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On the error term of the mean square formula for the Riemann zeta-function in the critical strip $3/4 < \sigma < 1$

by

Yuk-Kam Lau (Nancy and Hong Kong)

1. Introduction. For $1/2 < \sigma < 1$, define

$$E_\sigma(t) = \int_0^t |\zeta(\sigma + iu)|^2 \, du - \left(\zeta(2\sigma)t + (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}t^{2-2\sigma}\right)$$

where $\zeta(s)$ is the Riemann zeta-function. This is an analogue of $E(t)$ for the case $\sigma = 1/2$, which was extensively studied. Comparatively, $E_\sigma(t)$ is new and the following mean square formula was obtained by Matsumoto and Meurman [8] and [9] within this decade:

$$(1.1)\qquad \int_1^T E_\sigma(t)^2 \, dt = \begin{cases} A_1(\sigma)T^{5/2-2\sigma} + O(T) & \text{if } 1/2 < \sigma < 3/4, \\ A_0 T\log T + O(T) & \text{if } \sigma = 3/4, \\ O(T) & \text{if } 3/4 < \sigma < 1. \end{cases}$$

The O-term in the case $\sigma = 3/4$ in [8] was given with a slightly weaker estimate $O(T\sqrt{\log T})$ and Lam [5] improved it to $O(T)$. Then it is natural to wonder whether the O-terms are sharp. This question is meaningful, especially in the last case, because it provides the information how large $E_\sigma(t)$ $(3/4 < \sigma < 1)$ can be. In fact, if we denote the error term $O(T)$ by $F_\sigma(T)$, it is conjectured that $F_\sigma(T) = A(\sigma)T + o(T)$ (see [4] and [7] for more details).

Concerning the case $3/4 < \sigma < 1$, in addition to the mean square estimate in (1.1), Ivić and Matsumoto [2] proved that

$$E_\sigma(t) \ll t^{4\kappa(1-\sigma)/(1+4\kappa(1-\sigma))}\log T$$

where (κ, λ) is an exponent pair such that $\lambda = \kappa + 1/2$. It is the best upper bound to date. Furthermore, we found in [6] that

$$\int_1^T E_\sigma(t)\, dt = -2\pi\zeta(2\sigma-1)T + O(\sqrt{T}).$$

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This immediately implies that $F_\sigma(T) = \int_1^T E_\sigma(t)^2\,dt \gg T$ and partly answers the above-mentioned question (on whether the O-term in (1.1) is sharp). However, at the same time, it suggests that $E_\sigma(t)$ may consist of two parts: one is the constant $-2\pi\zeta(2\sigma-1)$ and the other has a small mean value $O(\sqrt{T})$. This was also pointed out in Matsumoto [7, p. 256]. Thus we carry out this splitting, or in other words, we consider

$$E^*_\sigma(t) = \int_0^t |\zeta(\sigma+iu)|^2\,du - \left(\zeta(2\sigma)t + (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}t^{2-2\sigma} - 2\pi\zeta(2\sigma-1)\right).$$

Then we have $E_\sigma(t) = -2\pi\zeta(2\sigma-1) + E^*_\sigma(t)$, $\int_1^T E^*_\sigma(t)\,dt \ll \sqrt{T}$ and $\int_1^T E^*_\sigma(t)^2\,dt \ll T$ by (1.1).

Our purpose here is to study the size of the last integral $\int_1^T E^*_\sigma(t)^2\,dt$. It is interesting because if the integral is $o(T)$, then the conjecture (for the last case) is settled; otherwise, in view of its mean value, this shows that $E^*_\sigma(t)$ is highly oscillating and so is $E_\sigma(t)$. We find that the latter case occurs, as is anticipated (see the remark below).

THEOREM. *Let $3/4 < \sigma < 1$ and T_0 be a sufficiently large constant. Then, for all $T \ge T_0$, we have $\int_1^T E^*_\sigma(t)^2\,dt \gg_\sigma T$ where the implied constant depends on σ only.*

REMARK. Let $\sigma_a(n) = \sum_{d|n} d^a$, and for $-1 < a < 0$ define

$$\Delta_a(x) = \sum_{n\le x}\sigma_a(n) - \zeta(1-a)x - \frac{\zeta(1+a)}{1+a}x^{1+a} + \frac{1}{2}\zeta(-a).$$

This is a generalization of the classical error term $\Delta(x)$ in Dirichlet's divisor problem. It is well known that there is an analogy between the error terms $\Delta(x)$ and $E(t)$. Such an analogy also appears between $\Delta_{1-2\sigma}(x)$ and $E_\sigma(t)$ (or more appropriately, $E^*_\sigma(t)$). (This was also discussed in [4] and [7].) Indeed, there is a mean square result for $\Delta_{1-2\sigma}(x)$, parallel to (1.1), obtained by Meurman [10] with a truncated Voronoi-type formula and delicate analysis. On the other hand, starting with another (finite) series representation, an asymptotic formula for $\int_1^T \Delta_{1-2\sigma}(x)^2\,dx$ with $3/4 < \sigma < 1$ can be derived. This was worked out in [1] and [12] and, therefore, a similar result for $E^*_\sigma(t)$ is expected. The series representation used in [1] and [12], however, highly depends on the arithmetic nature of the divisor function and there is no counterpart of that for $E_\sigma(t)$ yet. (Our result is perhaps not insignificant in view of this difficulty.)

Finally, let us outline our approach. Define $G_\sigma(t) = \int_1^t E^*_\sigma(u)\,du$. Then $G_\sigma(t+h) - G_\sigma(t) = \int_t^{t+h} E^*_\sigma(u)\,du$ can be regarded as an approximation of $hE^*_\sigma(t)$. Thus, it leads to evaluate the mean square $\int_T^{2T}(G_\sigma(t+h) - G_\sigma(t))^2\,dt$

which can be treated by Jutila's argument in [3] and techniques in [9]. However, in order to make small h admissible, we need a series representation of $G_\sigma(t)$ with a sufficiently small error term. To this end, we apply the argument in [8] to derive an averaged form of the series representation. However, we can only obtain a formula for $G_\sigma(2t)-G_\sigma(t)$ instead of $G_\sigma(t)$. This needs a little extra effort to handle (see Section 4).

In what follows, $3/4<\sigma<1$ and c_i $(i=1,2,\dots)$ denote unspecified positive constants which depend on σ only.

2. Averaged formula. We start with the formula

$$\int_T^{2T} E_\sigma^*(t)\,dt = \Sigma_1(t,X)|_T^{2T} + \pi^{-1/2}t^{5/2}I(t)|_T^{2T} + O(1)$$

where $X\in[AT,T]$ is not an integer with a constant $0<A<1$, and

$$I(t)=\int_X^\infty \frac{\widetilde{\Delta}_{1-2\sigma}(\xi)\cos(tV+2\pi\xi U-\pi\xi+\pi/4)}{\xi^3V^2U^{1/2}(U-1/2)^\sigma(U+1/2)^{\sigma+2}}\,d\xi.$$

The function $\Sigma_1(t,X)$ is defined at the beginning of [6, Section 2] while $\widetilde{\Delta}_{1-2\sigma}(\xi)$ is defined in [6, Lemma 3.2]. U and V are functions defined as in [6, Lemma 3.1] with k replaced by ξ.

This formula comes from [6, (3.3)–(3.5), (3.8) and (3.14)], and a simple refinement of [6, (3.9)]

$$\int_T^{2T} G_4^*(t)\,dt \ll T^{r-1/2}\log T \ll_\sigma 1.$$

The last upper bound relies on the fact that $r<1/2$ when $3/4<\sigma<1$. (Note that $r=-(4\sigma^2-7\sigma+2)/(4\sigma-1)$, see [6, Lemma 3.2].)

To deal with $I(t)$, it can be observed that $I(t)$ is essentially the same as the integral J in [8, p. 368] except that the exponent of V is replaced by 2 and sin by cos in our case. We proceed with the argument in [8, Section 6] by replacing $\widetilde{\Delta}_{1-2\sigma}(\xi)$ with its Voronoi-type series. The components corresponding to $J_2(n,b)$ and $O(T^{-\sigma-7/4})$ in [8, (6.1)] are treated in the same way. Then we carry out the smoothing process described in [8, Section 7], with $X=(L+\mu)^2$ $(0\le\mu\le M)$ and $L=M=\sqrt{T}/2$. It is easy to handle $\Sigma_1(t,X)$ but the remaining part in $I(t)$ needs a more complicated treatment, following the same lines of argument of the evaluation of K_n in [8, p. 372]. Finally, we can obtain

$$\int_T^{2T} E_\sigma^*(t)\,dt = \Sigma_1^*(t)|_T^{2T} - \Sigma_2^*(t)|_T^{2T} + O(1). \tag{2.1}$$

Here, corresponding to $\Sigma^*_{1,\sigma}(t,L,M)$ in [8], we have

$$\Sigma_1^*(t)=\sqrt{2}\left(\frac{t}{2\pi}\right)^{5/4-\sigma}\sum_{n\le T}(-1)^n w_1(n)\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}e_2(t,n)\sin f(t,n)$$

where $w_1(n)=1$ if $n\le T/4$ and $2(1-\sqrt{n/T})$ if $T/4<n\le T$,

$$e_2(t,n)=\left(1+\frac{\pi n}{2t}\right)^{-1/4}\left(\sqrt{\frac{2t}{\pi n}}\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2},$$
$$f(t,n)=2t\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}+(2\pi nt+\pi^2n^2)^{1/2}-\frac{\pi}{4}$$

with $\sigma_a(n)=\sum_{d|n}d^a$ and $\operatorname{arsinh}x=\log(x+\sqrt{x^2+1})$. Moreover, following the proof of [8, (8.2)] (i.e. [8, p. 375]), $\Sigma_2^*(t)$ analogous to $\Sigma^*_{2,\sigma}(t,L,M)$ satisfies

$$\int_T^{2T}\Sigma_2^*(t)^2\,dt\ll T \tag{2.2}$$

because $\Sigma_2^*(t)$ only differs from $\Sigma^*_{2,\sigma}(t,L,M)$ in the exponent of $\log(t/(2\pi n))$ (the exponent is -2 in $\Sigma_2^*(t)$) and the trigonometric function $\cos g(t,n)$ ($\sin g(t,n)$ in $\Sigma_2^*(t)$).

Define

$$S(t)=\Sigma_1^*(t)-\Sigma_2^*(t)\quad\text{and}\quad G_\sigma(t)=\int_1^t E_\sigma^*(u)\,du.$$

Then from (2.1) we get

$$G_\sigma(2T)-G_\sigma(T)=\int_T^{2T}E_\sigma^*(t)\,dt=S(2T)-S(T)+\mathcal{E} \tag{2.3}$$

where $|\mathcal{E}|\le c_1$. Our next section is to study the mean square of the difference $S(t+h)-S(t)$.

3. Mean square of $S(t+h)-S(t)$. We prove the following.

LEMMA 3.1. *Let B and B' be sufficiently large but fixed constants. Then*

$$K_1Th^{5-4\sigma}\le\int_T^{2T}(S(t+h)-S(t))^2\,dt\le K_2Th^{5-4\sigma}$$

uniformly for $B\le h\le\sqrt{T}/B'$, where K_1 and K_2 are positive constants depending on σ only.

Proof. Using $(t+h)^{5/4-\sigma} - t^{5/4-\sigma} \ll t^{1/4-\sigma}h$ and

$$e_2(t+h,n) - e_2(t,n) = h\,\frac{\partial}{\partial t}\bigg|_{t=\xi} e_2(t,n) \ll nht^{-2}$$

with $\xi \in [t, t+h]$, we have, for $t \in [T, 2T]$,

$$\Sigma_1^*(t+h) - \Sigma_1^*(t) = \sqrt{2}\left(\frac{t}{2\pi}\right)^{5/4-\sigma} \operatorname{Im}\sum_{n\le T} a_n k_n(t) e^{if(t,n)} + O(T^{-1/2}h)$$

where

$$a_n = (-1)^n w_1(n)\sigma_{1-2\sigma}(n) n^{\sigma-7/4},$$
$$k_n(t) = e_2(t,n)(\exp(i(f(t+h,n) - f(t,n))) - 1).$$

This yields

$$S(t+h) - S(t) = \sqrt{2}\left(\frac{t}{2\pi}\right)^{5/4-\sigma} \operatorname{Im}\sum_{n\le T} a_n k_n(t) e^{if(t,n)} + F(t,h) \tag{3.1}$$

where $F(t,h) = \Sigma_2^*(t+h) - \Sigma_2^*(t) + O(1)$. In view of (2.2), it suffices to evaluate

$$J = 2\int_T^{2T} \left(\frac{t}{2\pi}\right)^{5/2-2\sigma} \Big(\operatorname{Im}\sum_{n\le T} a_n k_n(t) e^{if(t,n)}\Big)^2\,dt.$$

Plainly,

$$\begin{aligned} J &= \int_T^{2T} \left(\frac{t}{2\pi}\right)^{5/2-2\sigma} \Big|\sum_{n\le T} a_n k_n(t) e^{if(t,n)}\Big|^2\,dt \\ &\quad - \operatorname{Re}\int_T^{2T} \left(\frac{t}{2\pi}\right)^{5/2-2\sigma} \Big(\sum_{n\le T} a_n k_n(t) e^{if(t,n)}\Big)^2\,dt \\ &= (2\pi)^{2\sigma-5/2}\sum_{n\le T} a_n^2 \int_T^{2T} t^{5/2-2\sigma}|k_n(t)|^2\,dt + O(T^{5/2-2\sigma}(|J_1| + |J_2|)) \end{aligned} \tag{3.2}$$

by the mean value theorem for integrals, where for some $T_i \in [T, 2T]$ $(i = 1,2,3,4)$,

$$J_1 = \int_{T_1}^{T_2} \sum_{m\ne n\le T} a_m a_n k_m(t)\overline{k_n(t)} e^{i(f(t,m)-f(t,n))}\,dt,$$
$$J_2 = \int_{T_3}^{T_4} \sum_{m,n\le T} a_m a_n k_m(t) k_n(t) e^{i(f(t,m)+f(t,n))}\,dt.$$

To bound J_1, we write $k(t) = k_m(t)\overline{k_n(t)}$ and $\phi(t) = f(t,m) - f(t,n)$. Then

$$J_1 = \sum_{m\ne n\le T} a_m a_n \bigg\{ k(t)\frac{e^{i\phi(t)}}{\phi'(t)}\bigg|_{T_1}^{T_2} - \int_{T_1}^{T_2}\left(\frac{k'(t)}{\phi'(t)} - k(t)\frac{\phi''(t)}{\phi'(t)^2}\right) e^{i\phi(t)}\,dt\bigg\}.$$

Noting that

$$f(t+h,n)-f(t,n)=2h\,\operatorname{arsinh}\sqrt{\frac{\pi n}{2\xi}}\quad\text{where } \xi\in[t,t+h], \tag{3.3}$$

we have $k_n(t)\ll\min(1,h\sqrt{n/t})$ as $\operatorname{arsinh}x\ll x$ for all $x\ge 0$. Applying Hilbert's inequality (see [11]) and $\min_{m\ne n}|\phi'(t)|\gg\sqrt{nt}$, we get

$$\begin{aligned}\sum_{m\ne n\le T}a_ma_nk(t)\frac{e^{i\phi(t)}}{\phi'(t)}\bigg|_{T_1}^{T_2}&\ll\sum_{n\le T}a_n^2|k_n(t)|^2\sqrt{nT}\\&\ll T^{1/2}\sum_{n\le T}\sigma_{1-2\sigma}(n)^2n^{2\sigma-3}\min(1,h\sqrt{n/T})^2\\&\ll T^{2\sigma-3/2}h^{4-4\sigma}.\end{aligned}$$

Using the fact that $k_n'(t)\ll\sqrt{n}\,ht^{-3/2}$, we have

$$\begin{aligned}&\int_{T_1}^{T_2}\sum_{m\ne n\le T}a_ma_nk'(t)\frac{e^{i\phi(t)}}{\phi'(t)}\,dt\\&\qquad\ll T\Big(\sum_{n\le T}a_n^2|k_n'(t)|^2\sqrt{nT}\Big)^{1/2}\Big(\sum_{n\le T}a_n^2|k_n(t)|^2\sqrt{nT}\Big)^{1/2}\\&\qquad\ll T^{2\sigma-3/2}h^{3-2\sigma}.\end{aligned}$$

Finally, from [9, p. 379], we have

$$\frac{\phi''(t)}{\phi'(t)^2}=-\frac{1}{2t\phi'(t)}\bigg(\cosh\Big(\frac12 f'(t,m)\Big)\bigg)^{-2}+O(t^{-1}).$$

Then we see that as $\cosh x\ge 1$,

$$\begin{aligned}&\int_{T_1}^{T_2}\sum_{m\ne n\le T}a_ma_nk(t)\frac{\phi''(t)}{\phi'(t)^2}e^{i\phi(t)}\,dt\\&\qquad\ll\sum_{n\le T}a_n^2|k_n(t)|^2\sqrt{nT}+\Big(\sum_{n\le T}|a_nk_n(t)|\Big)^2\ll T^{2\sigma-3/2}h^{4-4\sigma}.\end{aligned}$$

To sum up, $J_1\ll T^{2\sigma-3/2}h^{3-2\sigma}$. The estimation of J_2 is easier. Taking $k(t)=k_m(t)k_n(t)$ and $\phi(t)=f'(t,m)+f'(t,n)$, we then have $k(t)\ll\min(1,h\sqrt{n/T})$, $k'(t)\ll\sqrt{n}\,ht^{-3/2}$, $\phi'(t)\gg(\sqrt{m}+\sqrt{n})/\sqrt{T}$ and $\phi''(t)\ll(\sqrt{m}+\sqrt{n})T^{-3/2}$. The same treatment yields $J_2\ll T^{2\sigma-3/2}h$ and so (3.2) becomes

$$J=(2\pi)^{2\sigma-5/2}\sum_{n\le T}a_n^2\int_T^{2T}t^{5/2-2\sigma}|k_n(t)|^2\,dt+O(Th^{3-2\sigma}).$$

From this, we can deduce that

$$J \le c_2 T^{7/2-2\sigma} \sum_{n\le T} \sigma_{1-2\sigma}(n)^2 n^{2\sigma-7/2} \min(1, h\sqrt{n/T})^2 + c_3 T h^{3-2\sigma} \le c_4 T h^{5-4\sigma}. \tag{3.4}$$

Since $x/2 \le \operatorname{arsinh} x \le 2x$ when $0 \le x \le 1$, we see from (3.3) that $c_5 h\sqrt{n/T} \le f(t+h,n) - f(t,n) \le \pi/2$ when $n \le \delta T/h^2$ for some small constant $\delta > 0$. Therefore, provided $\delta B' \ge c_6$ (> 1), we get

$$J \ge c_7 T^{5/2-2\sigma} h^2 \sum_{n\le \delta T/h^2} \sigma_{1-2\sigma}(n)^2 n^{2\sigma-5/2} - c_8 T h^{3-2\sigma} \ge \left(c_9 - \frac{c_8}{B^{2-2\sigma}}\right) T h^{5-4\sigma} \ge c_{10} T h^{5-4\sigma} \tag{3.5}$$

provided $B > c_{11}$. From (2.2), we see that $\int_T^{2T} F(t,h)^2\,dt \ll T$, and thus from (3.1) and (3.4),

$$\int_T^{2T} (S(t+h) - S(t))^2\,dt = J + O(T + \sqrt{JT}) = J + O(Th^{5/2-2\sigma}).$$

This completes the proof of Lemma 3.1, with (3.4), (3.5) and a sufficiently large B.

4. Proof of Theorem. We choose an integer R such that $2^{R(5-4\sigma)} \ge 4K_2/K_1$ where K_1 and K_2 are defined as in Lemma 3.1. Let

$$h = \max(B, (12(c_1R)^2/K_2)^{1/(5-4\sigma)})$$

and T be any sufficiently large number (in particular, $T \ge 2^R(B'h)^2$). From (2.3), we have

$$\begin{aligned}
&(G_\sigma(2^r(t+h)) - G_\sigma(2^r t)) - (G_\sigma(2^{r-1}(t+h)) - G_\sigma(2^{r-1}t)) \\
&\quad = (G_\sigma(2^r(t+h)) - G_\sigma(2^{r-1}(t+h))) - (G_\sigma(2^r t) - G_\sigma(2^{r-1}t)) \\
&\quad = (S(2^r(t+h)) - S(2^{r-1}(t+h))) - (S(2^r t) - S(2^{r-1}t)) + \mathcal{E}_r' \\
&\quad = (S(2^r(t+h)) - S(2^r t)) - (S(2^{r-1}(t+h)) - S(2^{r-1}t)) + \mathcal{E}_r'
\end{aligned}$$

where $|\mathcal{E}_r'| \le 2c_1$. Summing over $r = 1, \ldots, R$, yields

$$\begin{aligned}
&(G_\sigma(2^R(t+h)) - G_\sigma(2^R t)) - (G_\sigma(t+h) - G_\sigma(t)) \\
&\qquad = (S(2^R(t+h)) - S(2^R t)) - (S(t+h) - S(t)) + \sum_{r=1}^{R} \mathcal{E}_r'.
\end{aligned}$$

Now, we square both sides and integrate over $[T, 2T]$. Using the inequality $2(a^2+b^2) \ge (a-b)^2 \ge a^2/2 - b^2$, we infer that

$$\begin{aligned}
&\int_T^{2T} (G_\sigma(2^R(t+h)) - G_\sigma(2^R t))^2\,dt + \int_T^{2T} (G_\sigma(t+h) - G_\sigma(t))^2\,dt \\
&\quad \ge 4^{-1} \int_T^{2T} ((S(2^R(t+h)) - S(2^R t)) - (S(t+h) - S(t)))^2\,dt - \frac{T}{2}\Big|\sum_{r=1}^{R} \mathcal{E}'_r\Big|^2 \\
&\quad \ge 4^{-1}\bigg(\frac{1}{2^{R+1}} \int_{2^R T}^{2^{R+1}T} (S(t+2^R h) - S(t))^2\,dt - \int_T^{2T} (S(t+h) - S(t))^2\,dt\bigg) \\
&\qquad - 2(c_1 R)^2 T \\
&\qquad \ge \frac{K_2}{4} T h^{5-4\sigma} - 2(c_1 R)^2 T \ge (c_1 R)^2 T
\end{aligned}$$

by Lemma 3.1 with our choices of R and h. Using the Cauchy–Schwarz inequality, we see that

$$\int_T^{2T} (G_\sigma(t+h) - G_\sigma(t))^2\,dt = \int_T^{2T} \Big| \int_t^{t+h} E^*_\sigma(u)\,du \Big|^2\,dt \le h^2 \int_T^{2T+h} E^*_\sigma(u)^2\,du.$$

As R and h are fixed constants, we conclude that $\int_T^{2^{R+2}T} E^*_\sigma(u)^2\,du \gg T$ and hence the result.

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Institut Élie Cartan
Université Henri Poincaré (Nancy 1)
54506 Vandœuvre-lès-Nancy Cedex, France

Current address:
Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong
E-mail: yklau@maths.hku.hk

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