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The class of Erdős–Turán sets

by

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1. Introduction. Let $\mathcal{P}$ denote the set of all infinite subsets of the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$. An element A of $\mathcal{P}$ will be identified to the sequence $(a_n)_{n\in\mathbb{N}^*}$ of its elements, denoted by the corresponding lower case letter, indexed by the set $\mathbb{N}^*$ of positive integers and taken in strictly increasing order, i.e. $a_1 < a_2 < \cdots < a_n < \cdots$.

For any subset A of $\mathbb{N}$ and any $n \in \mathbb{N}$, let

$$r(A, n) = |\{(a, b) \in A \times A : a + b = n\}| = |\{(i, j) \in \mathbb{N}^* \times \mathbb{N}^* : a_i + a_j = n\}|,$$

where $|E|$ denotes the cardinality of the set E. Further, let

$$s(A) = \sup\{r(A, n) : n \in \mathbb{N}\},$$

considered as an element of $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. We say that A is a *basis* of $\mathbb{N}$ if $r(A, n) \geq 1$ for all $n \in \mathbb{N}$, i.e. if the set $A + A = \{a + b : (a, b) \in A \times A\}$ coincides with $\mathbb{N}$.

In 1941, Erdős and Turán [3] made a conjecture which amounts to the following one.

1.1. The Erdős–Turán conjecture

(ET) If A is a basis of $\mathbb{N}$, then $s(A) = \infty$.

But they only established that $r(A, n)$ cannot become constant for large enough n and that $s(A) \geq 2$.

In 1956, Erdős and Fuchs [2] obtained more results and noted the following stronger conjecture.

1.2. The General Erdős–Turán conjecture

(GET) If $A \in \mathcal{P}$ is such that $a_n \leq cn^2$ for some constant $c > 0$ and all $n \in \mathbb{N}^*$, then $s(A) = \infty$.

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Indeed, it is well known [4, 5] that if A is a basis of $\mathbb{N}$, then it satisfies the condition $a_n \leq cn^2$ ($n \in \mathbb{N}^*$) for some $c > 0$, so that the validity of (GET) implies that of (ET).

Clearly, we may assume that the constant c is an integer. Thus (GET) may be stated as follows: if A is a sequence whose terms are less than or equal to the corresponding ones in the sequence $C = \{cn^2 : n \in \mathbb{N}^*\}$, then we have $s(A) = \infty$. This raises the natural and more general question of determining all the sequences of natural numbers that may replace C in the latter statement of (GET). We call such sets the Erdős–Turán sets, study some of their properties and establish several related statements which are equivalent to (GET).

We define an order relation $\ll$ in $\mathcal{P}$ by $A \ll B$ if and only if $a_n \leq b_n$ for all $n \in \mathbb{N}^*$. Then an element C of $\mathcal{P}$ is called an *Erdős–Turán set* if for all $A \in \mathcal{P}$ such that $A \ll C$, we have $s(A) = \infty$; the class of all such sets C is denoted by $\mathcal{C}(\mathrm{ET})$. Thus, writing $\mathbb{S} = \{n^2 : n \in \mathbb{N}^*\}$ for the set of squares in $\mathbb{N}^*$ and writing

$$c.A = \{ca_n : n \in \mathbb{N}^*\}$$

for the homothetic image of any $A \in \mathcal{P}$ by any $c \in \mathbb{N}^*$, the conjecture (GET) amounts to asserting that all homothetic images $c.\mathbb{S}$ of $\mathbb{S}$ lie in $\mathcal{C}(\mathrm{ET})$. So, a more general problem than that of proving or disproving (GET) consists in determining exactly the class $\mathcal{C}(\mathrm{ET})$. Hence the interest of thoroughly investigating its properties in order to better characterize its members. This is the essential goal of the present paper.

We first show, by a constructive argument, that the condition $s(C) = \infty$, which is obviously necessary, is not sufficient for C to lie in $\mathcal{C}(\mathrm{ET})$. In passing, we establish several properties of the important function $A \mapsto s(A)$. We then prove that the class $\mathcal{C}(\mathrm{ET})$ is stable under intersection by segments of the type $[t, \infty[$ in $\mathbb{N}$, under translation and under homothetic transformation. The latter property is however far from trivial and requires the introduction and study of some notions of intrinsic interest such as that of two close elements A and B in $\mathcal{P}$, by which we mean that the sequence $(|a_n - b_n|)_{n\in\mathbb{N}^*}$ is bounded in $\mathbb{N}$. One key result in this respect is that if A lies in $\mathcal{C}(\mathrm{ET})$, then so also does every element of $\mathcal{P}$ which is close to A. It then follows from the stability of $\mathcal{C}(\mathrm{ET})$ under homothetic transformation that (GET) is equivalent to the statement that $\mathbb{S}$ lies in $\mathcal{C}(\mathrm{ET})$. In other words, to establish the validity of (GET), it is enough to show that if a set $A \in \mathcal{P}$ satisfies the condition $a_n \leq n^2$ for all $n \in \mathbb{N}^*$, then $s(A) = \infty$; and the conclusion will then hold for all the sets $A \in \mathcal{P}$ satisfying the more general condition $a_n \leq cn^2$ for an arbitrary constant $c > 0$ and for all $n \in \mathbb{N}^*$. Moreover, using the properties of $\mathcal{C}(\mathrm{ET})$, we can replace $\mathbb{S}$ by any member of an infinite family of sets $E_r = \{[n + rn(n-1)] : n \in \mathbb{N}^*\}$, where r is a real number, $0 < r \leq 1$, and $[x]$ is the integer part of a real number x.

We then describe a large and significant subclass of $\mathcal{C}(\mathrm{ET})$, the restricted class of Erdős–Turán sets $\mathcal{C}(\mathrm{RET})$, by first introducing the caliber of an element $A \in \mathcal{P}$. This notion, which is of intrinsic interest, is defined by

$$\mathrm{cal}(A) = \liminf_{n\to\infty} \frac{a_n}{n^2};$$

and $\mathcal{C}(\mathrm{RET})$ consists precisely of all the sets $A \in \mathcal{P}$ such that $\mathrm{cal}(A) = 0$. We establish several properties for it, and use the results to give other equivalent formulations of the conjecture (GET). For instance, we show that

$$s(A) \geq \frac{1}{2\,\mathrm{cal}(A)}$$

for all $A \in \mathcal{P}$, and since the caliber is an increasing function with respect to the order relation $\ll$ in $\mathcal{P}$, it follows that $\mathcal{C}(\mathrm{RET})$ is a subset of $\mathcal{C}(\mathrm{ET})$. Furthermore, (GET) is true if and only if $\mathcal{C}(\mathrm{RET})$ is a proper subset of $\mathcal{C}(\mathrm{ET})$. We then establish that for (GET) to be valid, it is enough to prove, for some (arbitrarily chosen) real numbers $a > 0$, $b \geq 0$ and $0 \leq \nu < 2$, that for any $A \in \mathcal{P}$ such that $a_n \leq an^2 - bn^\nu$ for large enough n, we have $s(A) = \infty$. In other words, we can replace, in the statement of (GET), the squares by the integral parts $[an^2 - bn^\nu]$, where ν can be taken as close to 2 (but smaller than 2) as desired. Moreover, the function

$$\theta(A) = 1/\sqrt{\mathrm{cal}(A)}$$

is subadditive with respect to the union operation, i.e. it satisfies $\theta(A\cup B) \leq \theta(A)+\theta(B)$ for any $A, B \in \mathcal{P}$. It follows that if the union of two sets $A, B \in \mathcal{P}$ lies in $\mathcal{C}(\mathrm{RET})$, then so does one of them, at least. This raises the question of the validity of the analogous property for $\mathcal{C}(\mathrm{ET})$, about which we show that if it did not hold then (GET) would be true.

2. The class of Erdős–Turán sets

2.1. Notations and definitions. For any subset A of $\mathbb{N}$ and any $t \in \mathbb{N}$, we write $A[t] = A \cap [0, t]$ for an initial segment of A, and $A[t[= A \cap [t, \infty[$ for a terminal segment of A. We also write $t + A = \{t + a : a \in A\}$ for a translate of A, and when $t \neq 0$, we set $t.A = \{ta : a \in A\}$ for a homothetic (image) of A.

For any $A, B \in \mathcal{P}$, we write $A \ll B$ if $a_n \leq b_n$ for all $n \in \mathbb{N}^*$. This defines a partial order on $\mathcal{P}$. Let $\mathbb{S} = \{n^2 : n \in \mathbb{N}^*\}$.

2.2. Lemma. *For any $A, B \in \mathcal{P}$ and any $t \in \mathbb{N}$, we have*

(1) *If $B \subset A$, then $A \ll B$. In particular, $\mathbb{N} \ll A$ for all $A \in \mathcal{P}$.*
(2) *$A \ll A[t[$ and $A \ll t + A$. Also, if $t > 0$, then $A \ll t.A$.*

Proof. Straightforward. ∎

2.3. Proposition. *For any $A \in \mathcal{P}$ and any s in the interval $[2, \infty]$ of $\overline{\mathbb{N}}$, there exists $B \in \mathcal{P}$ such that $A \ll B$ and $s(B) = s$.*

Proof. We divide the proof into two parts, according as $s = \infty$ or s is an integer ≥ 2.

1. If $s = \infty$ then, given A, we define B as follows. For any $n \in \mathbb{N}^*$, let $m \in \mathbb{N}$ be the unique integer such that $2^m \leq n < 2^{m+1}$ and let $p = n - 2^m$, so that n is expressed, uniquely, as $n = 2^m + p$ with $0 \leq p < 2^m$, and let $b_n = a_{2^{m+1}} + p$. Since the sequence (a_n) is strictly increasing, we have $a_{n+k} \geq a_n + k$ for any $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$, by a simple induction on k. Now, if $n = 2^m + p$ with $0 \leq p \leq 2^m - 2$, then $b_n = a_{2^{m+1}} + p$ and $b_{n+1} = a_{2^{m+1}} + p + 1 > b_n$, while if $p = 2^m - 1$, i.e. if $n = 2^{m+1} - 1$, then $b_n = a_{2^{m+1}} + 2^m - 1$ and $b_{n+1} = a_{2^{m+2}} \geq a_{2^{m+1}} + 2^{m+1} > b_n$. Therefore the sequence (b_n) is also strictly increasing. Moreover, for any $n = 2^m + p$ with $0 \leq p < 2^m$, we have $b_n = a_{2^{m+1}} + p \geq a_{2^{m+1}} > a_{2^m + p} = a_n$. Therefore $A \ll B$. Finally, if $n = 2a_{2^{m+1}} + p$ with $m, p \in \mathbb{N}$ and $0 \leq p < 2^m$, then $n = a_{2^{m+1}} + q + a_{2^{m+1}} + p - q = b_{2^m+q} + b_{2^m+p-q}$ for $0 \leq q \leq p$, so that $r(B, n) \geq p + 1$. In particular, taking $p = 2^m - 1$ and letting m vary in $\mathbb{N}$, we see that $r(B, 2a_{2^{m+1}} + 2^m - 1) \geq 2^m$ is also unbounded. Therefore $s(B) = \infty$.

2. If s is an integer ≥ 2 then, given A, we define B inductively as follows. For $1 \leq k \leq s$, let $b_k = a_s + k - 1$. The first s terms clearly satisfy $b_k < b_{k+1}$ (for $1 \leq k \leq s - 1$) and $a_k \leq a_s \leq b_k$ (for $1 \leq k \leq s$). Now assume by induction that, for some $n \geq s$, the first n terms of B have been constructed and satisfy $b_k < b_{k+1}$ (for $1 \leq k \leq n - 1$) and $a_k \leq b_k$ (for $1 \leq k \leq n$). Let $B_n = \{b_1, \dots, b_n\}$, choose b_{n+1} to be any integer $\geq a_{n+1}$ and $> 2b_n$, e.g. $b_{n+1} = \max(a_{n+1}, 2b_n + 1)$, and set $B_{n+1} = B_n \cup \{b_{n+1}\}$. Then, for $m \in \mathbb{N}$, we have $r(B_{n+1}, m) = r(B_n, m) + d$, where $d = 2$ if $m = b_{n+1} + b_k$ for some $1 \leq k \leq n$, $d = 1$ if $m = 2b_{n+1}$ and $d = 0$ otherwise. But if $m \geq b_{n+1} > 2b_n$, then $r(B_n, m) = 0$. Therefore $r(B_{n+1}, m) \leq \max(r(B_n, m), 2)$ for all $m \in \mathbb{N}$. Thus $s(B_{n+1}) = s(B_n)$ for all $n \geq s$. The resulting sequence (b_n) is strictly increasing, and $B = \{b_n : n \in \mathbb{N}^*\} = \bigcup_{n=s}^{\infty} B_n$ satisfies $A \ll B$ and $s(B) = \sup\{s(B_n) : n \geq s\} = s(B_s) = s$, since B_s is the interval $[a_s, a_s + s - 1]$ of $\mathbb{N}$. ∎

2.4. Remarks. It follows from 2.3, by taking $A = \mathbb{N}$, that the range of the function $P \mapsto s(P)$ from $\mathcal{P}$ into $\overline{\mathbb{N}}$ is the interval $[2, \infty]$. Then it further follows that for any s, t in the interval $[2, \infty]$ of $\overline{\mathbb{N}}$, there exist $A, B \in \mathcal{P}$ such that $A \ll B$, $s(A) = s$ and $s(B) = t$.

The conjecture (GET) is equivalent to the assertion that if an element A of $\mathcal{P}$ satisfies $A \ll c.\mathbb{S}$ for some $c \in \mathbb{N}^*$, then $s(A) = \infty$.

This leads to the more general question of characterizing all the sets $C \in \mathcal{P}$ which can replace the sets $c.\mathbb{S}$ in the previous statement. Hence the following notion.

2.5. Definition. An infinite subset C of $\mathbb{N}$ is said to belong to the class $\mathcal{C}(\mathrm{ET})$ of Erdős–Turán if, for any $A \in \mathcal{P}$, the relation $A \ll C$ implies that $s(A) = \infty$.

The conjecture (GET) can thus be restated as follows.

(GET) For any $c \in \mathbb{N}^*$, the set $c.\mathbb{S}$ lies in $\mathcal{C}(\mathrm{ET})$.

2.6. Lemma. *Let $B, C \in \mathcal{P}$. We have*

(1) *If $C \in \mathcal{C}(\mathrm{ET})$, then $s(C) = \infty$. The converse is false, in view of* 2.3.
(2) $\mathbb{N} \in \mathcal{C}(\mathrm{ET})$.
(3) *If $C \in \mathcal{C}(\mathrm{ET})$ and $B \ll C$, then $B \in \mathcal{C}(\mathrm{ET})$.*
(4) *If $C \in \mathcal{C}(\mathrm{ET})$ and $C \subset B$ then $B \in \mathcal{C}(\mathrm{ET})$.*
(5) *If $C[t[\in \mathcal{C}(\mathrm{ET})$ for some $t \in \mathbb{N}$, then $C \in \mathcal{C}(\mathrm{ET})$.*
(6) *If $t + C \in \mathcal{C}(\mathrm{ET})$ for some $t \in \mathbb{N}$, then $C \in \mathcal{C}(\mathrm{ET})$.*
(7) *If $t.C \in \mathcal{C}(\mathrm{ET})$ for some $t \in \mathbb{N}^*$, then $C \in \mathcal{C}(\mathrm{ET})$.*

Proof. These properties are direct consequences of those in 2.2. ■

Our next goal is to establish the converses for the last three properties. We start with some useful preliminaries.

2.7. Lemma. *For any subset A of $\mathbb{N}$ and any $t \in \mathbb{N}$, we have $s(t + A) = s(A)$.*

Proof. For any $n \in \mathbb{N}$, the sets $R(A, n) = \{(a, b) \in A \times A : a + b = n\}$ and $R(t+A, 2t+n) = \{(c, d) \in (t+A) \times (t+A) : c+d = 2t+n\}$ are in one-to-one correspondence under the bijection $(a, b) \mapsto (t + a, t + b)$. Therefore $r(A, n) = |R(A, n)| = |R(t+A, 2t+n)| = r(t+A, 2t+n)$. Moreover, for any $m \in \mathbb{N}$ such that $m < 2t$, we have $r(t + A, m) = 0$. Hence

$$s(t + A) = \sup\{r(t + A, 2t + n) : n \in \mathbb{N}\} = \sup\{r(A, n) : n \in \mathbb{N}\} = s(A). \ ■$$

2.8. Lemma. *For any subset A of $\mathbb{N}$ and any $t \in \mathbb{N}^*$, we have $s(t.A) = s(A)$.*

Proof. For any $n \in \mathbb{N}$, the sets $R(A, n) = \{(a, b) \in A \times A : a + b = n\}$ and $R(t.A, tn) = \{(c, d) \in (t.A) \times (t.A) : c + d = tn\}$ are in one-to-one correspondence under the bijection $(a, b) \mapsto (ta, tb)$. Therefore $r(A, n) = |R(A, n)| = |R(t.A, tn)| = r(t.A, tn)$. Moreover, for any $m \in \mathbb{N}$ such that $t \nmid m$, we have $r(t.A, m) = 0$. Hence

$$s(t.A) = \sup\{r(t.A, tn) : n \in \mathbb{N}\} = \sup\{r(A, n) : n \in \mathbb{N}\} = s(A). \ ■$$

2.9. Lemma. *For any subsets A, F of $\mathbb{N}$, if F is finite, then $s(A) \le s(A \cup F) \le s(A) + 2|F|$.*

Proof. Since $(A \cup F) \times (A \cup F) = (A \times A) \cup (F \times (A \cup F)) \cup ((A \cup F) \times F)$, for any $n \in \mathbb{N}$ the set $R(A \cup F, n) = \{(a, b) \in (A \cup F) \times (A \cup F) : a + b = n\}$ is the union of the three sets $R(A, n) = \{(a, b) \in A \times A : a + b = n\}$ and

$R(F, A\cup F; n) = \{(a,b) \in F \times (A \cup F) : a+b=n\}$ and $R(A\cup F, F; n) = \{(a,b) \in (A\cup F) \times F : a+b=n\}$. Moreover the last two sets are in one-to-one correspondence under the bijection $(a,b) \mapsto (b,a)$ and, since e.g. $R(F, A\cup F; n) \subset \{(a, n-a) : a \in F\}$, they satisfy $|R(A\cup F, F; n)| = |R(F, A\cup F; n)| \le |F|$. Hence

$$r(A\cup F, n) = |R(A\cup F, n)| \le |R(A,n)| + 2|R(F, A\cup F; n)| \le r(A,n) + 2|F|.$$

Therefore

$$\begin{aligned} s(A\cup F) &= \sup\{r(A\cup F, n) : n \in \mathbb{N}\} \\ &\le \sup\{r(A,n) + 2|F| : n \in \mathbb{N}\} = s(A) + 2|F|. \end{aligned}$$

On the other hand, since $A \subset A\cup F$, we clearly have $s(A) \le s(A\cup F)$. ■

2.10. Example. The inequality $s(A\cup F) \le s(A) + 2|F|$ is optimal, as shown by the following example. Let h, s be two positive integers such that $2h \le s$. Consider, in $\mathbb{N}$, the intervals $H = [1, 2h+s]$ and $F = [h+1, 2h]$, and let $A = H \setminus F = [1,h] \cup [2h+1, 2h+s]$. Then, by simple combinatorial arguments, we get

$$r(A,n) = \begin{cases} n-1 & \text{if } 1 \le n \le h+1, \\ 2h-n+1 & \text{if } h+2 \le n \le 2h, \\ 2(n-2h-1) & \text{if } 2h+1 \le n \le 3h+1, \\ 2h & \text{if } 3h+2 \le n \le 4h+1, \\ n-2h-1 & \text{if } 4h+2 \le n \le 2h+s+1, \\ 2h+2s-n+1 & \text{if } 2h+s+2 \le n \le 3h+s, \\ n-4h-1 & \text{if } 3h+s+1 \le n \le 4h+s+1, \\ 4h+2s-n+1 & \text{if } 4h+s+2 \le n \le 4h+2s, \\ 0 & \text{if } n \ge 4h+2s+1. \end{cases}$$

In particular, $r(A,n) \le s$ for all $n \in \mathbb{N}$ and $r(A, 2h+s+1) = r(A, 4h+s+1) = s$. Thus $s(A) = s$. On the other hand, it is easy to see that

$$r(H,n) = \begin{cases} n-1 & \text{if } 1 \le n \le 2h+s+1, \\ 4h+2s-n+1 & \text{if } 2h+s+2 \le n \le 4h+2s, \\ 0 & \text{if } n \ge 4h+2s+1. \end{cases}$$

Therefore $s(A\cup F) = s(H) = 2h+s = s(A) + 2|F|$. ■

2.11. Corollary. *For any subsets A, F of $\mathbb{N}$ such that F is finite, we have $s(A) = \infty$ if and only if $s(A\cup F) = \infty$.*

2.12. Remark. A set $A \in \mathcal{P}$ is called an *asymptotic basis* of $\mathbb{N}$ if $r(A,n) > 0$ for all large enough integers n. The original form of the (ET) conjecture [3] is the following:

(OET) If A is an asymptotic basis then $s(A) = \infty$.

Corollary 2.11 shows that (OET) is equivalent to (ET), upon replacing the asymptotic basis A by the basis $B = \mathbb{N}[n_0] \cup A$ of $\mathbb{N}$.

2.13. Lemma. *Let $A, Q \in \mathcal{P}$ be such that $Q \ll 1 + A$ and $Q \neq \mathbb{N}$, and let m be the smallest positive integer such that $q_m \geq m$. Then there exists $P \in \mathcal{P}$ such that $P \ll A$ and $s(P) \leq s(Q) + 2(m-1)$.*

Proof. As noted in 2.2, for all $n \in \mathbb{N}^*$, we have $q_n \geq n-1$, and since $Q \neq \mathbb{N}$, this inequality should be strict for some $n \in \mathbb{N}^*$. Thus $\{n \in \mathbb{N}^* : q_n \geq n\}$ is not empty and has a smallest element m. We then have $q_n = n-1$ for $1 \leq n \leq m-1$, and $q_n \geq n$ for $n \geq m$. The latter inequality holds inductively, since the sequence (q_n) is strictly increasing; and if $m = 1$, the former equality never occurs. Now define P as follows: let $p_n = q_n = n-1$ for $1 \leq n \leq m-1$, and $p_n = q_n - 1$ for $n \geq m$. From the definition of m, the sequence (p_n) is strictly increasing, so that $P \in \mathcal{P}$. Since $Q \ll 1 + A$, for $n \geq m$ we have $p_n = q_n - 1 \leq a_n$, while for $1 \leq n \leq m-1$ we have $p_n = n - 1 \leq a_n$ (by 2.2). Thus $P \ll A$. Moreover, $P = F \cup R$, where $F = \{n \in \mathbb{N} : 0 \leq n \leq m-2\}$ is a finite, possibly empty, set and $R = \{q_n - 1 : n \geq m\}$, so that $1 + R = Q[q_m[$. Therefore, by 2.9 and 2.7, we have $s(P) \leq s(R) + 2|F|$ and $s(R) = s(1+R) = s(Q[q_m[) \leq s(Q)$, while $|F| = m-1$. Thus $s(P) \leq s(Q) + 2(m-1)$. ∎

2.14. Proposition. *If $C \in \mathcal{C}(\mathrm{ET})$ then $t + C \in \mathcal{C}(\mathrm{ET})$ for all $t \in \mathbb{N}$.*

Proof. By induction on t, it is enough to show that if $C \in \mathcal{C}(\mathrm{ET})$ then $1+C \in \mathcal{C}(\mathrm{ET})$. By 2.13, for any $Q \in \mathcal{P}$ such that $Q \ll 1+C$, either $Q = \mathbb{N}$ or there exist $m \in \mathbb{N}^*$ and $P \in \mathcal{P}$ such that $P \ll C$ and $s(Q) \geq s(P) - 2(m-1)$. Since $C \in \mathcal{C}(\mathrm{ET})$ and $P \ll C$, we have $s(P) = \infty$. Therefore $s(Q) = \infty$ for all $Q \ll 1 + C$. Thus $1 + C \in \mathcal{C}(\mathrm{ET})$. ∎

2.15. Lemma. *Let $A, Q \in \mathcal{P}$ and $m \in \mathbb{N}^*$ be such that $Q \ll A[a_m[$. Then there exists $P \in \mathcal{P}$ such that $P \ll a_m + A$ and $s(P) \leq s(Q) + 2(m-1)$.*

Proof. Define P as follows: let $p_n = a_n$ for $1 \leq n \leq m-1$, and $p_n = a_m + q_{n-m+1}$ for $n \geq m$. Since the sequences (a_n) and (q_n) are strictly increasing, and since $p_{m-1} = a_{m-1} < p_m = a_m + q_1$, the sequence (p_n) is also strictly increasing, so that $P \in \mathcal{P}$. Since $Q \ll A[a_m[$, we have $q_k \leq a_{m+k-1}$ for all $k \in \mathbb{N}^*$, so that $q_{n-m+1} \leq a_n$ and thus $p_n = a_m + q_{n-m+1} \leq a_m + a_n$ for all $n \geq m$. But also, $p_n = a_n \leq a_m + a_n$ for $1 \leq n \leq m-1$. Therefore $P \ll a_m + A$. Moreover, $P = F \cup R$, where $F = \{a_n : 1 \leq n \leq m-1\}$ is a finite, possibly empty, set and $R = a_m + Q$. Therefore, by 2.9 and 2.7, we have $s(P) \leq s(R) + 2|F|$ and $s(R) = s(Q)$, while $|F| = m-1$. Thus $s(P) \leq s(Q) + 2(m-1)$. ∎

2.16. Proposition. *If $C \in \mathcal{C}(\mathrm{ET})$ then $C[t[\in \mathcal{C}(\mathrm{ET})$ for all $t \in \mathbb{N}$.*

Proof. Let $c_m = \min C[t[$, so that $C[t[= C[c_m[$ with $m \in \mathbb{N}^*$. By 2.15, for any $Q \in \mathcal{P}$ such that $Q \ll C[t[= C[c_m[$, there exists $P \in \mathcal{P}$ such that $P \ll c_m + C$ and $s(Q) \geq s(P) - 2(m-1)$. Since $C \in \mathcal{C}(\mathrm{ET})$, by 2.14 we have $c_m + C \in \mathcal{C}(\mathrm{ET})$, so that, since $P \ll c_m + C$, we have $s(P) = \infty$. Therefore $s(Q) = \infty$ for all $Q \ll C[t[$. Thus $C[t[\in \mathcal{C}(\mathrm{ET})$. ∎

2.17. Corollary. *Let $B, C \in \mathcal{P}$ be such that $C[t[= B[u[$ for some $t, u \in \mathbb{N}$. Then $C \in \mathcal{C}(\mathrm{ET})$ if and only if $B \in \mathcal{C}(\mathrm{ET})$.*

Proof. Assume that $C \in \mathcal{C}(\mathrm{ET})$. Then, by 2.16, $C[t[\in \mathcal{C}(\mathrm{ET})$, i.e. $B[u[\in \mathcal{C}(\mathrm{ET})$. Hence $B \in \mathcal{C}(\mathrm{ET})$ by 2.6. The equivalence follows by symmetry. ∎

2.18. Proposition. *The following two statements are equivalent*:

(1) *For any $C \in \mathcal{C}(\mathrm{ET})$ and any $t \in \mathbb{N}^*$, we have $t.C \in \mathcal{C}(\mathrm{ET})$.*
(2) *For any $C \in \mathcal{C}(\mathrm{ET})$, we have $2.C \in \mathcal{C}(\mathrm{ET})$.*

Proof. Statement (2) is clearly a special case of (1). Now assume that (2) holds. Then, for any $C \in \mathcal{C}(\mathrm{ET})$, we have, by induction, $2^n.C \in \mathcal{C}(\mathrm{ET})$ for all $n \in \mathbb{N}$. Hence, for any $t \in \mathbb{N}^*$, we have $2^t.C \in \mathcal{C}(\mathrm{ET})$ and $t.C \ll 2^t.C$, and thus, by 2.6, we conclude that $t.C \in \mathcal{C}(\mathrm{ET})$. This shows that if (2) holds then so does (1), which completes the proof. ∎

In order to establish the truth of the statements in 2.18, we will need to make a detour in the next two sections.

3. Close sequences

3.1. Definition. For $A, B \in \mathcal{P}$, we set $\delta(A, B) = \sup\{|a_n - b_n| : n \in \mathbb{N}^*\}$ in $\overline{\mathbb{N}}$. If $\delta(A, B) \leq d$ for some $d \in \mathbb{R}^+$, we say that A and B are *d-close*. More generally, if $\delta(A, B) < \infty$, we say that A and B are *close*.

3.2. Proposition. *Let $A, B \in \mathcal{P}$ be d-close for some $d \in \mathbb{R}^+$. Then, for any $m \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that*

$$r(B, n) \geq \frac{r(A, m)}{4d + 1}.$$

Proof. Let $m \in \mathbb{N}$ and $E(A, m) = \{(i, j) \in \mathbb{N}^* \times \mathbb{N}^* : a_i + a_j = m\}$. Then $r(A, m) = |E(A, m)|$. Let $\sigma : E(A, m) \to \mathbb{N}$ be the map defined by $\sigma(i, j) = b_i + b_j$. For every $n \in \sigma(E(A, m))$, there is a pair $(i, j) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $a_i + a_j = m$ and $b_i + b_j = n$. Since A and B are d-close, we have $|a_i - b_i| \leq d$ and $|a_j - b_j| \leq d$. Hence $a_i + a_j - 2d \leq b_i + b_j \leq a_i + a_j + 2d$, i.e. $m - 2d \leq n \leq m + 2d$. Thus $\sigma(E(A, m)) \subset I$ where $I = [m - 2d, m + 2d] \cap \mathbb{N}$, so that $E(A, m) = \bigcup_{n \in I} \sigma^{-1}(n)$ is a finite union of pairwise disjoint sets. Moreover, for every $n \in I$, we have $\sigma^{-1}(n) = \{(i, j) \in E(A, m) : b_i + b_j = n\}$

$\subset E(B,n)$, so that $|\sigma^{-1}(n)| \leq r(B,n)$. Therefore

$$r(A,m) = |E(A,m)| = \sum_{n\in I} |\sigma^{-1}(n)| \leq \sum_{n\in I} r(B,n)$$
$$\leq |I| \cdot \max\{r(B,n) : n \in I\}.$$

Since I is a set of integers contained in the interval $[m-2d, m+2d]$ of length $4d$, we have $|I| \leq 4d+1$ and there exists some $n_0 \in I$ such that $r(B,n_0) = \max\{r(B,n) : n \in I\}$. Thus $r(A,m) \leq (4d+1)r(B,n_0)$, and the result follows. ∎

3.3. Corollary. *Let $A, B \in \mathcal{P}$ and $d \in \mathbb{R}^+$. If A and B are d-close, then*

$$\frac{s(A)}{4d+1} \leq s(B) \leq (4d+1)s(A).$$

Proof. In view of 3.2, and by the definition of $s(B)$, for every $m \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that

$$s(B) \geq r(B,n) \geq \frac{r(A,m)}{4d+1}.$$

Thus $r(A,m) \leq (4d+1)s(B)$ for all $m \in \mathbb{N}$, and therefore $s(A) \leq (4d+1)s(B)$. Hence the first inequality. Exchanging A and B, we also get the second inequality. ∎

3.4. Corollary. *Let $A, B \in \mathcal{P}$. If A and B are close, then $s(A) = \infty$ if and only if $s(B) = \infty$.*

Proof. This follows immediately from 3.3, since A and B are d-close for some $d \in \mathbb{R}^+$. ∎

3.5. Corollary. *Let $A, B \in \mathcal{P}$ and $d \in \mathbb{R}^+$. If A and B are d-close and $s(A)$ and $s(B)$ are finite, then $|s(A) - s(B)| \leq 4d \cdot \min(s(A), s(B))$.*

Proof. Assume that $s(A) \leq s(B)$. Then, by 3.3, we have $s(B) \leq (4d+1)s(A)$, i.e. $s(B) - s(A) \leq 4d \cdot s(A)$. Hence the result. ∎

3.6. Remark. The inequalities established in 3.3 and 3.5 hold with $d = \delta(A,B)$, and they even hold trivially when $\delta(A,B) = \infty$. Hence the following statements:

(i) For any $A, B \in \mathcal{P}$, we have $s(B) \leq (4\delta(A,B)+1)s(A)$ and $s(A) \leq (4\delta(A,B)+1)s(B)$.
(ii) For any $A, B \in \mathcal{P}$ such that $s(A)$ and $s(B)$ are finite, we have

$$|s(A) - s(B)| \leq 4\min(s(A), s(B)) \cdot \delta(A,B).$$

3.7. Lemma. *Let $A, B \in \mathcal{P}$. If A and B are close, then $A \in \mathcal{C}(\mathrm{ET})$ if and only if $B \in \mathcal{C}(\mathrm{ET})$.*

Proof. Assume that A and B are close and that $A \in \mathcal{C}(\mathrm{ET})$. Then A and B are d-close for some $d \in \mathbb{N}$ (e.g. $d = \delta(A, B)$). So, for all $n \in \mathbb{N}^*$, we have $|b_n - a_n| \le d$, i.e. $a_n - d \le b_n \le a_n + d$. Therefore $B \ll d + A$. Since $A \in \mathcal{C}(\mathrm{ET})$, by 2.14 we have $d + A \in \mathcal{C}(\mathrm{ET})$; and, since $B \ll d + A$, we conclude, by 2.6(3), that $B \in \mathcal{C}(\mathrm{ET})$. ∎

We need to extend some of the previous properties to increasing sequences in $\mathbb{N}$.

3.8. Definition. We denote by $\mathcal{I}(\mathbb{N})$ the set of all sequences $x = (x_n)_{n \in \mathbb{N}^*}$ in $\mathbb{N}$ such that $x_n \le x_{n+1}$ for all $n \in \mathbb{N}^*$. For each $x \in \mathcal{I}(\mathbb{N})$, we write $X = \{x_n : n \in \mathbb{N}^*\}$ for the set of terms of the sequence $x = (x_n)$. Note that the set $\mathcal{P}$ is identified to a subset of $\mathcal{I}(\mathbb{N})$.

For any $x = (x_n)$ and $y = (y_n)$ in $\mathcal{I}(\mathbb{N})$, we write $x \ll y$ if $x_n \le y_n$ for all $n \in \mathbb{N}^*$. This defines an order relation on $\mathcal{I}(\mathbb{N})$ which extends the previously defined one on $\mathcal{P}$.

For $x, y \in \mathcal{I}(\mathbb{N})$, we set $\delta(x, y) = \sup\{|x_n - y_n| : n \in \mathbb{N}^*\}$ in $\overline{\mathbb{N}}$. If $\delta(x, y) \le d$ for some $d \in \mathbb{R}^+$, we say that x and y are *d-close*. If $\delta(x, y) < \infty$, we say that x and y are *close*.

3.9. Remark. For $x \in \mathcal{I}(\mathbb{N})$, the corresponding set X is finite if and only if the sequence $x = (x_n)$ is bounded. Now, if x and y are d-close in $\mathcal{I}(\mathbb{N})$ for some $d \in \mathbb{R}^+$, then $x_n - d \le y_n \le x_n + d$ for all $n \in \mathbb{N}^*$, so that (x_n) is bounded if and only if (y_n) is bounded. Therefore, for two close elements x and y of $\mathcal{I}(\mathbb{N})$, the corresponding sets X and Y are either both finite or both infinite.

3.10. Proposition. *Let $x, y \in \mathcal{I}(\mathbb{N})$ be d-close for some $d \in \mathbb{R}^+$. Then, for any $m \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that*

$$r(Y, n) \ge \frac{r(X, m)}{(4d+1)^3}.$$

Proof. Let $m \in \mathbb{N}$ and $r = r(X, m)$. Then there exists a subset E of $\mathbb{N}^* \times \mathbb{N}^*$ such that $|E| = r$ and, for $(i, j) \in E$, we have $x_i + x_j = m$ and the pairs (x_i, x_j) are pairwise distinct. Let $\sigma : E \to \mathbb{N}$ be the map defined by $\sigma(i, j) = y_i + y_j$. For every $n \in \sigma(E)$, there is a pair $(i, j) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $x_i + x_j = m$ and $y_i + y_j = n$. Since x and y are d-close, we have $|x_i - y_i| \le d$ and $|x_j - y_j| \le d$. Hence $x_i + x_j - 2d \le y_i + y_j \le x_i + x_j + 2d$, i.e. $m - 2d \le n \le m + 2d$. Thus $\sigma(E) \subset I$ where $I = [m - 2d, m + 2d] \cap \mathbb{N}$. Consequently, $E = \bigcup_{n \in I} S_n$ is a finite union of the pairwise disjoint sets $S_n = \sigma^{-1}(n) = \{(i, j) \in E : y_i + y_j = n\}$. Therefore

$$r = |E| = \sum_{n \in I} |S_n| \le |I| \cdot \max\{|S_n| : n \in I\}.$$

Since I is a set of integers contained in the interval $[m-2d, m+2d]$ of length $4d$, we have $|I| \leq 4d+1$ and there exists some $n_0 \in I$ such that $|S_{n_0}| = \max\{|S_n| : n \in I\}$. Hence $r \leq (4d+1)|S_{n_0}|$.

Now consider any $n \in I$ and let $\phi : S_n \to Y \times Y$ be the map defined by $\phi(i,j) = (y_i, y_j)$. Since $|\phi(S_n)|$ gives a count of distinct pairs $(y_i, y_j) \in Y \times Y$ such that $y_i + y_j = n$, we see that $|\phi(S_n)| \leq r(Y,n)$. On the other hand, if we denote by $\mathcal{R}$ the equivalence relation on S_n defined by $(i,j)\mathcal{R}(k,l)$ if and only if $\phi(i,j) = \phi(k,l)$, i.e. $(y_i, y_j) = (y_k, y_l)$, then ϕ induces a bijection from the quotient set $S_n/\mathcal{R}$ onto the image set $\phi(S_n)$, so that $|S_n/\mathcal{R}| = |\phi(S_n)| \leq r(Y,n)$. Furthermore, if $(y_k, y_l) = (y_i, y_j)$ with (i,j) and (k,l) in E, then, since $|x_i - y_i| \leq d$ and $|x_k - y_k| \leq d$ and $y_i = y_k$, we get $|x_i - x_k| \leq 2d$, and similarly $|x_j - x_l| \leq 2d$. Thus, for a given $(i,j) \in E$, the number of possible values for x_k, under the condition $(y_k, y_l) = (y_i, y_j)$, does not exceed $4d+1$, and similarly for x_l, so that the number of pairs (x_k, x_l) does not exceed $(4d+1)^2$. Since, by the definition of E, the map $(k,l) \mapsto (x_k, x_l)$, restricted to E, is injective, for any $(i,j) \in E$ the number of $(k,l) \in E$ such that $(y_k, y_l) = (y_i, y_j)$ is equal to the number of corresponding pairs (x_k, x_l), and thus does not exceed $(4d+1)^2$. Therefore the number of elements in any equivalence class modulo $\mathcal{R}$, in S_n, does not exceed $(4d+1)^2$, and so $|S_n/\mathcal{R}| \geq |S_n|/(4d+1)^2$. It then follows from what precedes that $|S_n|/(4d+1)^2 \leq |\phi(S_n)| \leq r(Y,n)$, so that $|S_n| \leq (4d+1)^2 r(Y,n)$, and consequently

$$r(X,m) = r \leq (4d+1)|S_{n_0}| \leq (4d+1)^3 r(Y,n_0).$$

Hence the result. ∎

3.11. Corollary. *Let $x, y \in \mathcal{I}(\mathbb{N})$ and $d \in \mathbb{R}^+$. If x and y are d-close, then*

$$\frac{s(X)}{(4d+1)^3} \leq s(Y) \leq (4d+1)^3 s(X).$$

Proof. In view of 3.10, and by definition of $s(Y)$, for every $m \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that $s(Y) \geq r(Y,n) \geq r(X,m)/(4d+1)^3$. Thus $r(X,m) \leq (4d+1)^3 s(Y)$ for all $m \in \mathbb{N}$, and therefore $s(X) \leq (4d+1)^3 s(Y)$. Hence the first inequality. Exchanging x and y, we get the second one. ∎

3.12. Corollary. *Let $x, y \in \mathcal{I}(\mathbb{N})$. If x and y are close, then $s(X) = \infty$ if and only if $s(Y) = \infty$.*

Proof. This follows immediately from 3.11, since x and y are d-close for some $d \in \mathbb{R}^+$. ∎

3.13. Remark. In order to establish that the homothetic image of an element of $\mathcal{C}(\mathrm{ET})$ is also in $\mathcal{C}(\mathrm{ET})$, it is enough, in view of 2.18, to show that for any $C \in \mathcal{C}(\mathrm{ET})$, we have $2.C \in \mathcal{C}(\mathrm{ET})$. To this end, we need a special class of increasing sequences in $\mathbb{N}$ that we study next.

4. Two-step sequences and equivalent formulations of (GET)

4.1. Definition. A *two-step sequence* is an element $x = (x_n)_{n\in\mathbb{N}^*}$ of $\mathcal{I}(\mathbb{N})$ such that $x_{2k-1} = x_{2k} < x_{2k+1}$ for all $k \in \mathbb{N}^*$, i.e. $x_1 = x_2 < x_3 = x_4 < \cdots < x_{2k-1} = x_{2k} < \cdots$.

4.2. Proposition. *Let $C \in \mathcal{P}$ and let $c = (c_n)_{n\in\mathbb{N}^*}$ be the strictly increasing sequence of its elements. The following two conditions are equivalent*:

(1) $2.C \in \mathcal{C}(\mathrm{ET})$.

(2) *For any two-step sequence x such that $x \ll c$, we have $s(X) = \infty$.*

When these conditions are satisfied, we also have $C \in \mathcal{C}(\mathrm{ET})$.

Proof. (i) Assume that $2.C \in \mathcal{C}(\mathrm{ET})$. Let x be a two-step sequence such that $x \ll c$. Define a sequence $a = (a_n)_{n\in\mathbb{N}^*}$ by $a_{2k-1} = 2x_{2k-1}$ and $a_{2k} = 2x_{2k} + 1$ for all $k \in \mathbb{N}^*$. Then, since x is a two-step sequence, we have

$$a_{2k-1} = 2x_{2k-1} = 2x_{2k} < a_{2k} = 2x_{2k} + 1 < 2(x_{2k} + 1) \le 2x_{2k+1} = a_{2k+1}$$

for all $k \in \mathbb{N}^*$, so that the sequence a is strictly increasing, i.e. the set A of its terms lies in $\mathcal{P}$. Moreover, since $x \ll c$ and c is strictly increasing, we also have $a_{2k-1} = 2x_{2k-1} \le 2c_{2k-1}$ and $a_{2k} = 2x_{2k} + 1 = 2x_{2k-1} + 1 \le 2c_{2k-1} + 1 < 2(c_{2k-1} + 1) \le 2c_{2k}$ for all $k \in \mathbb{N}^*$, so that $A \ll 2.C$. Therefore, since $2.C \in \mathcal{C}(\mathrm{ET})$, we have $s(A) = \infty$. Furthermore, from the definition of a, we have $2x_n \le a_n \le 2x_n + 1$ for all $n \in \mathbb{N}^*$, so that $a = (a_n)$ and $2x = (2x_n)$ are 1-close. Thus, in view of 3.12, we also have $s(2.X) = \infty$. Hence, by 2.8, we conclude that $s(X) = s(2.X) = \infty$. Thus (1) implies (2).

(ii) Assume that for any two-step sequence x such that $x \ll c$, we have $s(X) = \infty$. Let $A \in \mathcal{P}$ be such that $A \ll 2.C$. Let $u = (u_n)_{n\in\mathbb{N}^*}$ be the sequence defined by $u_n = [a_n/2]$ for all $n \in \mathbb{N}^*$, where $[r]$ denotes the integral part of a real number r. For all $n \in \mathbb{N}^*$, since $a_n < a_{n+1}$, we have $u_n \le u_{n+1}$ and, since $a_{n+2} \ge a_n + 2$, we also have

$$u_{n+2} \ge [(a_n + 2)/2] = [a_n/2] + 1 = u_n + 1.$$

Therefore $u \in \mathcal{I}(\mathbb{N})$ and of any three consecutive terms u_n, u_{n+1} and u_{n+2} of u, at most two can be equal. Let $x = (x_n)_{n\in\mathbb{N}^*}$ be the sequence defined as follows:

$$x_1 = x_2 = u_1, \qquad x_{2k+1} = x_{2k+2} = \min U[(x_{2k} + 1)[\quad \text{for } k \in \mathbb{N}^*.$$

In other words, x is the two-step sequence which takes as values the consecutive elements of U. In particular, $X = U$. Note that, from the definition, for any $n \in \mathbb{N}^*$, the set $\{x_1, \ldots, x_n\}$ consists of the first $[(n+1)/2]$ distinct values of U, while since no more than two consecutive terms of u can be equal, the set $\{u_1, \ldots, u_n\}$ consists of at least the first $[(n+1)/2]$ distinct values of U, so that $\{x_1, \ldots, x_n\} \subset \{u_1, \ldots, u_n\}$, i.e. $x_n \le u_n$ for all

$n \in \mathbb{N}^*$, hence $x \ll u$. Moreover, for all $n \in \mathbb{N}^*$, since $a_n \le 2c_n$ and since $u_n \le a_n/2$, we have $u_n \le c_n$, so that $x_n \le u_n \le c_n$. Thus $x \ll c$, and therefore, by the assumption, we have $s(X) = \infty$, i.e. $s(U) = \infty$. Hence, by 2.8, $s(2.U) = s(U) = \infty$. Furthermore, from the definition of u, we have $2u_n \le a_n \le 2u_n + 1$ for all $n \in \mathbb{N}^*$, so that a and $2u$ are 1-close. Therefore, in view of 3.12, since $s(2.U) = \infty$, also $s(A) = \infty$. This shows that, under the given assumption, for any $A \in \mathcal{P}$ such that $A \ll 2.C$, we have $s(A) = \infty$. Hence $2.C \in \mathcal{C}(\mathrm{ET})$. Thus (2) implies (1).

Finally, in view of 2.6(7), if $2.C \in \mathcal{C}(\mathrm{ET})$, then $C \in \mathcal{C}(\mathrm{ET})$ as well. ■

4.3. Definition. Let $x = (x_n)$ be any sequence of elements of $\mathbb{N}$. The *strict cover* of x is the sequence $\widehat{x} = (\widehat{x}_n)$ defined by

$$\widehat{x}_1 = x_1, \qquad \widehat{x}_{n+1} = \max(x_{n+1}, \widehat{x}_n + 1) \quad \text{for all } n \in \mathbb{N}^*.$$

The reason for this name lies in the following universal property of $\widehat{x}$.

4.4. Lemma. *Let x be any sequence in $\mathbb{N}$ and $\widehat{x}$ be the strict cover of x. Then $\widehat{x}$ is a strictly increasing sequence*; *it satisfies $x \ll \widehat{x}$*; *and for any strictly increasing sequence y in $\mathbb{N}$, if $x \ll y$ then $\widehat{x} \ll y$. In other words, $\widehat{x}$ is the least* (*for the order relation $\ll$*) *strictly increasing sequence y in $\mathbb{N}$ such that $x \ll y$.*

Proof. From the definition of $\widehat{x}$, for all $n \in \mathbb{N}^*$, we have

$$\widehat{x}_{n+1} = \max(x_{n+1}, \widehat{x}_n + 1),$$

so that $\widehat{x}_{n+1} \ge \widehat{x}_n + 1$ and $x_{n+1} \le \widehat{x}_{n+1}$. Therefore the sequence $\widehat{x}$ is strictly increasing and, taking into account that $\widehat{x}_1 = x_1$, we have $x \ll \widehat{x}$. Now let $y = (y_n)$ be a strictly increasing sequence in $\mathbb{N}$ such that $x \ll y$. Thus $x_n \le y_n$ for all $n \in \mathbb{N}^*$; in particular, $\widehat{x}_1 = x_1 \le y_1$. Assume, by induction, that $\widehat{x}_n \le y_n$. Then, from the definition, if $\widehat{x}_n \ge x_{n+1}$, we have $\widehat{x}_{n+1} = \widehat{x}_n + 1 \le y_n + 1 \le y_{n+1}$, while if $\widehat{x}_n < x_{n+1}$, we have $\widehat{x}_{n+1} = x_{n+1} \le y_{n+1}$. Therefore $\widehat{x} \ll y$. ■

4.5. Lemma. *Let x be a two-step sequence in $\mathbb{N}$ and $\widehat{x}$ be its strict cover. Consider the difference sequence $d = \widehat{x} - x$, defined by $d_n = \widehat{x}_n - x_n$ for all $n \in \mathbb{N}^*$. Then the following properties hold for all $k, n \in \mathbb{N}^*$:*

(1) $d_n \ge 0$.
(2) $d_{2n} = d_{2n-1} + 1$.
(3) $d_{2n+1} = 0$, *or* $d_{2n+1} = d_{2n-1} + x_{2n} - x_{2n+2} + 2$.
(4) $d_{2n+1} \le d_{2n-1} + 1$.
(5) $d_{2n+2k-1} \le d_{2n-1} + k$.
(6) *If* $d_{2n-1} = 0$ *and* $d_{2n+2i-1} > 0$ *for* $1 \le i \le k$, *then* $x_{2n+2i} - x_{2n} < 2i$ *for* $1 \le i \le k$.

Proof. (1) This amounts to $x \ll \widehat{x}$ and follows from 4.4.

(2) Since x is a two-step, we have $x_{2n-1} = x_{2n}$ and

$$\widehat{x}_{2n} = \max(x_{2n}, \widehat{x}_{2n-1} + 1) = \max(x_{2n-1}, \widehat{x}_{2n-1} + 1) = \widehat{x}_{2n-1} + 1,$$

since $x_{2n-1} \le \widehat{x}_{2n-1}$. Hence $d_{2n} = \widehat{x}_{2n} - x_{2n} = \widehat{x}_{2n-1} + 1 - x_{2n-1} = d_{2n-1} + 1$.

(3) We have $\widehat{x}_{2n+1} = \max(x_{2n+1}, \widehat{x}_{2n} + 1)$. Thus, either $\widehat{x}_{2n+1} = x_{2n+1}$ and $d_{2n+1} = 0$, or $\widehat{x}_{2n+1} = \widehat{x}_{2n} + 1$ and, since x is two-step,

$$\begin{aligned} d_{2n+1} &= \widehat{x}_{2n} + 1 - x_{2n+1} = \widehat{x}_{2n} + 1 - x_{2n+2} \\ &= (\widehat{x}_{2n} - x_{2n}) + x_{2n} - x_{2n+2} + 1 = d_{2n} + x_{2n} - x_{2n+2} + 1. \end{aligned}$$

Therefore, in view of (2), $d_{2n+1} = d_{2n-1} + x_{2n} - x_{2n+2} + 2$.

(4) Since x is two-step, we have $x_{2n+2} = x_{2n+1} > x_{2n}$, i.e. $x_{2n} - x_{2n+2} \le -1$. Hence, by (3), either $d_{2n+1} = 0 \le d_{2n-1} < d_{2n-1} + 1$, or $d_{2n+1} = d_{2n-1} + x_{2n} - x_{2n+2} + 2 \le d_{2n-1} + 1$. Thus, in all cases, $d_{2n+1} \le d_{2n-1} + 1$.

(5) This follows from (4) by induction on k.

(6) Assume that $d_{2n-1} = 0$ and $d_{2n+2i-1} > 0$ for $1 \le i \le k$. Then, by (3), for $1 \le i \le k$, we have $d_{2n+2i-1} = d_{2n+2i-3} + x_{2n+2i-2} - x_{2n+2i} + 2$, so that $x_{2n+2i} - x_{2n+2i-2} = d_{2n+2i-3} - d_{2n+2i-1} + 2$. Adding up the equalities corresponding to $j = 1, \ldots, i$, we get

$$\begin{aligned} x_{2n+2i} - x_{2n} &= \sum_{j=1}^{i} (x_{2n+2j} - x_{2n+2j-2}) = \sum_{j=1}^{i} (d_{2n+2j-3} - d_{2n+2j-1} + 2) \\ &= 2i - d_{2n+2i-1} < 2i \end{aligned}$$

for $1 \le i \le k$. ∎

4.6. PROPOSITION. *Let x be a two-step sequence in $\mathbb{N}$ and $\widehat{x}$ be its strict cover. Then either $s(X) = \infty$ or x and $\widehat{x}$ are close.*

Proof. We use the notations of 4.5 and we distinguish two possible cases.

(i) First we assume that, for any $k \in \mathbb{N}^*$, there exists an $n \in \mathbb{N}^*$ such that $d_{2n-1} = 0$ and $d_{2n+2i-1} > 0$ for $1 \le i \le k$. We set $e_i = x_{2n+2i} - x_{2n}$, so that, by 4.5(6), we have $e_i < 2i$ for $1 \le i \le k$. Moreover, since $x_{2n+2i+2} = x_{2n+2i+1} > x_{2n+2i}$, we also have $0 < e_1 < e_2 < \cdots < e_k$. Let $E = \{e_1, \ldots, e_k\}$, and consider the map $\sigma : E \times E \to \mathbb{N}[2e_k]$ defined by $\sigma(p, q) = p + q$. Then $E \times E = \bigcup_{n=0}^{2e_k} \sigma^{-1}(n)$, a union of pairwise disjoint sets whose cardinalities are $|\sigma^{-1}(n)| = r(E, n)$ for $0 \le n \le 2e_k$. Therefore

$$k^2 = |E \times E| = \sum_{n=0}^{2e_k} r(E, n) \le (2e_k + 1)s(E) \le 4ks(E),$$

since $e_k < 2k$. Thus $s(E) \ge k/4$, and since $\{x_{2n+2i} : 1 \le i \le k\} = x_{2n} + E$ is a subset of X, we conclude, using 2.7, that $s(X) \ge s(x_{2n} + E) = s(E) \ge k/4$. By the assumption made in this case, the inequality holds for all $k \in \mathbb{N}^*$. Hence $s(X) = \infty$.

(ii) We now assume that there exists some $k \in \mathbb{N}^*$ such that for any $n \in \mathbb{N}^*$, there is, in $\mathbb{N}$, some $i \le k-1$ such that $d_{2n-2i-1} = 0$. This is always possible when the assumption in (i) does not hold, since we at least have $d_1 = 0$. Then, by 4.5(5), we get $d_{2n-1} \le d_{2n-2i-1} + i = i \le k-1$ for all $n \in \mathbb{N}^*$. Therefore, in view of 4.5(2), we also get $d_{2n} \le k$ for all $n \in \mathbb{N}^*$. It follows, using 4.5 again, that $0 \le d_n = \widehat{x}_n - x_n \le k$ for all $n \in \mathbb{N}^*$, so that x and $\widehat{x}$ are k-close. ∎

4.7. Corollary. *Let $C \in \mathcal{C}(\mathrm{ET})$ and let x be a two-step sequence in $\mathbb{N}$ such that $x \ll c$. Then $s(X) = \infty$.*

Proof. Let $\widehat{x}$ be the strict cover of x. By 4.4, the sequence $\widehat{x}$ is strictly increasing, like c, and since $x \ll c$, we also have $\widehat{x} \ll c$, i.e. $\widehat{X} \ll C$. It follows, as $C \in \mathcal{C}(\mathrm{ET})$, that $s(\widehat{X}) = \infty$. Thus, if x and $\widehat{x}$ are close, we conclude, by 3.12, that $s(X) = \infty$. Otherwise the same conclusion follows from 4.6. ∎

4.8. Proposition. *If $C \in \mathcal{C}(\mathrm{ET})$, then $t.C \in \mathcal{C}(\mathrm{ET})$ for any $t \in \mathbb{N}^*$.*

Proof. Let $C \in \mathcal{C}(\mathrm{ET})$. In view of 2.18, it is enough to show that $2.C \in \mathcal{C}(\mathrm{ET})$. This condition is, by 4.2, equivalent to asserting that for any two-step sequence x such that $x \ll c$, we have $s(X) = \infty$. But the validity of the latter condition follows from 4.7. Hence the result. ∎

4.9. Proposition. *For any $C \in \mathcal{P}$, the following statements are equivalent*:

(1) *$C \in \mathcal{C}(\mathrm{ET})$.*
(2) *There exists $B \in \mathcal{C}(\mathrm{ET})$ such that $C \ll B$.*
(3) *There exists $B \in \mathcal{C}(\mathrm{ET})$ such that $B \subset C$.*
(4) *There exists $t \in \mathbb{N}$ such that $C[t[\in \mathcal{C}(\mathrm{ET})$ (resp. $C[t[\in \mathcal{C}(\mathrm{ET})$ for all $t \in \mathbb{N}$).*
(5) *There exists $t \in \mathbb{N}$ such that $t + C \in \mathcal{C}(\mathrm{ET})$ (resp. $t + C \in \mathcal{C}(\mathrm{ET})$ for all $t \in \mathbb{N}$).*
(6) *There exists $t \in \mathbb{N}^*$ such that $t.C \in \mathcal{C}(\mathrm{ET})$ (resp. $t.C \in \mathcal{C}(\mathrm{ET})$ for all $t \in \mathbb{N}^*$).*
(7) *There exist $B \in \mathcal{C}(\mathrm{ET})$ and $t, u \in \mathbb{N}$ such that $C[t[= B[u[$.*
(8) *There exists $B \in \mathcal{C}(\mathrm{ET})$ such that $\delta(B, C) < \infty$ (i.e. B and C are close).*

Proof. These equivalences immediately follow from 2.6, 2.14, 2.16, 2.17, 3.7 and 4.8. ∎

4.10. Corollary. *The conjecture* (GET) *is equivalent to the following simplified form*:

(SGET) *The set $\mathbb{S} = \{n^2 : n \in \mathbb{N}^*\}$ lies in $\mathcal{C}(\mathrm{ET})$.*

4.11. Definition. For a given real number $r \in]0,1]$, consider the sequences (x_n) in $\mathbb{R}$ and (e_n) in $\mathbb{N}$ defined by $x_n = n + rn(n-1)$ and $e_n = [x_n] = n + [rn(n-1)]$ (integral part), for $n \in \mathbb{N}^*$. Let $E_r = \{e_n : n \in \mathbb{N}^*\}$.

In particular, $E_1 = \mathbb{S}$; and $E_{1/2} = \{n(n+1)/2 : n \in \mathbb{N}^*\} = T$ is the set of triangular numbers in $\mathbb{N}^*$.

4.12. Lemma. *The sequences (x_n) and (e_n) have the following properties*:

(1) $e_1 = x_1 = 1$.
(2) $x_{n+1} - x_n = 1 + 2rn > 1$ *for all* $n \in \mathbb{N}^*$.
(3) $e_{n+1} - e_n = 1 + [rn(n+1)] - [rn(n-1)] \geq 1$ *for all* $n \in \mathbb{N}^*$, *so that the sequence* (e_n) *is strictly increasing.*
(4) $x_n + (1-r)n(n-1) = n^2$ *for all* $n \in \mathbb{N}^*$.
(5) $e_n \leq x_n \leq n^2 < te_n + t$ *for any* $t \in \mathbb{R}^+$ *such that* $tr \geq 1$ *and for all* $n \in \mathbb{N}^*$.

Proof. Parts (1) to (4) and the first two inequalities in (5) are immediately verified. For the last inequality in (5), note that since $e_n = [x_n]$, we have $x_n < e_n + 1$. Hence

$$t(e_n + 1) > tx_n = tn + trn(n-1) \geq n + n(n-1) = n^2,$$

since $t \geq 1/r \geq 1$. ∎

4.13. Corollary. *For any real number $r \in]0,1]$ and any $t \in \mathbb{N}^*$ such that $tr \geq 1$, we have $E_r \ll \mathbb{S} \ll t + t.E_r$.*

4.14. Theorem. *The following conjectures are equivalent*:

(GET) $c.\mathbb{S} \in \mathcal{C}(\mathrm{ET})$ *for all* $c \in \mathbb{N}^*$.
(SGET) $\mathbb{S} \in \mathcal{C}(\mathrm{ET})$.
(EGET) $E_r \in \mathcal{C}(\mathrm{ET})$ *for some* $r \in]0,1]$ (*resp. for all* $r \in]0,1]$).
(TGET) $T \in \mathcal{C}(\mathrm{ET})$, *where* $T = \{n(n+1)/2 : n \in \mathbb{N}^*\}$.

Proof. This follows from 4.10 and the fact that, by 4.13 and 4.9, we have

$$\mathbb{S} \in \mathcal{C}(\mathrm{ET}) \Rightarrow E_r \in \mathcal{C}(\mathrm{ET}) \Rightarrow t + t.E_r \in \mathcal{C}(\mathrm{ET}) \Rightarrow \mathbb{S} \in \mathcal{C}(\mathrm{ET}). \blacksquare$$

4.15. Remark. Note that $s(\mathbb{S}) = \infty$, which is a necessary (but not sufficient) condition for (GET) to hold, follows from well known results about the number of representations of a positive integer as a sum of two squares [6, Theorem 278].

5. The caliber function and the restricted class of Erdős–Turán sets. For any subset A of $\mathbb{N}$ and any $x \in \mathbb{R}^+$, let $A(x) = |A[x]| = |A \cap [0,x]|$.

5.1. Lemma. *Let $A = \{a_1 < a_2 < \cdots\}$ be a subset of $\mathbb{N}$.*

(1) *For any $x \in \mathbb{R}^+$ and $n \in \mathbb{N}^*$, we have $A(x) = n$ if and only if $a_n \le x < a_{n+1}$.*
(2) *For any $n \in \mathbb{N}$, we have $r(A, n) \le A(n)$. For any $x \in \mathbb{R}^+$, we have $s(A[x]) \le A(x)$.*
(3) *For $A, B \in \mathcal{P}$, we have $A \ll B$ if and only if $B(n) \le A(n)$ for all $n \in \mathbb{N}$. In particular, $A = B$ if and only if $A(n) = B(n)$ for all $n \in \mathbb{N}$.*
(4) *For any $t \in \mathbb{N}$ and $x \in \mathbb{R}^+$, if $x \ge t$, then $(t + A)(x) = A(x-t)$. Moreover, if $t \ge 1$, then $(t.A)(x) = A(x/t)$ and $(A[t[)(x) = A(x) - A(t-1)$.*
(5) *For any subsets A, B of $\mathbb{N}$, we have $(A \cup B)(x) \le A(x) + B(x)$, with equality holding if $A \cap B = \emptyset$.*

Proof. The proofs are mostly straightforward. We just prove (3). If $A \ll B$ and if $B(n) = k \ge 1$, then $b_k \le n < b_{k+1}$, so that $a_k \le b_k \le n$ and thus $A(n) \ge k = B(n)$; the inequality holds trivially if $B(n) = 0$. Conversely, if $B(n) \le A(n)$ for all $n \in \mathbb{N}$, then for $k \in \mathbb{N}^*$, we have $B(b_k) = k \le A(b_k)$, so that $a_1, \ldots, a_k \in A[b_k]$, i.e. $a_k \le b_k$; hence $A \ll B$. ∎

5.2. Lemma. *If $A \subset \mathbb{N}$ and $x \in \mathbb{R}^+$, then*

$$\sum_{0 \le n \le x} r(A, n) \le A(x)^2 \le \sum_{0 \le n \le 2x} r(A, n).$$

Proof. Let $\sigma : A \times A \to \mathbb{N}$ be the function defined by $\sigma(a, b) = a + b$. Then $r(A, n) = |\sigma^{-1}(n)|$ for all $n \in \mathbb{N}$. Since $\sigma(A[x] \times A[x]) \subset \mathbb{N}[2x]$, we see that $A[x] \times A[x] \subset \bigcup_{0 \le n \le 2x} \sigma^{-1}(n)$. Moreover the sets $\sigma^{-1}(n)$ are pairwise disjoint and, for $0 \le n \le x$, we have $\sigma^{-1}(n) \subset A[x] \times A[x]$, so that $\bigcup_{0 \le n \le x} \sigma^{-1}(n) \subset A[x] \times A[x]$. Therefore

$$\sum_{0 \le n \le x} r(A, n) \le |A[x] \times A[x]| \le \sum_{0 \le n \le 2x} r(A, n),$$

which gives the desired inequalities. ∎

5.3. Corollary. *For any subset A of $\mathbb{N}$ and any $x \in \mathbb{R}^+$, we have*

$$s(A) \ge \frac{A(x)^2}{2x + 1}.$$

Proof. By 5.2, and since $r(A, n) \le s(A)$ for all $n \in \mathbb{N}$, we have $A(x)^2 \le \sum_{0 \le n \le 2x} r(A, n) \le (2x + 1)s(A)$. Hence the inequality. ∎

5.4. Corollary. *For any $A \in \mathcal{P}$, we have*

$$s(A) \ge \sup\left\{\frac{n^2}{2a_n + 1} : n \in \mathbb{N}^*\right\}.$$

Proof. Indeed, if $A = \{a_1 < a_2 < \cdots\}$, then, for any $n \in \mathbb{N}^*$ and any $x \in \mathbb{N}$, we have $A(x) \geq n$ if and only if $a_n \leq x$. In particular, $A(a_n) = n$. Thus, applying 5.3 with $x = a_n$, we get $s(A) \geq n^2/(2a_n + 1)$ for all $n \in \mathbb{N}^*$. Hence the result. ■

5.5. Definition. For any $A \in \mathcal{P}$, the *caliber* of A is the element of $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$ defined by

$$\mathrm{cal}(A) = \liminf_{n\to\infty} \frac{a_n}{n^2}.$$

5.6. Examples. 1. The set $E_r = \{n + [rn(n-1)] : n \in \mathbb{N}^*\}$, defined in 4.11, where $r \in \,]0,1]$, has $\mathrm{cal}(E_r) = r$. Indeed, if we let $x_n = n + rn(n-1)$, then the elements of E_r are $e_n = [x_n]$, so that $x_n - 1 < e_n \leq x_n$ for all $n \in \mathbb{N}^*$; hence

$$\lim_{n\to\infty} \frac{e_n}{n^2} = \lim_{n\to\infty} \frac{x_n}{n^2} = \lim_{n\to\infty} \frac{x_n - 1}{n^2} = r.$$

2. Let $a, b \in \mathbb{N}^*$ be two relatively prime integers, and denote by $\mathbb{P}(a,b)$ the set of all prime numbers p such that $p \equiv b \pmod{a}$. We then have $\mathrm{cal}(\mathbb{P}(a,b)) = 0$. Indeed, according to the prime number theorem in arithmetic progressions [1], the number $\pi(x; a, b)$ of elements $p \leq x$ in $\mathbb{P}(a,b)$ satisfies

$$\pi(x; a, b) \sim \frac{1}{\varphi(a)} \frac{x}{\log x} \quad \text{as } x \to \infty,$$

where φ is Euler's totient function. It then follows, upon replacing x by the nth element p_n of $\mathbb{P}(a,b)$, that

$$\lim_{n\to\infty} \frac{p_n}{\varphi(a) n \log p_n} = 1,$$

which implies that $\lim_{n\to\infty}(\log p_n - \log n - \log\log p_n) = \log \varphi(a)$, and therefore

$$\lim_{n\to\infty} \frac{\log p_n}{\log n} = 1.$$

Hence $p_n \sim \varphi(a) n \log n$ as $n \to \infty$, and the result follows.

In particular, when $a = 1$, we get $\mathrm{cal}(\mathbb{P}) = 0$, where $\mathbb{P}$ is the set of all prime numbers. It follows from 5.9 below that $s(\mathbb{P}) = \infty$. But $r(\mathbb{P}, n) \leq 2$ for odd $n \in \mathbb{N}^*$. Hence the number of representations of the even natural numbers as a sum of two primes is unbounded.

5.7. Lemma. *For any $A \in \mathcal{P}$, we have*

$$\mathrm{cal}(A) = \liminf_{n\to\infty} \frac{a_n}{n^2} = \liminf_{n\to\infty} \frac{n}{A(n)^2}.$$

Proof. The first equality is just the definition of the caliber of A. For an integer $n \geq a_1$, if $k = A(n)$, then $a_k \leq n < a_{k+1}$ and therefore

$$\frac{a_k}{k^2} \leq \frac{n}{A(n)^2} < \frac{a_{k+1}}{k^2}.$$

Since $k = A(n)$ increases to infinity as $n \to \infty$, it follows that

$$\liminf_{k\to\infty} \frac{a_k}{k^2} \leq \liminf_{n\to\infty} \frac{n}{A(n)^2} \leq \liminf_{k\to\infty} \frac{a_{k+1}}{k^2}.$$

Moreover, since

$$\frac{a_{k+1}}{k^2} = \frac{(k+1)^2}{k^2}\frac{a_{k+1}}{(k+1)^2}$$

and the sequence $((k+1)^2/k^2)$ is convergent to 1, we have

$$\liminf_{k\to\infty} \frac{a_{k+1}}{k^2} = \liminf_{k\to\infty} \frac{a_{k+1}}{(k+1)^2} = \liminf_{k\to\infty} \frac{a_k}{k^2}.$$

Therefore all the inferior limits considered here are equal. Hence the second equality. ∎

5.8. Proposition. *Let* $A, B \in \mathcal{P}$ *and* $t \in \mathbb{N}$.

(1) *If* $A \ll B$ *then* $\operatorname{cal}(A) \leq \operatorname{cal}(B)$.
(2) *If* $B \subset A$ *then* $\operatorname{cal}(A) \leq \operatorname{cal}(B)$.
(3) *We have* $\operatorname{cal}(A[t[) = \operatorname{cal}(A)$.
(4) *We have* $\operatorname{cal}(t + A) = \operatorname{cal}(A)$.
(5) *For* $t > 0$, *we have* $\operatorname{cal}(t.A) = t.\operatorname{cal}(A)$.
(6) *If* $A[t[= B[u[$ *for some* $t, u \in \mathbb{N}$, *then* $\operatorname{cal}(A) = \operatorname{cal}(B)$.
(7) *If* A *and* B *are close then* $\operatorname{cal}(A) = \operatorname{cal}(B)$.
(8) *If* $a_n \leq b_n + cn^\nu$ *for some real constants* $c \geq 0$ *and* $0 \leq \nu < 2$ *and for all* $n \in \mathbb{N}^*$, *then* $\operatorname{cal}(A) \leq \operatorname{cal}(B)$.
(9) *If* $|a_n - b_n| \leq cn^\nu$ *for some real constants* $c \geq 0$ *and* $0 \leq \nu < 2$ *and for all* $n \in \mathbb{N}^*$, *then* $\operatorname{cal}(A) = \operatorname{cal}(B)$.

Proof. These properties follow directly from the definitions, using 2.2 in some of them. ∎

5.9. Lemma. *For any* $A \in \mathcal{P}$, *we have*

$$s(A) \geq \frac{1}{2\operatorname{cal}(A)}.$$

Proof. By 5.4, we have

$$s(A) \geq \frac{n^2}{2a_n + 1}$$

for all $n \in \mathbb{N}^*$. Hence

$$s(A) \geq \limsup \frac{n^2}{2a_n + 1}.$$

Moreover,

$$\limsup \frac{n^2}{2a_n+1} = \frac{1}{\liminf((2a_n+1)/n^2)}, \qquad \liminf \frac{2a_n+1}{n^2} = 2\,\mathrm{cal}(A).$$

Hence the result. ∎

5.10. Corollary. *For any $A \in \mathcal{P}$, if $\mathrm{cal}(A) = 0$ then $s(A) = \infty$.*

5.11. Remark. The converse of 5.10 is false. Indeed, for the set $\mathbb{S}$ of the squares in $\mathbb{N}^*$, we have $\mathrm{cal}(\mathbb{S}) = 1$ while $s(\mathbb{S}) = \infty$, in view of 4.15. We may even have $\mathrm{cal}(A) = \infty$ and $s(A) = \infty$, as in the case of the set $A = \{n^3 : n \in \mathbb{N}^*\}$ of the cubes in $\mathbb{N}^*$, in view of a result essentially due to Fermat [6, Ch. 21]. A natural question is whether this $A \in \mathcal{C}(\mathrm{ET})$.

5.12. Definition. An infinite subset A of $\mathbb{N}$ is said to belong to the restricted class $\mathcal{C}(\mathrm{RET})$ of Erdős–Turán if $\mathrm{cal}(A) = 0$.

5.13. Theorem. *Let $A, B \in \mathcal{P}$ and $t \in \mathbb{N}$.*

(1) *If $B \in \mathcal{C}(\mathrm{RET})$ and $A \ll B$, then $A \in \mathcal{C}(\mathrm{RET})$.*
(2) *If $B \in \mathcal{C}(\mathrm{RET})$ and $B \subset A$ then $A \in \mathcal{C}(\mathrm{RET})$.*
(3) *We have $A \in \mathcal{C}(\mathrm{RET})$ if and only if $A[t[\in \mathcal{C}(\mathrm{RET})$.*
(4) *We have $A \in \mathcal{C}(\mathrm{RET})$ if and only if $t + A \in \mathcal{C}(\mathrm{RET})$.*
(5) *For $t > 0$, we have $A \in \mathcal{C}(\mathrm{RET})$ if and only if $t.A \in \mathcal{C}(\mathrm{RET})$.*
(6) *If $A[t[= B[u[$ for some $t, u \in \mathbb{N}$, then $A \in \mathcal{C}(\mathrm{RET})$ if and only if $B \in \mathcal{C}(\mathrm{RET})$.*
(7) *If A and B are close then $A \in \mathcal{C}(\mathrm{RET})$ if and only if $B \in \mathcal{C}(\mathrm{RET})$.*
(8) *If $A \in \mathcal{C}(\mathrm{RET})$, then $s(A) = \infty$.*
(9) *If $a_n = o(n^2)$, then $A \in \mathcal{C}(\mathrm{RET})$.*
(10) *For any $d \in \mathbb{N}^*$ and any $r \in \mathbb{N}$, the arithmetic progression $d.\mathbb{N}+r \in \mathcal{C}(\mathrm{RET})$.*
(11) *If $B \in \mathcal{C}(\mathrm{RET})$ and $a_n \le b_n + cn^\nu$ for some real constants $c \ge 0$ and $0 \le \nu < 2$ and for all $n \in \mathbb{N}^*$, then $A \in \mathcal{C}(\mathrm{RET})$.*
(12) *If $|a_n - b_n| \le cn^\nu$ for some real constants $c \ge 0$ and $0 \le \nu < 2$ and for all $n \in \mathbb{N}^*$, then $A \in \mathcal{C}(\mathrm{RET})$ if and only if $B \in \mathcal{C}(\mathrm{RET})$.*

Proof. These properties follow directly from the ones in 5.8–5.10. ∎

5.14. Lemma. *We have $\mathcal{C}(\mathrm{RET}) \subset \mathcal{C}(\mathrm{ET})$.*

Proof. Indeed, in view of 5.13, if $A \in \mathcal{C}(\mathrm{RET})$ then, for any $B \in \mathcal{P}$ such that $B \ll A$, we have $B \in \mathcal{C}(\mathrm{RET})$ and thus $s(B) = \infty$, so that $A \in \mathcal{C}(\mathrm{ET})$. ∎

5.15. Corollary. *Every subset A of $\mathbb{N}$ which contains an infinite arithmetic progression lies in $\mathcal{C}(\mathrm{ET})$. More generally, for any $A \in \mathcal{P}$, if there is* *a subsequence* $(a_{n_k})_{k\in\mathbb{N}^*}$ *of* (a_n) *such that*

$$\lim_{k\to\infty} a_{n_k}/n_k^2 = 0,$$

then $A \in \mathcal{C}(\mathrm{RET})$ *and thus* $A \in \mathcal{C}(\mathrm{ET})$.

The question as to whether $\mathcal{C}(\mathrm{RET})$ is a proper subset of $\mathcal{C}(\mathrm{ET})$ is answered by the following characterization.

5.16. Theorem. *We have* $\mathcal{C}(\mathrm{RET}) \subsetneq \mathcal{C}(\mathrm{ET})$ *if and only if the conjecture* (GET) *is true.*

Proof. By 4.10, (GET) is equivalent to the condition $\mathbb{S} \in \mathcal{C}(\mathrm{ET})$. Thus, since $\mathrm{cal}(\mathbb{S}) = 1$, if (GET) is true then $\mathbb{S} \in \mathcal{C}(\mathrm{ET}) \setminus \mathcal{C}(\mathrm{RET})$, so that $\mathcal{C}(\mathrm{RET})$ is a proper subset of $\mathcal{C}(\mathrm{ET})$.

Conversely, assume that $\mathcal{C}(\mathrm{RET}) \subsetneq \mathcal{C}(\mathrm{ET})$, i.e. there exists $A \in \mathcal{C}(\mathrm{ET})$ such that $\mathrm{cal}(A) = l > 0$. Let $t \in \mathbb{N}^*$ be such that $tl > 1$ and let $B = t.A$. By 4.8, $B \in \mathcal{C}(\mathrm{ET})$ and, by 5.8, $\mathrm{cal}(B) = tl > 1$, i.e. $\liminf b_n/n^2 > 1$. So there exists $n_0 \in \mathbb{N}^*$ such that $\inf\{b_n/n^2 : n \geq n_0\} > 1$, i.e. for any $n \geq n_0$, we have $n^2 < b_n$. Thus $\mathbb{S}[n_0^2[\ll B[b_{n_0}[$. Since $B \in \mathcal{C}(\mathrm{ET})$, we conclude, by 2.6 and 2.16, that $\mathbb{S} \in \mathcal{C}(\mathrm{ET})$, i.e. (GET) is true. ∎

5.17. Proposition. *If* $B \in \mathcal{C}(\mathrm{ET}) \setminus \mathcal{C}(\mathrm{RET})$ *and* $A \in \mathcal{P}$ *are such that* $a_n \leq b_n + cn^2$ *for some real constant* $c \geq 0$ *and all* $n \in \mathbb{N}^*$, *then* $A \in \mathcal{C}(\mathrm{ET})$.

Proof. Since $B \notin \mathcal{C}(\mathrm{RET})$, we have $\liminf b_n/n^2 = l > 0$. So there exists $n_0 \in \mathbb{N}^*$ such that $\inf\{b_n/n^2 : n \geq n_0\} \geq l/2$, i.e. for $n \geq n_0$, we have $n^2 \leq (2/l)b_n$. Hence $a_n \leq b_n + cn^2 \leq db_n$ for all $n \geq n_0$, where d is any fixed integer $\geq 1 + 2c/l$. Thus $A[a_{n_0}[\ll d.B[b_{n_0}[$. Since $B \in \mathcal{C}(\mathrm{ET})$, we conclude, by 2.6, 2.16 and 4.8, that $A \in \mathcal{C}(\mathrm{ET})$. ∎

5.18. Corollary. *For any* $A, B \in \mathcal{P}$, *if* $B \in \mathcal{C}(\mathrm{ET})$ *and* $a_n \leq b_n + cn^\nu$ *for some real constants* $c \geq 0$ *and* $0 \leq \nu < 2$ *and for all* $n \in \mathbb{N}^*$, *then* $A \in \mathcal{C}(\mathrm{ET})$.

Proof. Assume that $B \in \mathcal{C}(\mathrm{ET})$ and that $a_n \leq b_n + cn^\nu$ for all $n \in \mathbb{N}^*$, with $\nu < 2$. Then either $B \in \mathcal{C}(\mathrm{RET})$ and we conclude by 5.13 that $A \in \mathcal{C}(\mathrm{RET})$, hence $A \in \mathcal{C}(\mathrm{ET})$; or else $B \in \mathcal{C}(\mathrm{ET}) \setminus \mathcal{C}(\mathrm{RET})$ and we conclude by 5.17 (since $a_n \leq b_n + cn^\nu \leq b_n + cn^2$ for all n) that $A \in \mathcal{C}(\mathrm{ET})$. Thus the result holds in any case. ∎

5.19. Corollary. *For any* $A, B \in \mathcal{P}$, *if* $|a_n - b_n| \leq cn^\nu$ *for some real constants* $c \geq 0$ *and* $0 \leq \nu < 2$ *and for all* $n \in \mathbb{N}^*$, *then* $A \in \mathcal{C}(\mathrm{ET})$ *if and only if* $B \in \mathcal{C}(\mathrm{ET})$.

5.20. Notations and remarks. Given $a, b, \nu \in \mathbb{R}^+$ such that $a > 0$ and $0 \leq \nu < 2$, let (x_n) be the sequence in $\mathbb{R}$ defined by $x_n = an^2 - bn^\nu$ for $n \in \mathbb{N}^*$, and let $B(a, b, \nu) = \{[x_n] : n \in \mathbb{N}^*\} \cap \mathbb{N}^*$ be the set consisting of the positive integral parts $[x_n]$ of its terms.

Note that only finitely many of the reals x_n may be negative, since if n_1 is the smallest positive integer $\geq (b/a)^{1/(2-\nu)}$, then $x_n \geq 0$ for $n \geq n_1$. Moreover,

$$x_{n+1} - x_n = n(2a + a/n - bn^{\nu-1}\varepsilon_n),$$

where $\varepsilon_n = (1+1/n)^\nu - 1 \sim \nu/n$ as $n \to \infty$, so that $n^{\nu-1}\varepsilon_n \sim \nu n^{\nu-2}$ has limit 0, because $\nu < 2$. It follows that $\lim_{n\to\infty}(x_{n+1} - x_n) = \infty$. Therefore, for large enough n, the integers $[x_n]$ form a strictly increasing sequence in $\mathbb{N}$. Hence $B(a,b,\nu) = \{b_1 < b_2 < \cdots\}$ lies in $\mathcal{P}$, and there exist $n_0 \in \mathbb{N}^*$ and $t \in \mathbb{N}$ such that $b_n = [x_{n+t}]$ for $n \geq n_0$.

5.21. Proposition. *For any real numbers $a > 0$, $b \geq 0$ and $0 \leq \nu < 2$, the following two statements are equivalent*:

(1) $B(a,b,\nu) \in \mathcal{C}(\mathrm{ET})$.
(2) *If $A \in \mathcal{P}$ is such that $a_n \leq x_n = an^2 - bn^\nu$ for large enough n, then $s(A) = \infty$.*

Proof. Assume that the first statement holds and set $B = B(a,b,\nu)$. Let $A \in \mathcal{P}$ be such that, for large enough n, the integers $a_n \leq x_n$, i.e. $a_n \leq [x_n]$. Then there exists an $N \in \mathbb{N}^*$ such that $a_{n+t} \leq [x_{n+t}] = b_n$ for $n \geq N$, i.e. $A[a_{N+t}[\ll B[b_N[$. Since, by assumption, $B \in \mathcal{C}(\mathrm{ET})$, by 2.16 it follows that $B[b_N[\in \mathcal{C}(\mathrm{ET})$ and therefore $s(A[a_{N+t}[) = \infty$. Since $A[a_{N+t}[\subset A$, also $s(A) = \infty$. This shows that (1) implies (2).

Assume now that the second statement holds. Let $A \in \mathcal{P}$ be such that $A \ll B$. Then $a_n \leq b_n = [x_{n+t}] \leq x_{n+t}$ for $n \geq n_0$. Thus, setting $m = n_0 + t$, for $n \geq m$ we have $a_{n-t} \leq x_n$. If $a_n \leq m + n$ for all $n \in \mathbb{N}^*$, then we get $A \ll m + \mathbb{N}$, and since $m + \mathbb{N} \in \mathcal{C}(\mathrm{ET})$, by 2.6 and 2.14, we conclude that $s(A) = \infty$. So we may assume that there exists some $k \in \mathbb{N}^*$ such that $a_k > m + k$, which implies, by induction, that $a_n > m + n$ for all $n \geq k$. Let $N = k + m$ and define $A' \in \mathcal{P}$ by $a'_n = n - 1$ for $1 \leq n \leq N - 1$, and $a'_n = a_{n-t}$ for $n \geq N$. This is well defined, since

$$a'_{N-1} = N - 2 < m + N - t < a_{N-t} = a'_N,$$

because $N - t \geq k$. Moreover, for $n \geq N \geq m$, we have $a'_n = a_{n-t} \leq x_n$. Therefore, by the assumption, $s(A') = \infty$. But $A' = A[a_{N-t}[\cup F$, where $F = \mathbb{N}[N-2]$ is a finite set, so that, by 2.11, we also have $s(A[a_{N-t}[) = \infty$ and thus $s(A) = \infty$. Since this is valid for any $A \in \mathcal{P}$ such that $A \ll B$, we see that $B \in \mathcal{C}(\mathrm{ET})$. This shows that (2) implies (1). ∎

5.22. Definition. For any real numbers $a > 0$, $b \geq 0$ and $0 \leq \nu < 2$, denote by $(\mathrm{GET}, a, b, \nu)$ either one of the two equivalent statements in 5.21, e.g.

$(\mathrm{GET}, a, b, \nu)$ If $A \in \mathcal{P}$ is such that $a_n \leq an^2 - bn^\nu$ for large enough n, then $s(A) = \infty$.

5.23. Theorem. *The conjecture* (GET) *is equivalent to any statement* $(\mathrm{GET}, a, b, \nu)$, *where* a, b, ν *are real numbers such that* $a > 0$, $b \geq 0$ *and* $0 \leq \nu < 2$.

Proof. Assume that (GET) is true. Let $A \in \mathcal{P}$ be such that $a_n \leq x_n = an^2 - bn^\nu$ for all $n \geq p$ and some $p \in \mathbb{N}^*$. Let c be an integer $\geq a$. Then $a_n \leq cn^2 - bn^\nu \leq cn^2$ for all $n \geq p$. Hence $A[a_p[\ll c.\mathbb{S}[p^2[$, and since $\mathbb{S} \in \mathcal{C}(\mathrm{ET})$ by (GET), by 2.16 and 4.8 we obtain $c.\mathbb{S}[p^2[\in \mathcal{C}(\mathrm{ET})$, so that $s(A[a_p[) = \infty$ and thus $s(A) = \infty$. Hence $(\mathrm{GET}, a, b, \nu)$ holds true.

Assume now that $(\mathrm{GET}, a, b, \nu)$ is true. Let $c \in \mathbb{N}^*$ be such that $ac > 1$ and let $C = c.B$, where $B = B(a, b, \nu)$. By 5.21, $B \in \mathcal{C}(\mathrm{ET})$. So, by 4.8, $C \in \mathcal{C}(\mathrm{ET})$. Moreover, for $n \geq n_0$, we have $c_n = cb_n = c[x_{n+t}] \geq c[x_n] > c(an^2 - bn^\nu - 1)$, so that $c_n - n^2 > (ac-1)n^2 - bcn^\nu - c$. Since $ac - 1 > 0$ and $\nu < 2$, it follows that $\lim_{n\to\infty}((ac-1)n^2 - bcn^\nu - c) = \infty$. Consequently, $n^2 < c_n$ for large enough n. Therefore there exists some $m \in \mathbb{N}^*$ such that $\mathbb{S}[m^2[\ll C[c_m[$. Since C lies in $\mathcal{C}(\mathrm{ET})$, then, by 2.16, so does $C[c_m[$. Thus $\mathbb{S}[m^2[\in \mathcal{C}(\mathrm{ET})$ and therefore $\mathbb{S} \in \mathcal{C}(\mathrm{ET})$, by 2.6. Hence, by 4.10, (GET) holds. ∎

5.24. Definition. For $A \in \mathcal{P}$, let

$$\theta(A) = \frac{1}{\sqrt{\mathrm{cal}(A)}},$$

defined in $\overline{\mathbb{R}}^+$.

Note that, $\mathrm{cal}(A) = 1/\theta(A)^2$, so that $A \in \mathcal{C}(\mathrm{RET})$ if and only if $\theta(A) = \infty$.

5.25. Lemma. *For* $A \in \mathcal{P}$, *we have*

$$\theta(A) = \limsup_{n\to\infty} \frac{n}{\sqrt{a_n}} = \limsup_{n\to\infty} \frac{A(n)}{\sqrt{n}}.$$

Proof. In view of 5.7, we have

$$\theta(A) = \frac{1}{\liminf \sqrt{a_n/n^2}} = \frac{1}{\liminf \sqrt{n/A(n)^2}}.$$

But, for any sequence (x_n) in $\overline{\mathbb{R}}^+$, we have $1/\liminf x_n = \limsup(1/x_n)$. Hence $\theta(A) = \limsup(n/\sqrt{a_n}) = \limsup(A(n)/\sqrt{n})$. ∎

5.26. Proposition. *For* $A, B \in \mathcal{P}$, *we have* $\max(\theta(A), \theta(B)) \leq \theta(A \cup B) \leq \theta(A) + \theta(B)$. *Equivalently,*

$$\frac{\mathrm{cal}(A)\,\mathrm{cal}(B)}{(\sqrt{\mathrm{cal}(A)} + \sqrt{\mathrm{cal}(B)})^2} \leq \mathrm{cal}(A \cup B) \leq \min(\mathrm{cal}(A), \mathrm{cal}(B)).$$

Proof. Let $C = A \cup B$. For any $n \in \mathbb{N}$, we have $C[n] = A[n] \cup B[n]$, so that $C(n) \leq A(n) + B(n)$, and therefore $C(n)/\sqrt{n} \leq A(n)/\sqrt{n} + B(n)/\sqrt{n}$.

Hence, in view of 5.25,

$$\theta(C) = \limsup \frac{C(n)}{\sqrt{n}} \leq \limsup \frac{A(n)}{\sqrt{n}} + \limsup \frac{B(n)}{\sqrt{n}} = \theta(A) + \theta(B).$$

On the other hand, since A and B are contained in C, in view of 5.8, $\mathrm{cal}(C) \leq \min(\mathrm{cal}(A), \mathrm{cal}(B))$, so that

$$\theta(C) = \frac{1}{\sqrt{\mathrm{cal}(C)}} \geq \max\left(\frac{1}{\sqrt{\mathrm{cal}(A)}}, \frac{1}{\sqrt{\mathrm{cal}(B)}}\right) = \max(\theta(A), \theta(B)). \ \blacksquare$$

5.27. Corollary. *For $A, B \in \mathcal{P}$, if $A \cup B$ lies in $\mathcal{C}(\mathrm{RET})$, then A or B lies in $\mathcal{C}(\mathrm{RET})$.*

Proof. Indeed, if neither A nor B is in $\mathcal{C}(\mathrm{RET})$, then $\theta(A) < \infty$ and $\theta(B) < \infty$, hence, by 5.26, $\theta(A \cup B) \leq \theta(A) + \theta(B) < \infty$ and therefore $A \cup B$ is not in $\mathcal{C}(\mathrm{RET})$. $\blacksquare$

5.28. Theorem. *If there exist $C \in \mathcal{C}(\mathrm{ET})$ and $A, B \in \mathcal{P} \setminus \mathcal{C}(\mathrm{ET})$ such that $C = A \cup B$, then the conjecture* (GET) *is true.*

Proof. Since A and B are not in $\mathcal{C}(\mathrm{ET})$, they are not in $\mathcal{C}(\mathrm{RET})$ either, in view of 5.14. Hence, by 5.27, $C = A \cup B$ is not in $\mathcal{C}(\mathrm{RET})$. Thus $C \in \mathcal{C}(\mathrm{ET}) \setminus \mathcal{C}(\mathrm{RET})$. Hence $\mathcal{C}(\mathrm{RET}) \subsetneq \mathcal{C}(\mathrm{ET})$ and, in view of 5.16, (GET) is true. $\blacksquare$

5.29. Remark. An interesting question is whether the converse of 5.28 is true. In particular, if (GET) holds, we would have $\mathbb{S} \in \mathcal{C}(\mathrm{ET}) \setminus \mathcal{C}(\mathrm{RET})$. It is therefore natural to ask whether $\mathbb{S}$ is the union of two infinite subsets A and B, neither of which lies in $\mathcal{C}(\mathrm{ET})$ (e.g. such that $s(A) < \infty$ and $s(B) < \infty$).

More generally, this raises the question of the truth of the following statement:

(S?) If $C = A \cup B$ with $A, B \in \mathcal{P}$, and if $C \in \mathcal{C}(\mathrm{ET})$, then A or B is in $\mathcal{C}(\mathrm{ET})$.

Note that if A and B are arbitrary subsets of $\mathbb{N}$ such that $A \cup B \in \mathcal{C}(\mathrm{ET})$ and if A is finite, then $B \in \mathcal{C}(\mathrm{ET})$. Indeed, since A is finite, we have $B[t[= C[t[$ for some $t \in \mathbb{N}$, so that, by 2.17, $C \in \mathcal{C}(\mathrm{ET})$ implies that $B \in \mathcal{C}(\mathrm{ET})$.

Also note that if (S?) were true, it would imply an easy proof of 2.18(2). Indeed, let $A \in \mathcal{P}$ be such that $A \ll 2.C$. For $r = 0, 1$, let $I_r = \{n \in \mathbb{N}^* : a_n \equiv r \pmod 2\}$, let $A_r = \{a_n : n \in I_r\}$ and $C_r = \{c_n : n \in I_r\}$, and define $B_r = \{[a_n/2] : n \in I_r\}$. Then $A = A_0 \cup A_1 \ll 2.C = 2.C_0 \cup 2.C_1$, with A_r and C_r indexed by the same set I_r, so that $A_r \ll 2.C_r$ (for $r = 0, 1$). We thus have $A_0 = 2.B_0 \ll 2.C_0$ and $A_1 = 1 + 2.B_1 \ll 2.C_1$, so that $B_r \ll C_r$ (for $r = 0, 1$). Now, since $C = C_0 \cup C_1 \in \mathcal{C}(\mathrm{ET})$, and assuming the answer to the above question positive, we should have C_0 or C_1 in $\mathcal{C}(\mathrm{ET})$, which would imply that $s(B_0)$ or $s(B_1)$ is ∞, i.e. $s(A_0) = s(2.B_0) = s(B_0) = \infty$ or $s(A_1) = s(1+2B_1) = s(B_1) = \infty$ (by 2.8 and 2.7), and therefore $s(A) = \infty$. Since this is valid for any $A \ll 2.C$, in $\mathcal{P}$, we conclude that $2.C \in \mathcal{C}(\mathrm{ET})$. ∎

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