ACTA ARITHMETICA
155.2 (2012)

Linear polynomials in numbers of bounded degree

by

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1. Introduction. Given natural numbers n, Δ, a *hypersurface of type* $S(n,\Delta)$ will be a hypersurface in $\mathbb{C}^n$ defined over the rationals, and of total degree at most Δ. Such a surface is the set of zeros of a nonzero polynomial with rational coefficients, and of total degree $\le \Delta$.

Recently Philippon and Schlickewei [1] proved a result about simultaneous approximation by algebraic n-tuples of bounded degree. Their result is as follows.

Theorem A. *Let n, d be natural numbers, and set*

$$c = \frac{n+1}{n}((n+1)!)^{1/n}, \tag{1.1}$$

$$\Delta = \lfloor ((n+1)!d)^{1/n} \rfloor. \tag{1.2}$$

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{C}^n$ have algebraic components, and lie on no hypersurface of type $S(n,\Delta)$. Then given

$$B > cd^{(n+1)/n}, \tag{1.3}$$

there are only finitely many points $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ with

$$[\mathbb{Q}(\beta_1, \ldots, \beta_n); \mathbb{Q}] \le d \tag{1.4}$$

and

$$|\alpha_i - \beta_i| < H(\boldsymbol{\beta})^{-B} \quad (i = 1, \ldots, n), \tag{1.5}$$

where $H(\boldsymbol{\beta})$ is the absolute Weil height of the projective point $(1 : \beta_1 : \ldots : \beta_n)$.

We will recall the definition of this height in Section 2. In the case of simultaneous approximation by rational n-tuples, there is a "dual" result on linear forms. For Theorem A there appears to be no simple duality. We will only be able to prove the following.

2010 *Mathematics Subject Classification*: Primary 11J68.
Key words and phrases: diophantine approximation, bounded degree.

DOI: 10.4064/aa155-2-7

By a *hypersurface of type* $S_h(n,d)$ we will understand a homogeneous hypersurface in $\mathbb{C}^{n+1}$ defined over the rationals, and of degree at most d. Such a hypersurface is the zero set of a nonzero homogeneous polynomial $f(X_0, X_1, \ldots, X_n)$ with rational coefficients, and of total degree at most d.

Theorem B. *Suppose* $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ *has algebraic components, and does not lie on a surface of type* $S_h(n,d)$. *Then given*

$$B > d\binom{d+n}{n} + d, \tag{1.6}$$

there are only finitely many points $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ *with* (1.4) *and*

$$|\alpha_0 + \alpha_1\beta_1 + \cdots + \alpha_n\beta_n| < H(\boldsymbol{\beta})^{-B}. \tag{1.7}$$

Note that the condition (1.6) is independent of the degree of $\mathbb{Q}(\alpha_0, \alpha_1, \ldots, \alpha_n)$. But there is little doubt that it is more restrictive than need be.

Corollary. *Suppose* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *has algebraic components, and if* $n > 1$, *does not lie on a hypersurface of type* $S_h(n-1,d)$. *Then given*

$$B > d\binom{d+n-1}{n-1} + 2d, \tag{1.8}$$

there are only finitely many $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ *with* (1.4) *and*

$$|\alpha_1\beta_1 + \cdots + \alpha_n\beta_n| < H(\boldsymbol{\beta})^{-B}. \tag{1.9}$$

2. Proofs. For a number field K, let $M(K)$ be the set of its places, and $M_\infty(K)$ the set of its archimedean places. For $v \in M(K)$ let $|\cdot|_v$ denote the absolute value induced by v normalized to extend the standard or a p-adic absolute value of $\mathbb{Q}$. Further if $D = \deg K$ and D_v is the local degree associated with v, set $\|\cdot\|_v = |\cdot|_v^{D_v/D}$. When $\boldsymbol{\beta} \in K^n$, then we define

$$H(\boldsymbol{\beta}) = \prod_{v \in M(K)} \|\boldsymbol{\beta}\|_v$$

where

$$\|\boldsymbol{\beta}\|_v = \max(1, \|\beta_1\|_v, \ldots, \|\beta_n\|_v).$$

Suppose $k = \mathbb{Q}(\beta_1, \ldots, \beta_n)$ is a number field of degree d. Let $x \mapsto x^{(i)}$ ($i = 1, \ldots, d$) be the embeddings of k into $\mathbb{C}$. When P is a subset of $\{1, \ldots, d\}$, put

$$x^{(P)} = \prod_{i \in P} x^{(i)}.$$

This is understood to be 1 when P is empty. It will be convenient to set

$\beta_0 = 1$. Given $\alpha_0, \alpha_1, \ldots, \alpha_n$, we have

$$(2.1)\qquad \prod_{i=1}^{d}(\alpha_0\beta_0^{(i)} + \cdots + \alpha_n\beta_n^{(i)}) = \sum_{\substack{j_0,\ldots,j_n\in\mathbb{Z}_{\geq 0}\\ j_0+\cdots+j_n=d}} \alpha_0^{j_0}\cdots\alpha_n^{j_n} q_{j_0\cdots j_n}$$

with

$$(2.2)\qquad q_{j_0\cdots j_n} = \sum\nolimits^* \beta_0^{(P_0)}\cdots\beta_n^{(P_n)},$$

where $\sum^*$ is the sum over all partitions of $\{1,\ldots,d\}$ into (not necessarily nonempty) subsets $P_0,\ldots,P_n$ with $|P_\ell| = j_\ell$ $(\ell = 0,\ldots,n)$. The numbers $q_{j_0\cdots j_n}$ are easily seen to be rational. The point $\mathbf{q}$ with coordinates $q_{j_0\cdots j_n}$ (where $j_0 + \cdots + j_n = d$) lies in $\mathbb{Q}^N$ with $N = \binom{d+n}{n}$.

LEMMA 2.1. *$H(\mathbf{q}) \leq d!H(\boldsymbol{\beta})^d$.*

Proof. Set $K = \mathbb{Q}(\boldsymbol{\beta}^{(1)},\ldots,\boldsymbol{\beta}^{(d)}) = \mathbb{Q}(\beta_0^{(1)},\ldots,\beta_n^{(1)},\ldots,\beta_0^{(d)},\ldots,\beta_n^{(d)})$. For $v \in M(K)$,

$$\|\beta_\ell^{(P_\ell)}\|_v = \prod_{i\in P_\ell}\|\beta_\ell^{(i)}\|_v \leq \prod_{i\in P_\ell}\|\boldsymbol{\beta}^{(i)}\|_v,$$

hence

$$\|\beta_0^{(P_0)}\cdots\beta_n^{(P_n)}\|_v \leq \prod_{i=1}^{d}\|\boldsymbol{\beta}^{(i)}\|_v.$$

The sum $\sum^*$ in (2.2) has $\leq d!$ summands, so that

$$(2.3)\qquad \|q_{j_0\cdots j_n}\|_v \leq c_v^{D_v/D}\prod_{i=1}^{d}\|\boldsymbol{\beta}^{(i)}\|_v,$$

where $c_v = d!$ when $v \in M_\infty(K)$, and $c_v = 1$ otherwise.

The estimate (2.3) also holds for $\|\mathbf{q}\|_v$. We obtain

$$H(\mathbf{q}) = \prod_{v\in M(K)}\|\mathbf{q}\|_v \leq d!\prod_{i=1}^{d}\prod_{v\in M(K)}\|\boldsymbol{\beta}^{(i)}\|_v = d!\prod_{i=1}^{d}H(\boldsymbol{\beta}^{(i)}) = d!H(\boldsymbol{\beta})^d. \ \blacksquare$$

LEMMA 2.2. *Suppose $\alpha = (\alpha_0,\alpha_1,\ldots,\alpha_n)$ has algebraic components and does not lie on a surface of type $S_h(n,d)$. Then given $B > dN + d$, the points $\boldsymbol{\beta}$ with* (1.4) *and* (1.7) *give rise to only finitely many points $\mathbf{q} \in \mathbb{Q}^N$ as described above.*

Proof. We may suppose that $x^{(1)} = x$ for $x \in k$. Then

$$H(\boldsymbol{\beta}) \geq \prod_{u\in M_\infty(\mathbb{K})}\|\boldsymbol{\beta}\|_u = \prod_{i=1}^{d}|\boldsymbol{\beta}^{(i)}|^{1/d} \geq \prod_{i=2}^{d}|\boldsymbol{\beta}^{(i)}|^{1/d}$$

where $|\boldsymbol{\beta}^{(i)}| = \max(1, |\beta_1^{(i)}|, \ldots, |\beta_n^{(i)}|)$. We clearly have

$$|\alpha_0 + \alpha_1\beta_1^{(i)} + \cdots + \alpha_n\beta_n^{(i)}| \le c(\boldsymbol{\alpha})|\boldsymbol{\beta}^{(i)}|,$$

in particular for $i = 2, \ldots, d$. In conjunction with (1.7), this yields

$$\Big|\prod_{i=1}^{d}(\alpha_0 + \alpha_1\beta_1^{(i)} + \cdots + \alpha_n\beta_n^{(i)})\Big| < c(\boldsymbol{\alpha})^{d-1}H(\boldsymbol{\beta})^{-B+d},$$

and therefore, by virtue of (2.1) and Lemma 2.1,

$$\Big|\sum_{j_0+\cdots+j_n=d} \alpha_0^{j_0}\cdots\alpha_n^{j_n} q_{j_0\cdots j_n}\Big| < c(\boldsymbol{\alpha})^{d-1}d^{B-d}H(\mathbf{q})^{-(B-d)/d}. \tag{2.4}$$

Before proceeding further, consider an inequality

$$|\alpha_1 q_1 + \cdots + \alpha_N q_N| < H(q)^{-C} \tag{2.5}$$

where $\mathbf{q} = (q_1, \ldots, q_N) \in \mathbb{Q}^N \setminus \{\mathbf{0}\}$. Say $q_i := a_i/b$ with $\gcd(b, a_1, \ldots, a_N) = 1$, so that $H(\mathbf{q}) = \max(|b|, |a_1|, \ldots, |a_N|)$. Then (2.5) gives

$$|\alpha_1 a_1 + \cdots + \alpha_N a_N| < |b|H(\mathbf{q})^{-C} \le H(\mathbf{q})^{1-C} \le \max(|a_1|, \ldots, |a_N|)^{1-C},$$

provided $C \ge 1$. If $\alpha_1, \ldots, \alpha_N$ are algebraic and linearly independent over $\mathbb{Q}$, it follows from the Subspace Theorem that if $C > N$, then there are only finitely many such $(a_1, \ldots, a_N)$. Given $a_1, \ldots, a_N$, the left hand side of (2.5) becomes $|a/b|$ with $a = \alpha_1 a_1 + \cdots + \alpha_n a_N$, and the right hand side for large $|b|$ becomes $|b|^{-C}$. Therefore $|b|$ is bounded, and (2.5) has only finitely many solutions.

Now $\boldsymbol{\alpha}$ as in Theorem B and Lemma 2.2 has the numbers $\alpha_0^{j_0}\cdots\alpha_n^{j_n}$ with $j_0 + \cdots + j_n = d$ linearly independent over $\mathbb{Q}$. Returning to (2.4), we may conclude that when $B > dN + d$, hence $(B-d)/d > N$, then (2.4) leads to finitely many points $\mathbf{q}$ ([1]). ■

Proof of Theorem B. Let ℓ, t with $1 \le \ell \le n$ and $1 \le t \le d$ be given. Set $j_0 = d - t$, $j_\ell = t$, and $j_m = 0$ for $m \notin \{0, \ell\}$. Then

$$q_{\ell t} := q_{j_0\cdots j_n} = \sum\nolimits^{*} 1^{(P_0)}\beta_\ell^{(P_\ell)}$$

where the sum $\sum^*$ is over the partitions of $\{1, \ldots, d\}$ into sets P_0, P_ℓ with $|P_0| = d - t$, $|P_\ell| = t$. Therefore

$$q_{\ell t} = \sum \beta_\ell^{(u_1)}\cdots\beta_\ell^{(u_t)} = s_t(\beta_\ell^{(1)}, \ldots, \beta_\ell^{(d)}),$$

with the sum over the subsets $\{u_1, \ldots, u_t\}$ of $\{1, \ldots, d\}$, and s_t the tth elementary symmetric polynomial. Therefore the symmetric polynomials in

([1]) The components of $\mathbf{q}$ satisfy certain polynomial equations independent of $\boldsymbol{\beta}$. Therefore presumably a better result than the one given by the Subspace Theorem should apply.

$\beta_\ell^{(1)}, \ldots, \beta_\ell^{(d)}$ are determined by $\mathbf{q}$. For given $\mathbf{q}$, there are at most d possibilities for β_ℓ ($\ell = 1, \ldots, n$), hence at most d^n possibilities for $\boldsymbol{\beta}$. Theorem B now is a consequence of Lemma 2.2. ∎

Proof of the Corollary. We may suppose that $\beta_1 \neq 0$. Assume first that $n = 1$. Since $H(\beta_1) = H(1/\beta_1) \geq 1/|\beta_1|^{1/d}$, (1.9) gives $H(\beta_1)^{-B} \geq |\alpha_1 \beta_1| \geq |\alpha_1| H(\beta_1)^{-d}$. Therefore $H(\beta_1)$ is bounded, and there are only finitely many choices for β_1.

When $n > 1$, write $\beta_\ell = \beta_1 \gamma_\ell$ ($\ell = 2, \ldots, n$). Since $H(\boldsymbol{\beta}) \geq H(\beta_1) \geq 1/|\beta_1|^{1/d}$, (1.9) yields

$$(2.6) \qquad |\alpha_1 + \alpha_2 \gamma_2 + \cdots + \alpha_n \gamma_n| \leq |\beta_1|^{-1} H(\boldsymbol{\beta})^{-B} \leq H(\boldsymbol{\beta})^{d-B}.$$

By (1.8), and by the case $n-1$ of Theorem B, there are only finitely many $\gamma_2, \ldots, \gamma_n$ with (2.6). Given $\gamma_2, \ldots, \gamma_n$, set $\gamma = \alpha_1 + \alpha_2 \gamma_2 + \cdots + \alpha_n \gamma_n$, so that (1.9) becomes $|\gamma \beta_1| < H(\beta)^{-B}$. Here $\gamma \neq 0$, for otherwise we have $\prod_{i=1}^{d} (\alpha_1 + \alpha_2 \gamma_2^{(i)} + \cdots + \alpha_n \gamma_n^{(i)}) = 0$, and $(\alpha_1, \ldots, \alpha_n)$ lies on a hypersurface of type $S_h(n-1, d)$. By the case $n = 1$, with γ in place of α_1, we obtain only finitely many choices for β_1. The Corollary follows. ∎

References

[1] P. Philippon and H. P. Schlickewei, *Simultaneous approximation to algebraic numbers by algebraic numbers of bounded degree*, to appear.

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Received on 28.7.2011
and in revised form on 20.9.2011 (6776)