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On a theorem of V. Bernik in the metric theory of Diophantine approximation

by

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1. Introduction. We begin by introducing some notation: $\#S$ will denote the number of elements in a finite set S; the Lebesgue measure of a measurable set $S \subset \mathbb{R}$ will be denoted by $|S|$; P_n will be the set of integral polynomials of degree $\le n$. Given a polynomial P, $H(P)$ will denote the height of P, i.e. the maximum of the absolute values of its coefficients; $P_n(H) = \{P \in P_n : H(P) = H\}$. The symbol of Vinogradov $\ll$ in the expression $A \ll B$ means $A \le CB$, where C is a constant. The symbol $\asymp$ means both $\ll$ and $\gg$. Given a point $x \in \mathbb{R}$ and a set $S \subset \mathbb{R}$, let $\operatorname{dist}(x, S) = \inf\{|x - s| : x \in S\}$. Throughout, Ψ will be a positive function.

K. Mahler's problem. In 1932 K. Mahler [9] introduced a classification of real numbers x into the so-called classes of A, S, T and U numbers according to the behavior of $w_n(x)$ defined as the supremum of $w > 0$ for which

$$|P(x)| < H(P)^{-w}$$

holds for infinitely many $P \in P_n$. By Minkowski's theorem on linear forms, one readily shows that $w_n(x) \ge n$ for all $x \in \mathbb{R}$. Mahler [8] has proved that for almost all $x \in \mathbb{R}$ (in the sense of Lebesgue measure), $w_n(x) \le 4n$, thus almost all $x \in \mathbb{R}$ are in the S-class. Mahler has also conjectured that for almost all $x \in \mathbb{R}$ one has the equality $w_n(x) = n$. For about 30 years the progress in Mahler's problem was limited to $n = 2$ and 3 and to partial results for $n > 3$. It was V. Sprindžuk who has proved Mahler's conjecture in full (see [11]).

A. Baker's conjecture. Let $W_n(\Psi)$ be the set of $x \in \mathbb{R}$ such that there are infinitely many $P \in P_n$ satisfying

$$|P(x)| < \Psi(H(P)). \tag{1}$$

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A. Baker [1] has improved Sprindžuk's theorem by showing that

$$|W_n(\Psi)| = 0 \text{ if } \sum_{h=1}^{\infty} \Psi^{1/n}(h) < \infty \text{ and } \Psi \text{ is monotonic.}$$

He has also conjectured a stronger statement, proved over 20 years later by V. Bernik [5], that $|W_n(\Psi)| = 0$ if the sum

$$\sum_{h=1}^{\infty} h^{n-1}\Psi(h) \tag{2}$$

converges and Ψ is monotonic. Afterwards V. Beresnevich [2] has shown that $|\mathbb{R} \setminus W_n(\Psi)| = 0$ if (2) diverges and Ψ is monotonic. Here we prove

THEOREM 1. *Let $\Psi : \mathbb{R} \to \mathbb{R}^+$ be an arbitrary function* (*not necessarily monotonic*) *such that the sum* (2) *converges. Then* $|W_n(\Psi)| = 0$.

Theorem 1 is no longer improvable as, by [2], the convergence of (2) is crucial. Notice that for $n = 1$ the theorem is simple and known (see, for example, [7, p. 121]). Therefore, from now on we assume that $n \geq 2$.

2. Subcases of Theorem 1. Let $\delta > 0$. We define the following three sets $W_{\mathrm{big}}(\Psi)$, $W_{\mathrm{med}}(\Psi)$ and $W_{\mathrm{small}}(\Psi)$ consisting of $x \in \mathbb{R}$ such that there are infinitely many $P \in P_n$ simultaneously satisfying (1) and one of the following inequalities:

$$1 \leq |P'(x)|, \tag{3}$$

$$H(P)^{-\delta} \leq |P'(x)| < 1, \tag{4}$$

$$|P'(x)| < H(P)^{-\delta}, \tag{5}$$

respectively. Obviously $W_n(\Psi) = W_{\mathrm{big}}(\Psi) \cup W_{\mathrm{med}}(\Psi) \cup W_{\mathrm{small}}(\Psi)$. Hence to prove Theorem 1 it suffices to show that each of these sets has zero measure.

Since the sum (2) converges, $H^{n-1}\Psi(H)$ tends to 0 as $H \to \infty$. Therefore,

$$\Psi(H) = o(H^{-n+1}) \quad \text{as } H \to \infty. \tag{6}$$

3. The case of a big derivative. The aim of this section is to prove that $|W_{\mathrm{big}}(\Psi)| = 0$. Let $B_n(H)$ be the set of $x \in \mathbb{R}$ such that there exists a polynomial $P \in P_n(H)$ satisfying (3). Then

$$W_{\mathrm{big}}(\Psi) = \bigcap_{N=1}^{\infty} \bigcup_{H=N}^{\infty} B_n(H). \tag{7}$$

Now $|W_{\mathrm{big}}(\Psi)| = 0$ if $|W_{\mathrm{big}}(\Psi) \cap I| = 0$ for any open interval $I \subset \mathbb{R}$ satisfying

$$0 < c_0(I) = \inf\{|x| : x \in I\} < \sup\{|x| : x \in I\} = c_1(I) < \infty. \tag{8}$$

Therefore we can fix an interval I satisfying (8).

By (7) and the Borel–Cantelli Lemma, $|W_{\mathrm{big}}(\Psi)\cap I| = 0$ whenever

$$\sum_{H=1}^{\infty} |B_n(H)\cap I| < \infty. \tag{9}$$

By the convergence of (2), condition (9) will follow on showing that

$$|B_n(H)\cap I| \ll H^{n-1}\Psi(H) \tag{10}$$

with the implicit constant in (10) independent of H.

Given a $P\in P_n(H)$, let $\sigma(P)$ be the set of $x\in I$ satisfying (3). Then

$$B_n(H)\cap I = \bigcup_{P\in P_n(H)} \sigma(P). \tag{11}$$

LEMMA 1. *Let I be an interval with endpoints a and b. Define $I'' = [a, a+4\Psi(H)] \cup [b-4\Psi(H), b]$ and $I' = I\setminus I''$. Then for all sufficiently large H, for any $P\in P_n(H)$ such that $\sigma(P)\cap I' \neq \emptyset$, for any $x_0\in\sigma(P)\cap I'$ there exists $\alpha\in I$ such that $P(\alpha)=0$, $|P'(\alpha)| > |P'(x_0)|/2$ and $|x_0-\alpha| < 2\Psi(H)\,|P'(\alpha)|^{-1}$.*

The proof of this lemma nearly coincides with the one of Lemma 1 in [2] and is left to the reader. There will be some changes to constants and notation and one will also have to use (6).

Given a polynomial $P\in P_n(H)$ and a real number α such that $P'(\alpha)\neq 0$, define $\sigma(P;\alpha) = \{x\in I : |x-\alpha| < 2\Psi(H)|P'(\alpha)|^{-1}\}$. Let I' and I'' be as in Lemma 1. For every polynomial $P\in P_n(H)$, we define the set

$$Z_I(P) = \{\alpha\in I : P(\alpha)=0 \text{ and } |P'(\alpha)|\geq 1/2\}.$$

By Lemma 1, for any $P\in P_n(H)$ we have the inclusion

$$\sigma(P)\cap I' \subset \bigcup_{\alpha\in Z_I(P)} \sigma(P;\alpha). \tag{12}$$

Given $k\in\mathbb{Z}$ with $0\leq k\leq n$, define

$$P_n(H,k) = \{P = a_nx^n+\cdots+a_0 \in P_n(H) : a_k = 0\}$$

and for $R\in P_n(H,k)$ let

$$P_n(H,k,R) = \{P\in P_n(H) : P-R = a_kx^k\}.$$

It is easily observed that

$$P_n(H) = \bigcup_{k=0}^{n} \bigcup_{R\in P_n(H,k)} P_n(H,k,R) \tag{13}$$

and

$$\#P_n(H,k) \ll H^{n-1} \quad \text{for every } k. \tag{14}$$

Taking into account (11), (13), (14) and the fact that $|I''|\ll\Psi(H)$, it now becomes clear that to prove (10) it is sufficient to show that for every fixed

k and fixed $R \in P_n(H,k)$,

$$\Big| \bigcup_{P \in P_n(H,k,R)} \sigma(P) \cap I' \Big| \ll \Psi(H). \tag{15}$$

Let k and R be fixed. Define the rational function $\widetilde{R}(x) = x^{-k}R(x)$. By (8), there exists a collection of intervals $[w_{i-1}, w_i) \subset I$ $(i = 1, \ldots, s)$, which do not intersect pairwise and cover I, such that $\widetilde{R}'(x)$ is monotonic and does not change sign on every interval $[w_{i-1}, w_i)$. It is clear that s depends on n only. Let $Z_{I,R} = \bigcup_{P \in P_n(H,k,R)} Z_I(P)$, $k_i = \#(Z_{I,R} \cap [w_{i-1}, w_i))$ and $Z_{I,R} \cap [w_{i-1}, w_i) = \{\alpha_i^{(1)}, \ldots, \alpha_i^{(k_i)}\}$, where $\alpha_i^{(j)} < \alpha_i^{(j+1)}$. Given a $P \in P_n(H,k,R)$, we obviously have the identity

$$\frac{x^k P'(x) - kx^{k-1}P(x)}{x^{2k}} = \left(\frac{P(x)}{x^k}\right)' = \widetilde{R}'(x).$$

Taking x to be $\alpha \in Z_I(P)$ leads to $P'(\alpha)/\alpha^k = \widetilde{R}'(\alpha)$. By (8), $|P'(\alpha)| \asymp |\widetilde{R}'(\alpha)|$. Now, by Lemma 1, $|\sigma(P;\alpha)| \ll \Psi(H)\,|P'(\alpha)|^{-1} \ll \Psi(H)\,|\widetilde{R}'(\alpha)|^{-1}$.

Using (12), we get

$$\Big| \bigcup_{P \in P_n(H,k,R)} \sigma(P) \cap I' \Big| \ll \Psi(H) \sum_{i=1}^{s} \sum_{j=1}^{k_i} \frac{1}{|\widetilde{R}(\alpha_i^{(j)})|}.$$

Now to show (15) it suffices to prove that for every i $(1 \le i \le s)$,

$$\sum_{j=1}^{k_i} |\widetilde{R}'(\alpha_i^{(j)})|^{-1} \ll 1. \tag{16}$$

Fix an index i $(1 \le i \le s)$. If $k_i \ge 2$ then we can consider two sequential roots $\alpha_i^{(j)}$ and $\alpha_i^{(j+1)}$ of two rational functions $\widetilde{R} + a_k^{i,j}$ and $\widetilde{R} + a_k^{i,j+1}$ respectively. For convenience let us assume that $\widetilde{R}'$ is increasing and positive on $[w_{i-1}, w_i)$. Then $\widetilde{R}$ is strictly monotonic on $[w_{i-1}, w_i)$, and we have $a_k^{i,j} \ne a_k^{i,j+1}$. It follows that $|a_k^{i,j} - a_k^{i,j+1}| \ge 1$. Using the Mean Value Theorem and the monotonicity of $\widetilde{R}'$, we get

$$\begin{aligned} 1 &\le |a_0^{i,j} - a_0^{i,j+1}| = |\widetilde{R}'(\alpha_i^{(j)}) - \widetilde{R}'(\alpha_i^{(j+1)})| = |\widetilde{R}'(\widetilde{\alpha}_i^{(j)})| \cdot |\alpha_i^{(j)} - \alpha_i^{(j+1)}| \\ &\le |\widetilde{R}'(\alpha_i^{(j+1)})| \cdot |\alpha_i^{(j)} - \alpha_i^{(j+1)}|, \end{aligned}$$

where $\widetilde{\alpha}_i^{(j)}$ is a point between $\alpha_i^{(j)}$ and $\alpha_i^{(j+1)}$. This implies $|\widetilde{R}'(\alpha_i^{(j+1)})|^{-1} \le \alpha_i^{(j+1)} - \alpha_i^{(j)}$, whence we readily get

$$\sum_{j=1}^{k_i-1} |\widetilde{R}'(\alpha_i^{(j+1)})|^{-1} \le \sum_{j=1}^{k_i-1} (\alpha_i^{(j+1)} - \alpha_i^{(j)}) = \alpha_i^{(k_i)} - \alpha_i^{(1)} \le w_i - w_{i-1}.$$

The last inequality and $|\widetilde{R}'(\alpha_i^{(1)})| \asymp |P'(\alpha_i^{(1)})| \gg 1$ yield (16). It is easily verified that (16) holds for every i with $k_i \geq 2$ and is certainly true when $k_i = 1$ or $k_i = 0$. This completes the proof of the case of a big derivative.

4. The case of a medium derivative. As above we fix an interval I satisfying (8). Then $|W_{\mathrm{med}}(\Psi)| = 0$ will follow from $|W_{\mathrm{med}}(\Psi) \cap I| = 0$. We will use the following

LEMMA 2 (see Lemma 2 in [3]). *Let $\alpha_0, \ldots, \alpha_{k-1}, \beta_1, \ldots, \beta_k \in \mathbb{R} \cup \{\infty\}$ be such that $\alpha_0 > 0$, $\alpha_j > \beta_j \geq 0$ for $j = 1, \ldots, k-1$ and $0 < \beta_k < \infty$. Let $f : (a, b) \to \mathbb{R}$ be a $C^{(k)}$ function such that $\inf_{x \in (a,b)} |f^{(k)}(x)| \geq \beta_k$. Then the set of $x \in (a, b)$ satisfying*

$$|f(x)| \leq \alpha_0, \quad \beta_j \leq |f^{(j)}(x)| \leq \alpha_j \quad (j = 1, \ldots, k-1)$$

is a union of at most $k(k+1)/2 + 1$ intervals, each with length at most $\min_{0 \leq i < j \leq k} 3^{(j-i+1)/2} (\alpha_i/\beta_j)^{1/(j-i)}$. Here we assume $\frac{c}{0} = \infty$ for $c > 0$.

Given a polynomial $P \in P_n(H)$, we redefine $\sigma(P)$ to be the set of solutions of (4). Since $P^{(n)}(x) = n! a_n$, we can apply Lemma 2 to P with $k = n$ and

$$\alpha_0 = \Psi(H), \quad \alpha_1 = 1, \quad \beta_1 = \inf_{x \in \sigma(P)} |P'(x)| \geq H^{-\delta}, \quad \beta_n = 1,$$

$$\alpha_2 = \cdots = \alpha_{n-1} = \infty, \quad \beta_2 = \cdots = \beta_{n-1} = 0.$$

Then we conclude that $\sigma(P)$ is a union of at most $n(n+1)/2+1$ intervals of length $\ll \alpha_0/\beta_1$. There is no loss of generality in assuming that the sets $\sigma(P)$ are intervals, as otherwise, we would treat the intervals of $\sigma(P)$ separately. We can also ignore those P for which $\sigma(P)$ is empty. For every P we define a point $\gamma_P \in \sigma(P)$ such that $\inf_{x \in \sigma(P)} |P'(x)| \geq \frac{1}{2} |P'(\gamma_P)|$. The existence is easily seen. Now we have

$$|\sigma(P)| \ll \Psi(H) |P'(\gamma_P)|^{-1}. \tag{17}$$

It also follows from the choice of γ_P that

$$H(P)^{-\delta} \leq |P'(\gamma_P)| < 1. \tag{18}$$

Now define expansions of $\sigma(P)$ as follows:

$$\sigma_1(P) := \{x \in I : \operatorname{dist}(x, \sigma(P)) < (H|P'(\gamma_P)|)^{-1}\},$$

$$\sigma_2(P) := \{x \in I : \operatorname{dist}(x, \sigma(P)) < H^{-1+2\delta}\}.$$

By (4), $\sigma_1(P) \subset \sigma_2(P)$. Moreover, it is easy to see that

$$\sigma_1(P) \subset \sigma_2(Q) \quad \text{for any } Q \in P_n(H) \text{ with } \sigma_1(Q) \cap \sigma_1(P) \neq \emptyset. \tag{19}$$

It is also readily verified that $|\sigma_1(P)| \asymp (H|P'(\gamma_P)|)^{-1}$, and therefore, by (17),

$$|\sigma(P)| \ll |\sigma_1(P)| \, H\Psi(H).$$

Take any $x \in \sigma_2(P)$. Using the Mean Value Theorem, (18) and the fact that $|x - \gamma_P| \ll H^{-1+2\delta}$, we get

$$|P'(x)| \le |P'(\gamma_P)| + |P''(\widetilde{x})(x - \gamma_P)| \ll 1 + H \cdot H^{-1+2\delta} \ll H^{2\delta},$$

where $\widetilde{x}$ is between x and γ_P. Similarly we estimate $|P(x)|$:

$$(20) \qquad |P(x)| \ll H^{-1+4\delta}, \quad |P'(x)| \ll H^{2\delta} \quad \text{for any } x \in \sigma_2(P).$$

Now for every pair (k, m) of integers with $0 \le k < m \le n$ we define

$$P_n(H, k, m) = \{R = a_n x^n + \cdots + a_0 \in P_n(H) : a_k = a_m = 0\}$$

and for a given polynomial $R \in P_n(H, k, m)$ we define

$$P_n(H, k, m, R) = \{P = R + a_m x^m + a_k x^k \in P_n(H)\}.$$

The intervals $\sigma(P)$ will be divided into 2 classes of essential and non-essential intervals. The interval $\sigma(P)$ will be essential if for any choice of (k, m, R) such that $P \in P_n(H, k, m, R)$ for any $Q \in P_n(H, k, m, R)$ other than P we have $\sigma_1(P) \cap \sigma_1(Q) = \emptyset$. For fixed k, m and R summing the measures of essential intervals gives

$$\sum |\sigma(P)| \le H\Psi(H) \sum |\sigma_1(P)| \le H\Psi(H)|I| \ll H\Psi(H).$$

As $\# P_n(H, k, m) \ll H^{n-2}$ and there are only $n(n+1)/2$ different pairs (k, m) we obtain the following estimate:

$$\sum_{\text{essential intervals } \sigma(P) \text{ with } P \in P_n(H)} |\sigma(P)| \ll H^{n-1}\Psi(H).$$

Thus, by the Borel–Cantelli Lemma and the convergence of (2), the set of points x of $W_{\mathrm{med}}(\Psi) \cap I$ which belong to infinitely many essential intervals is of measure zero.

Now let $\sigma(P)$ be non-essential. Then, by definition and (19), there is a choice of k, m, R such that $P \in P_n(H, k, m, R)$ and there is a $Q \in P_n(H, k, m, R)$ different from P such that

$$\sigma(P) \subset \sigma_1(P) \subset \sigma_2(P) \cap \sigma_2(Q).$$

On the set $\sigma_2(P) \cap \sigma_2(Q)$ both P and Q satisfy (20) and so does the difference $P(x) - Q(x) = b_m x^m + b_k x^k$. It is not difficult to see that $b_m \ne 0$ if H is large enough. Therefore using (20) we get

$$(21) \qquad \left| x^{m-k} + \frac{b_k}{b_m} \right| \ll \frac{H^{-1+4\delta}}{|b_m|} \le H^{-1+4\delta}, \qquad \max\{|b_m|, |b_k|\} \ll H^{2\delta}.$$

Now let x belong to infinitely many non-essential intervals. Without loss of generality we assume that x is transcendental as otherwise it belongs to a countable set, which is of measure zero. Therefore (21) is satisfied for infinitely many $b_m, b_k \in \mathbb{Z}$. Hence, the inequality

$$\left|x^{m-k} - \frac{p}{q}\right| < q^{-(1-5\delta)/2\delta}$$

holds for infinitely many $p, q \in \mathbb{Z}$. Taking $\delta = 1/10$ so that $(1-5\delta)/2\delta$ becomes $2+\delta$, and applying standard Borel–Cantelli arguments (see [7, p. 121]) we complete the proof of the case of a medium derivative for non-essential intervals.

5. The case of a small derivative. In this section we prove that $|W_{\mathrm{small}}(\Psi)| = 0$. We will make use of Theorem 1.4 in [6]. By taking $d = 1$, $\mathbf{f} = (x, x^2, \ldots, x^n)$, $U = \mathbb{R}$, $T_1 = \cdots = T_n = H$, $\theta = H^{-n+1}$, $K = H^{-\delta}$ in that theorem, we arrive at

THEOREM 2. *Let* $x_0 \in \mathbb{R}$ *and*

$$\delta' = \frac{\min(\delta, n-1)}{(n+1)(2n-1)}.$$

Then there exists a finite interval $I_0 \subset \mathbb{R}$ *containing* x_0 *and a constant* $E > 0$ *such that*

$$\Bigl|\bigcup_{P\in P_n,\, 0<H(P)\le H} \{x \in I_0 : |P(x)| < H^{-n+1},\ |P'(x)| < H^{-\delta}\}\Bigr| \le EH^{-\delta'}.$$

In particular Theorem 2 implies that, for any $\delta > 0$, the set of $x \in \mathbb{R}$ for which there are infinitely many polynomials $P \in P_n$ satisfying the system

$$|P(x)| < H(P)^{-n+1}, \qquad |P'(x)| < H(P)^{-\delta}, \tag{22}$$

has zero measure. Indeed, this set consists of points $x \in I_0$ which belong to infinitely many sets

$$\tau_m = \{x \in I_0 : (22) \text{ holds for some } P \in P_n \text{ with } 2^{m-1} < H(P) \le 2^m\}.$$

By Theorem 2, $|\tau_m| \ll 2^{-m\delta'}$ with $\delta' > 0$. Therefore, $\sum_{m=1}^{\infty} |\tau_m| < \infty$ and the Borel–Cantelli Lemma completes the proof of the claim.

In view of (6), this completes the proof of the case of a small derivative and the proof of Theorem 1.

6. Concluding remarks. An analogue of Theorem 1 when P is assumed to be irreducible over $\mathbb{Q}$ and primitive (i.e. with coprime coefficients) can also be sought. To make it more precise, let $P_n^*(H)$ be the subset of $P_n(H)$ consisting of primitive irreducible polynomials P of degree $\deg P = n$ and height $H(P) = H$. Now the set of primitive irreducible polynomials of degree n is $P_n^* = \bigcup_{H=1}^{\infty} P_n^*(H)$. Let $W_n^*(\Psi)$ be the set of $x \in \mathbb{R}$ such that there are infinitely many $P \in P^*$ satisfying (1).

THEOREM 3. *Let $\Psi : \mathbb{R} \to \mathbb{R}^+$ be an arbitrary function such that the sum*

$$\sum_{H=1}^{\infty} \frac{\#P_n^*(H)}{H}\,\Psi(H) \tag{23}$$

converges. Then $|W_n^*(\Psi)| = 0$.

For $n = 1$ the proof of Theorem 3 is a straightforward application of the Borel–Cantelli Lemma and we again refer to [7, p. 121]. For $n > 1$ the proof is deduced from the following two observations: 1) $W_n^*(\Psi) \subset W_n(\Psi)$ and 2) $\#P_n^*(H) \asymp H^n$. The second one guarantees the convergence of (2), which now implies $0 \le |W_n^*(\Psi)| \le |W_n(\Psi)| = 0$. The proof of the relation $\#P_n^*(H) \asymp H^n$ is elementary and is left to the reader. In fact, one can easily estimate the number of primitive reducible polynomials in P_n and take them off the set of all primitive polynomials in P_n which is well known to contain at least a constant times $\#P_n(H)$ elements.

The Duffin–Schaeffer conjecture. The conjecture states that for $n = 1$ if (23) diverges then $|\mathbb{R} \setminus W_n^*(\Psi)| = 0$. The multiple $\#P_1^*(H)$ in (23) becomes $\asymp \varphi(H)$, where φ is the Euler function.

The following problem can be regarded as the generalisation of the Duffin–Schaeffer conjecture to integral polynomials of higher degree:

Prove that $|\mathbb{R} \setminus W_n^*(\Psi)| = 0$ whenever (23) *diverges.*

Alternatively, for $n > 1$ one might investigate the measure of $\mathbb{R} \setminus W_n(\Psi)$. So far it is unclear if for $n > 1$, $|\mathbb{R} \setminus W_n^*(\Psi)| = 0$ is equivalent to $|\mathbb{R} \setminus W_n(\Psi)| = 0$, which is another intricate question.

A remark on manifolds. In the metric theory of Diophantine approximation on manifolds one usually studies sets of Ψ-approximable points lying on a manifold with respect to the measure induced on that manifold. Mahler's problem and its generalisations can be regarded as Diophantine approximation on the Veronese curve $(x, x^2, \ldots, x^n)$.

A point $\mathbf{f} \in \mathbb{R}^n$ is called *Ψ-approximable* if

$$\|\mathbf{a} \cdot \mathbf{f}\| < \Psi(|\mathbf{a}|_\infty)$$

for infinitely many $\mathbf{a} \in \mathbb{Z}^n$, where $|\mathbf{a}|_\infty = \max_{1\le i\le n} |a_i|$ for $\mathbf{a} = (a_1, \ldots, a_n)$, $\|x\| = \min\{|x - z| : z \in \mathbb{Z}\}$ and $\Psi : \mathbb{R} \to \mathbb{R}^+$.

Let $\mathbf{f} : U \to \mathbb{R}^n$ be a map defined on an open set $U \subset \mathbb{R}^d$. We say that $\mathbf{f}$ is *non-degenerate at* $\mathbf{x}_0 \in U$ if for some $l \in \mathbb{N}$ the map $\mathbf{f}$ is l times continuously differentiable on a sufficiently small ball centered at $\mathbf{x}_0$ and there are n linearly independent over $\mathbb{R}$ partial derivatives of $\mathbf{f}$ at $\mathbf{x}_0$ of orders up to l. We say that $\mathbf{f}$ is *non-degenerate* if it is non-degenerate almost everywhere on U. The non-degeneracy of a manifold is naturally defined via the non-degeneracy of its local parameterisation.

In 1998 D. Kleinbock and G. Margulis proved the Baker–Sprindžuk conjecture by showing that any non-degenerate manifold is strongly extremal. In particular, this implies an analogue of Mahler's problem for non-degenerate manifolds. A few years later an analogue of A. Baker's conjecture with monotonic Ψ (normally called a Groshev type theorem for convergence) has independently been proven by V. Beresnevich [3] and by V. Bernik, D. Kleinbock and G. Margulis [6] for non-degenerate manifolds. It is also remarkable that the proofs were given with different methods. The divergence counterpart (also for monotonic Ψ) has been established in [4]. In [6] a multiplicative version of the Groshev type theorem for convergence has also been given.

Theorem 1 of this paper can be readily generalised to non-degenerate curves: *Given a non-degenerate map* $\mathbf{f} : I \to \mathbb{R}^n$ *defined on an interval* I, *for any function* $\Psi : \mathbb{R} \to \mathbb{R}^+$ *such that the sum* (2) *converges for almost all* $x \in I$ *the point* $\mathbf{f}(x)$ *is not* Ψ*-approximable.* Even further, using the slicing technique of Pyartli [10] one can extend this to a class of n-differentiable non-degenerate manifolds which can be foliated by non-degenerate curves. In particular, this class includes arbitrary non-degenerate analytic manifolds. However with the technique at our disposal we are currently unable to prove the following

CONJECTURE. *Let* $\mathbf{f} : U \to \mathbb{R}^n$ *be a non-degenerate map, where* U *is an open subset of* $\mathbb{R}^d$. *Then for any function* $\Psi : \mathbb{R} \to \mathbb{R}^+$ *such that the sum* (2) *converges for almost all* $\mathbf{x} \in U$ *the point* $\mathbf{f}(\mathbf{x})$ *is not* Ψ*-approximable.*

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