such polynomials exist.) Second, on each curve $C_{i,D}$, we have a point $P_i = (x_i, y_i)$ satisfying $x_i y_i \neq 0$ for $i = 1,2,3$. Moreover, one has the morphism:

$$\varphi_i : C_{i,D} \ni (x,y) \mapsto (y, x^m) \in E_i : Y^2 = X^3 - a_i D(u,v,w).$$

On the curve E_i we have a $\mathbb{Q}(u,v,w)$-rational point $Q_i = (y_i, x_i^m)$ for $i = 1,2,3$. As $-a_i D(u,v,w)$ is not of the form $a' F(u,v,w)^6$ for $a' \in \mathbb{Z}$ and $F \in \mathbb{Z}(u,v,w)$, we deduce that E_i is not birationally equivalent to an elliptic curve defined over $\mathbb{Q}$. As the coordinates of the point Q_i are nonzero, Q_i is of infinite order on the curve E_i for $i = 1,2,3$. Using now Remark 2.2, we deduce that the point $P_i = (x_i, y_i)$ which lies on the curve $C_{i,D}$ for $i = 1,2,3$ corresponds to the divisor of infinite order in the group $\mathrm{Jac}(C_{i,D})(\mathbb{Q}(u,v,w))$. Thus the rank of $\mathrm{Jac}(C_{i,D})(\mathbb{Q}(u,v,w))$ is positive for $i = 1,2,3$. ∎

3. A generalization of Theorem 2.1. In this section we will prove the following theorem.

THEOREM 3.1. *Let $a,b,c \in \mathbb{Z} \setminus \{0\}$. Let $m \in \mathbb{N}_+$ be such that $(3,m) = 1$ and let $0 < k < m$. Then the system*

$$\frac{y_1^3 - x_1^m}{a x_1^k} = \frac{y_2^3 - x_2^m}{b x_2^k} = \frac{y_3^3 - x_3^m}{c x_3^k} \tag{7}$$

has infinitely many rational parametric solutions depending on three parameters.

Proof. The method of proof is similar to that for Theorem 2.1. We consider the variety $\mathcal{V}$ defined by (7). We are interested in nontrivial points on $\mathcal{V}$, i.e. points which satisfy $y_i^3 \neq x_i^m$ and $x_i y_i \neq 0$ for $i = 1, 2, 3$. Take $\alpha, \beta \in \mathbb{Z}$ such that $m\beta - 3\alpha = 1$. Put

$$(8)\qquad \begin{aligned} x_1 &= u^3 T^\beta, & x_2 &= v^3 T^\beta, & x_3 &= w^3 T^\beta, \\ y_1 &= p u^k T^\alpha, & y_2 &= q v^k T^\alpha, & y_3 &= r w^k T^\alpha, \end{aligned}$$

where u, v, w are rational parameters and p, q, r, T are variables.

Now, note that if $T = (bp^3 - aq^3)/(bu^{3(m-k)} - av^{3(m-k)})$ then the first equation defining the variety $\mathcal{V}$ is satisfied. On the other hand, if $T = (cq^3 - br^3)/(cv^{3(m-k)} - bw^{3(m-k)})$ then the second equation defining $\mathcal{V}$ is satisfied. Consequently, to finish the proof, it is enough to show that the set of $\mathbb{Q}(u, v, w)$-rational points on the curve $\mathcal{C}'$: $A'p^3 + B'q^3 + C'r^3 = 0$, where

$$A' = bw^{3(m-k)} - cv^{3(m-k)}, \qquad B' = cu^{3(m-k)} - aw^{3(m-k)},$$
$$C' = av^{3(m-k)} - bu^{3(m-k)},$$

is infinite. But this is obvious, as $\mathcal{C}'$ is obtained from the curve $\mathcal{C}$ in the proof of Theorem 2.1, where instead of m we take $m-k$. As we have proved that $\mathcal{C}$, for any given nonzero m (not necessarily satisfying the condition $(3, m) = 1$) has infinitely many $\mathbb{Q}(u, v, w)$-rational points, the same is true for $\mathcal{C}'$. ∎

4. Parametric solutions of the Diophantine equation $(y_1^4 - x_1^{2n})/a = (y_2^4 - x_2^{2n})/b$**,** n **odd.** Before stating our result, let us recall what the torsion part of the curve $E : y^2 = x^3 + px$, with a fixed $p \in \mathbb{Z} \setminus \{0\}$, looks like (see [7, p. 311]). If $p = 4$, then $\operatorname{Tors} E(\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$. If $-p$ is a square in $\mathbb{Z}$, then $\operatorname{Tors} E(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and finally if p does not satisfy any of these conditions, then $\operatorname{Tors} E(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$. As an immediate consequence, we deduce that if $P \in \mathbb{Z}[t] \setminus \mathbb{Z}$ and $\mathcal{E} : y^2 = x^3 + Px$ is an elliptic curve defined over $\mathbb{Q}(t)$ with a $\mathbb{Q}(t)$-rational point $P = (x, y)$ satisfying $xy \neq 0$, then the order of P in the group $\mathcal{E}(\mathbb{Q}(t))$ is not finite provided that $\mathcal{E}$ is not isomorphic to an elliptic curve defined over $\mathbb{Q}$. Thus, in that case the curve $\mathcal{E}$ over $\mathbb{Q}(t)$ has a positive rank.

Now we are ready to prove the following theorem.

THEOREM 4.1. *Let $a, b \in \mathbb{Z} \setminus \{0\}$ and n be an odd integer. Then the Diophantine equation*

$$(9)\qquad \frac{y_1^4 - x_1^{2n}}{a} = \frac{y_2^4 - x_2^{2n}}{b}$$

has infinitely many rational parametric solutions.

Proof. We consider the hypersurface given by (9). Note that instead of (9) we can consider the system

$$(10)\qquad b(y_1^2 - x_1^N) = aU(y_2^2 - x_2^N), \qquad U(y_1^2 + x_1^N) = y_2^2 + x_2^N,$$

where U is a parameter. In order to solve the above system, we take

$$(11) \qquad x_1 = T, \quad y_1 = T^{(n-1)/2}, \quad x_2 = -t^2T, \quad y_2 = qT^{(n-1)/2},$$

where t is a rational parameter and q, T are variables. From the first equation in (10), we get

$$T = \frac{b - aq^2U}{b + at^{2n}U},$$

and then from the second equation in (10),

$$(b + 2at^{2n}U + aU^2)q^2 = bt^{2n} + 2bU + at^{2n}U^2.$$

Thus we get the equation of a hyperelliptic quartic curve (defined over $\mathbb{Q}(t)$) of the form

$$\mathcal{C}_{a,b} : V^2 = (bt^{2n} + 2bU + at^{2n}U^2)(b + 2at^{2n}U + aU^2),$$

where $V = q(b + 2at^{2n}U + aU^2)$. Note that $\mathcal{C}_{a,b}$ has the $\mathbb{Q}(t)$-rational point $Q_{a,b} = (0, bt^n)$. Treating $Q_{a,b}$ as a point at infinity on the curve $\mathcal{C}_{a,b}$ and using the method described in [6, p. 77], we see that $\mathcal{C}_{a,b}$ is birationally equivalent to the elliptic curve with the Weierstrass equation

$$\mathcal{E}_{a,b} : Y^2 = X^3 + 4ab(at^{4n} - b)^2X.$$

A mapping from $\mathcal{E}_{a,b}$ to $\mathcal{C}_{a,b}$ is given by $\varphi(X, Y) = (U, V)$, where

$$U = \frac{2b(b^2 - 2abt^{4n} + a^2t^{8n} - t^{2n}X)}{(at^{4n} + b)X - t^nY},$$

$$V = \frac{t^{3n}X^3 - (at^{4n} - b)^2(f_1X^2 + f_2X + f_3 + f_4Y)}{((at^{4n} + b)X - t^nY)^2},$$

and

$$f_1 = 3t^n, \qquad f_2 = 4abt^{3n},$$
$$f_3 = 4abt^n(at^{4n} - b)^2, \qquad f_4 = -2(at^{4n} + b).$$

In order to show that the rank of $\mathcal{E}_{a,b}$ is positive, we notice that the point $P_{a,b} = (X^2, XY)$, where

$$X = \frac{b^2 - 6abt^{4n} + a^2t^{8n}}{2t^n(b + at^{4n})}, \qquad Y = X^2 + \frac{8abt^{2n}(at^{4n} - b)^2}{(at^{4n} + b)^2},$$

lies on our curve. As $4ab(at^{4n} - b)^2$ is not of the form $a'F(t)^4$ for $a' \in \mathbb{Z}$ and $F \in \mathbb{Z}[t]$, we deduce that $\mathcal{E}_{a,b}$ is not birationally equivalent to an elliptic curve defined over $\mathbb{Q}$. Invoking now the remark from the beginning of this section we deduce that $P_{a,b}$ is of infinite order in the group $\mathcal{E}_{a,b}(\mathbb{Q}(t))$. Now it is an easy task to obtain the statement of our theorem. First, for $m = 2, 3, \ldots$, we calculate $mP_{a,b}$ on the curve $\mathcal{E}_{a,b}$. Next, we calculate the corresponding point (U, V) on $\mathcal{C}_{a,b}$ and from the equation $V = q(b + 2at^{2n}U + aU^2)$ we get the value of q. Then we calculate the value of $T = T(t)$ and the values of x_i, y_i, $i = 1, 2$, given by (11) which give solutions of our equation.

For example, if $n = 3$, then the point $P_{a,b}$ leads (after necessary simplifications) to the solution of the equation $(y_1^4 - x_1^6)/a = (y_2^4 - x_2^6)/b$ of the form

$$\begin{aligned} x_1 &= t^3(-3a^2 + 6abt^{12} + b^2t^{24})y_2, \\ y_1 &= t^2y_2, \\ x_2 &= (-a^2 - 6abt^{12} + 3b^2t^{24})y_2, \\ y_2 &= a^4 - 28a^3bt^{12} + 6a^2b^2t^{24} - 28ab^3t^{36} + b^4t^{48}. \ \blacksquare \end{aligned}$$

REMARK 4.2. With the use of Theorem 4.1, we can easily prove that the set of rational parametric solutions of the Diophantine equation

$$x_1^4 + x_2^4 = y_1^{2n} + y_2^{2n},$$

where n is an odd integer, is infinite. Indeed, just take $b = -a$ in the solution obtained in Theorem 4.1. Thus our method shows that the set of integers which are simultaneously representable as a sum of two fourth powers and two $2m$th powers is infinite. This result is related to the Diophantine problem called *equal sums of unlike powers* investigated by Lander in [5]. The method presented by Lander cannot be used in order to construct integer solutions to the above Diophantine equation.

In particular, if $n = 3$, then we get the following solution of the equation $x_1^4 + x_2^4 = y_1^6 + y_2^6$:

$$\begin{aligned} x_1 &= (-1 + 6t^{12} + 3t^{24})y_2, \\ x_2 &= t^3(-3 - 6t^{12} + t^{24})y_2, \\ y_1 &= t^2y_2, \\ y_2 &= 1 + 28t^{12} + 6t^{24} + 28t^{36} + t^{48}, \end{aligned}$$

which has not been obtained before.

From Theorem 4.1, we deduce an interesting result similar to Theorem 2.3.

THEOREM 4.3. *Let $a_1, a_2 \in \mathbb{Z} \setminus \{0\}$ be such that $a_1 \neq a_2$ and let $m \in \mathbb{N}_+$ be an odd integer. Consider the superelliptic curves defined by the equations*

$$C_i : y^4 = x^{2m} + a_i, \qquad i = 1, 2.$$

Then there exists a polynomial $D \in \mathbb{Z}[t]$ such that the Jacobian variety associated with the curve $C_{i,D} : y^4 = x^{2m} + a_iD(t)$ has a positive rank over $\mathbb{Q}(t)$ for $i = 1, 2$.

Proof. From Theorem 4.1, we can deduce the existence of a polynomial (in fact, there are infinitely many such polynomials) $D \in \mathbb{Z}[t]$ such that the set of $\mathbb{Q}(t)$-rational points on the curve $C_{i,D} : y^4 = x^{2m} + a_i D(t)$ is nonempty for $i = 1, 2$. Moreover, on each curve $C_{i,D}$, we get a point $P_i = (x_i, y_i)$ satisfying $x_i y_i \neq 0$ for $i = 1, 2, 3$. Note the existence of the following morphism:

$$\varphi_i : C_{i,D} \ni (x, y) \mapsto (y, x^m) \in E_i' : Y^2 = X^4 - a_i D(t).$$

The curve E_i' over $\mathbb{Q}(t)$ is birationally equivalent to the elliptic curve whose Weierstrass equation is

$$E_i : Y^2 = X^3 - 4a_i D(t) X.$$

The mapping $\chi_i : E_i \to E_i'$ is given by

$$\chi_i(X, Y) = \left(\frac{Y}{2X}, \frac{X^2 - 4a_i D(t)}{4X}\right),$$

and its inverse is given by

$$\chi_i^{-1}(X, Y) = (2(X^2 + Y), 4X(X^2 + Y)).$$

First of all, note that the curve E_i is not birationally equivalent to one defined over $\mathbb{Q}$. This follows from the fact that $-4a_i D(t)$ is not of the form $a' F(t)^4$ for $a' \in \mathbb{Z}$ and $F \in \mathbb{Z}[t]$. Next, on E_i', we have a $\mathbb{Q}(t)$-rational point $Q_i = (y_i, x_i^m)$ for $i = 1, 2$. Thus the point $(X, Y) = \chi_i^{-1}(Q_i)$ lies on E_i and one can easily check that $XY \neq 0$. From the remark at the beginning of this section we deduce that the point $\chi_i^{-1}(Q_i)$ is of infinite order on E_i, and the same holds for the point Q_i on E_i'. Now, using Remark 2.2 we conclude that the rank of $\mathrm{Jac}(C_{i,D})(\mathbb{Q}(t))$ is positive for $i = 1, 2$. ∎

Acknowledgments. I am grateful to the anonymous referee for constructive suggestions to improve an earlier draft of this paper. I am also grateful to Professor A. Togbe for useful comments.

The author is a holder of a START scholarship funded by the Foundation for Polish Science (FNP).

References

[1] A. Choudhry and J. Wróblewski, *Quartic diophantine chains*, Acta Arith. 128 (2007), 339–348.

[2] M. Hindry and J. Silverman, *Diophantine Geometry. An Introduction*, Grad. Texts in Math. 201, Springer, New York, 2000.

[3] T. Jędrzejak and M. Ulas, *Characterization of torsion of the Jacobian of $y^2 = x^5 + Ax$ and some applications*, Acta Arith. 144 (2010), 183–191.

[4] —, —, *Higher twists of hyperelliptic curves*, submitted.

[5] L. J. Lander, *Equal sums of unlike powers*, Fibonacci Quart. 28 (1990), 141–150.

[6] L. J. Mordell, *Diophantine Equations*, Academic Press, London, 1969.

[7] J. Silverman, *The Arithmetic of Elliptic Curves*, Grad. Texts in Math. 106, Springer, New York, 1986.

[8] M. Ulas, *A note on higher twists of elliptic curves*, Glasgow Math. J. 52 (2010), 371–381.
[9] —, *Variations on higher twists of pairs of elliptic curves*, Int. J. Number Theory 6 (2010), 1183–1189.

Maciej Ulas
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
and
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: Maciej.Ulas@im.uj.edu.pl

Received on 29.12.2009
and in revised form on 7.4.2010 (6248)