ACTA ARITHMETICA
160.1 (2013)

Inhomogeneous Diophantine approximation with general error functions

by

LINGMIN LIAO (Créteil) and MICHAŁ RAMS (Warszawa)

1. Introduction. Let α be an irrational real number. Denote by $\|\cdot\|$ the distance to the nearest integer. A famous 1907 result of Minkowski [Min57] showed that if $y \notin \mathbb{Z} + \alpha\mathbb{Z}$, then for infinitely many $n \in \mathbb{Z}$, we have

$$\|n\alpha - y\| < \frac{1}{4|n|}.$$

If n is restricted to positive integers only, Khintchine [Khi26] proved in 1926 that for any real number y, there exist infinitely many $n \in \mathbb{N}$ satisfying the Diophantine inequalities

$$\|n\alpha - y\| < \frac{1}{\sqrt{5}\, n}. \tag{1.1}$$

We shall always restrict n to positive integers. Khintchine's result is equivalent to saying that the set

$$E(\alpha, c) := \{y \in \mathbb{R} : \|n\alpha - y\| < c/n \text{ for infinitely many } n\}$$

is the whole space $\mathbb{R}$ when the constant c equals $1/\sqrt{5}$. It was shown by Cassels [Cas50] in 1950 that the set $E(\alpha, c)$ is of full measure for any constant $c > 0$.

The generalization of this question to more general error functions was first considered by Bernik and Dodson [BD99] in 1999. Define

$$\omega(\alpha) := \sup\{\theta \geq 1 : \liminf_{n\to\infty} n^{\theta}\|n\alpha\| = 0\}.$$

(Observe that α is a Liouville number if $\omega(\alpha) = \infty$.) Bernik and Dodson proved that the Hausdorff dimension, denoted by $\dim_{\mathrm{H}}$, of the set

$$E_\gamma(\alpha) := \{y \in \mathbb{R} : \|n\alpha - y\| < 1/n^\gamma \text{ for infinitely many } n\} \quad (\gamma \geq 1),$$

2010 *Mathematics Subject Classification*: Primary 28A80; Secondary 37E05, 28A78.
Key words and phrases: inhomogeneous Diophantine approximation, Hausdorff dimension.

satisfies

$$\frac{1}{\omega(\alpha)\cdot\gamma} \le \dim_{\mathrm{H}} E_\gamma(\alpha) \le \frac{1}{\gamma}.$$

In 2003, Bugeaud [Bug03], and independently Schmeling and Troubetzkoy [TS03], improved the above result. They showed that for any irrational α,

$$\dim_{\mathrm{H}} E_\gamma(\alpha) = 1/\gamma.$$

Now let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a function decreasing to zero. Consider the set

$$E_\varphi(\alpha) := \{y \in \mathbb{R} : \|n\alpha - y\| < \varphi(n) \text{ for infinitely many } n\}.$$

This is the set of well-approximated numbers with a general error function φ. It easily follows from the Borel–Cantelli lemma that the Lebesgue measure of $E_\varphi(\alpha)$ is zero whenever the series $\sum_{n=1}^{\infty} \varphi(n)$ converges. But on the other hand, it seems hard to obtain a lower bound of the Lebesgue measure of $E_\varphi(\alpha) \cap [0,1]$ when the series $\sum_{n=1}^{\infty} \varphi(n)$ diverges. For results on this measure, we refer the readers to [Kur55], [LN12], [Kim12], and the references therein.

In this paper, we are concerned with the Hausdorff dimension of the set $E_\varphi(\alpha)$. As in [Dod92] and [Dic94], for an increasing function $\psi : \mathbb{N} \to \mathbb{R}^+$, we define the lower and upper orders at infinity by

$$\lambda(\psi) := \liminf_{n\to\infty} \frac{\log \psi(n)}{\log n} \quad \text{and} \quad \kappa(\psi) := \limsup_{n\to\infty} \frac{\log \psi(n)}{\log n}.$$

For simplicity, let us denote

$$u_\varphi := \frac{1}{\lambda(1/\varphi)} \quad \text{and} \quad l_\varphi := \frac{1}{\kappa(1/\varphi)}.$$

The results of Bugeaud, Schmeling and Troubetzkoy imply the inequality

$$l_\varphi \le \dim_{\mathrm{H}} E_\varphi(\alpha) \le u_\varphi.$$

The upper bound was considered a good candidate to be the precise formula for the dimension of $E_\varphi(\alpha)$. It would not be sharp without the monotonicity of φ: for any irrational α one can easily find a function φ with $u_\varphi = 1$ but with $E_\varphi(\alpha) = \emptyset$. But when φ is nonincreasing, we actually have

$$\dim_{\mathrm{H}} E_\varphi(\alpha) = u_\varphi$$

for all α of bounded type (see [FW06]). This result was further strengthened by Xu [Xu10] (see below).

However, in [FW06], Fan and Wu constructed an example which shows that the equality is not always true. In fact, they found a Liouville number α and constructed an error function φ such that

$$\dim_{\mathrm{H}} E_\varphi(\alpha) = l_\varphi < u_\varphi.$$

In the general case, the dimension formula seems a mystery.

Recently, Xu [Xu10] made a progress by proving the following theorem.

Theorem 1.1 (Xu). *For any α,*

$$\limsup_{n\to\infty} \frac{\log q_n}{-\log \varphi(q_n)} \le \dim_{\mathrm{H}} E_\varphi(\alpha) \le u_\varphi,$$

where q_n denotes the denominator of the nth convergent of the continued fraction of α.

As a corollary, Xu proved that

$$\dim_{\mathrm{H}} E_\varphi(\alpha) = u_\varphi$$

for any irrational number α with $\omega(\alpha) = 1$.

In this paper, we prove the following results.

Theorem 1.2. *For any α with $\omega(\alpha) = w \in [1, \infty]$, we have*

$$\min\left\{u_\varphi, \max\left\{l_\varphi, \frac{1+u_\varphi}{1+w}\right\}\right\} \le \dim_{\mathrm{H}} E_\varphi(\alpha) \le u_\varphi.$$

Corollary 1.3. *If $w \le 1/u_\varphi$, then*

$$\dim_{\mathrm{H}} E_\varphi(\alpha) = u_\varphi.$$

Example 1.4. Take $w = 2$, $u = 1/2$ and $l = 1/3$. We can construct an irrational α such that $q_n^2 \le q_{n+1} \le 2q_n^2$ for all n. Define

$$\varphi(n) = \max\{n^{-1/l}, q_k^{-1/l}\} \quad \text{if } q_{k-1}^{u/l} < n \le q_k^{u/l}.$$

Then by Corollary 1.3, we have

$$\lim_{n\to\infty} \frac{\log q_n}{-\log \varphi(q_n)} = l < u = \dim_{\mathrm{H}}(E_\varphi(\alpha)).$$

Thus the lower bound of Xu (Theorem 1.1) is not optimal.

The next two theorems show that the estimates in Theorem 1.2 are sharp.

Theorem 1.5. *For any irrational α and for any $0 \le l < u \le 1$ with $u > 1/w$, there exists a decreasing function $\varphi : \mathbb{N} \to \mathbb{R}^+$ with $l_\varphi = l$ and $u_\varphi = u$ such that*

$$\dim_{\mathrm{H}} E_\varphi(\alpha) = \max\left\{l, \frac{1+u}{1+w}\right\} < u.$$

Theorem 1.6. *Suppose $0 \le l < u \le 1$. There exists a decreasing function $\varphi : \mathbb{N} \to \mathbb{R}^+$ with $l_\varphi = l$ and $u_\varphi = u$ such that for any α which is not a Liouville number,*

$$\dim_{\mathrm{H}} E_\varphi(\alpha) = u.$$

2. Three steps dimension. The goal of this section is to prove Proposition 2.3 which will be the base of our dimension estimation (compare [Xu10, Section 3]).

Let us start with a technical lemma.

LEMMA 2.1. *Let $1 > a > b$ and $1 > c > d$. Then for any $\delta \in [0,1]$ we have*

$$\frac{\log(\delta a + (1-\delta)c)}{\log(\delta b + (1-\delta)d)} \geq \min\left(\frac{\log a}{\log b}, \frac{\log c}{\log d}\right).$$

Proof. Denote

$$s := \min\left(\frac{\log a}{\log b}, \frac{\log c}{\log d}\right).$$

Then

$$\frac{\log(\delta a + (1-\delta)c)}{\log(\delta b + (1-\delta)d)} \geq \frac{\log(\delta b^s + (1-\delta)d^s)}{\log(\delta b + (1-\delta)d)}.$$

By concavity of the function $x \mapsto x^s$, we have

$$\delta b^s + (1-\delta)d^s \leq (\delta b + (1-\delta)d)^s,$$

and the assertion follows. ■

We will also need the following result, which can be found in any standard textbook on geometric measure theory (see for example [F95, Proposition 4.9]). Given a probability measure μ on $\mathbb{R}$, the *lower density* of μ at a point $x \in \mathbb{R}$ is defined by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

LEMMA 2.2 (Frostman). *Let $E \subset \mathbb{R}$. Assume we can find a measure μ supported on E such that*

$$\underline{d}_\mu(x) \geq s \quad \textit{for } \mu\textit{-almost every } x \in E.$$

Then $\dim_{\mathrm{H}} E \geq s$.

Let α be an irrational number with $\omega(\alpha) > 1$. Recall that q_n is the denominator of the nth convergent of the continued fraction of α. Let $B \geq 1$ and suppose there exists a sequence $\{n_i\}$ of natural numbers such that

$$\frac{\log q_{n_i+1}}{\log q_{n_i}} \to B. \tag{2.1}$$

Let $\{m_i\}$ be a sequence of natural numbers such that $q_{n_i} < m_i \leq q_{n_i+1}$. By passing to subsequences, we suppose the limit

$$N := \lim_{i\to\infty} \frac{\log m_i}{\log q_{n_i}}$$

exists. Then obviously, $1 \leq N \leq B$.

Let $K > 1$. Denote

$$E_i := \left\{y \in \mathbb{R} : \|n\alpha - y\| < \tfrac{1}{2} q_{n_i}^{-K} \text{ for some } n \in (m_{i-1}, m_i]\right\},$$

$$E := \bigcap_{i=1}^{\infty} E_i \quad \text{and} \quad F := \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_i.$$

PROPOSITION 2.3. *Suppose $N > 1$. If $\{n_i\}$ is increasing sufficiently fast then*

$$\dim_{\mathrm{H}} E = \dim_{\mathrm{H}} F = S,$$

where

$$S = S(N, B, K) := \min\left(\frac{N}{K}, \max\left(\frac{1}{K}, \frac{1}{1 + B - N}\right)\right).$$

We remark that Proposition 2.3 is a generalization of Xu's result, so it could also be used to prove Xu's theorem (Theorem 1.1). However, we would need some modifications to include the case $N = 1$ in Proposition 2.3, which we skip.

Proof of Proposition 2.3. As $F \supset E$, we only need to get the lower bound for $\dim_{\mathrm{H}} E$ and the upper bound for $\dim_{\mathrm{H}} F$. For the former, we will use the Frostman Lemma, and for the latter, we will use a natural cover.

We will distinguish two cases: $B \geq K$ and $B < K$. Notice the following fact.

FACT. *If $B \geq K$ then*

$$\frac{N}{K} > \frac{1}{1 + B - N} \quad \text{and} \quad S = \max\left(\frac{1}{K}, \frac{1}{1 + B - N}\right).$$

If $B < K$, then

$$\frac{1}{K} < \frac{1}{1 + B - N} \quad \text{and} \quad S = \min\left(\frac{N}{K}, \frac{1}{1 + B - N}\right).$$

Indeed, the second statement follows by noting $1/K < 1/B$. For the first statement, if $N \geq K$ then it is obviously true because the right hand side is smaller than 1. Otherwise, we have

$$\frac{K - N}{N} < K - N,$$

hence

$$\frac{K}{N} < 1 + K - N.$$

Since $B \geq K$, we have

$$1 + B - N \geq 1 + K - N > K/N.$$

Distribution of the points. Now, let us study the distribution of the points $\{n\alpha \pmod 1\}$. Let $\{n_i\}$ be a fast increasing sequence satisfying (2.1). By passing to a subsequence, we can always assume that $\{n_i\}$ grows as fast as we wish; the exact rate of growth will be clear later. Denote

$$N_i := m_i - m_{i-1}.$$

Since $N > 1$, by passing to a subsequence, we can suppose that $N_i \geq q_{n_i}$. As N will not change when we change m_i to the closest greater or closest smaller multiple of q_{n_i}, it is enough to prove the statement for $q_{n_i} \mid N_i$.

The three steps theorem tells us how the points $\{n\alpha \pmod 1\}_{n=m_{i-1}+1}^{m_i}$ are distributed on the unit circle: there are q_{n_i} groups of points, each consisting of N_i/q_{n_i} points, the distances between points inside each group are equal to $\xi_i := \|q_{n_i}\alpha\|$ and the distances between groups are $\zeta_i := \|q_{n_i-1}\alpha\| - (N_i/q_{n_i} - 1)\|q_{n_i}\alpha\|$ (except for one distance which is equal to $\zeta_i + \xi_i$).

Let ε be fixed and small. In the first case, i.e., $B \geq K$, we have $\xi_i \leq q_{n_i}^{-K}$ for all i large enough, hence the intervals $[n\alpha - q_{n_i}^{-K}/2, n\alpha + q_{n_i}^{-K}/2]$ intersect each other (within each group). So E_i consists of $M_i := q_{n_i}$ intervals of length $y_i := (N_i/q_{n_i} - 1)\xi_i + q_{n_i}^{-K}$. By noting that $\|q_n\alpha\|$ is comparable with q_{n+1}^{-1}, we have

$$y_i = (N_i/q_{n_i} - 1)\xi_i + q_{n_i}^{-K} = q_{n_i}^{-\min(K,1+B-N)+\varepsilon}$$

for i large enough.

In the second case, i.e., $B < K$, for large i, E_i consists of N_i intervals of length $z_i := q_{n_i}^{-K}$.

As $q_{n_{i+1}} \gg q_{n_i+1}$, we can freely assume that for i large enough each component of E_i contains at least $M_{i+1}^{1-\varepsilon}$ (in the first case) or $N_{i+1}^{1-\varepsilon}$ (in the second case) components of E_{i+1}.

Calculations. We will distribute a probability measure μ in the most natural way: the measure assigned to each component of $F_i = E_1 \cap \cdots \cap E_i$ is the same. Here we distribute the measure only on those components of F_i that are components of E_i, i.e., at all stages we count only components completely contained in previous generation sets.

Case 1: $B \geq K$. At level i we have at least $M_i^{1-\varepsilon}$ components of F_i, each of size y_i, and inside each component of F_{i-1}, the components of F_i are at equal distance $c_i := \zeta_i - q_{n_i}^{-K}$.

Let $x \in E$. For $y_i \leq r < y_{i-1}$, consider

$$f(r) = \frac{\log \mu(B_r(x))}{\log r}. \tag{2.2}$$

Notice that the convex hull of components of F_i intersecting $B_r(x)$ has measure at most $3\mu(B_r(x))$ and length at most $6r$. Hence, it is enough to consider the case when the interval $B_r(x)$ is the convex hull of some components of F_i contained in one component of F_{i-1}. Denoting the number of components of F_i contained in one component of F_{i-1} by n, we get

$$f(ny_i + (n-1)c_i) \geq \frac{\log(nM_i^{-(1-\varepsilon)})}{\log(ny_i + (n-1)c_i)}. \tag{2.3}$$

As the right hand side of (2.3) is the ratio of logarithms of two functions, both linear in n and smaller than 1, by Lemma 2.1 the minimum of $f(r)$ in (y_i, y_{i-1}) is achieved at one of the endpoints. We have

$$f(y_i) \geq (1-\varepsilon)\frac{-\log M_i}{\log y_i} = \max\left(\frac{1}{K}, \frac{1}{1+B-N}\right) + O(\varepsilon) \tag{2.4}$$

and the same holds for $f(y_{i-1})$. Recalling the fact at the beginning of the proof, we get the lower bound by Lemma 2.2.

The upper bound is simpler: for any i, F is contained in $\bigcup_{n>i} E_n$. Hence, we can use the components of all $E_n, n > i$, as a cover for F. For any s the sum of the sth powers of the diameters of the components of E_n is bounded by $M_n y_n^s$, and for $s > \max\left(\frac{1}{K}, \frac{1}{1+B-N}\right) + O(\varepsilon)$ it is exponentially decreasing in n. The upper bound then follows by the definition of Hausdorff dimension.

Case 2: $B < K$. Once again to obtain the lower bound we will consider the function $f(r)$ given by (2.2). However, in this case the components of F_i are not uniformly distributed inside a component of F_{i-1} but they are in groups. There are at least s_i groups at distance c_i from each other, each group is of size y_i and contains at least $N_i^{1-\varepsilon}$ components. Inside each group the components of size z_i are at distance $d_i := \xi_i - q_{n_i}^{-K}$ from each other.

We need to consider $z_i \leq r < z_{i-1}$. This range can be divided into two subranges. The inequality (2.3) works for $y_i \leq r < z_{i-1}$, while for $z_i \leq r < y_i$ the same reasoning gives

$$f(nz_i + (n-1)d_i) \geq \frac{\log(nN_i^{-(1-\varepsilon)})}{\log(nz_i + (n-1)d_i)}. \tag{2.5}$$

As in the first case, Lemma 2.1 implies that the minimum of $f(r)$ in each subrange is achieved at one of the endpoints. We have

$$f(z_i) \geq (1-\varepsilon)\frac{-\log N_i}{\log z_i} = \frac{N}{K} + O(\varepsilon)$$

and the same for $f(z_{i-1})$, while $f(y_i)$ is still given by (2.4). Together with Lemma 2.2 and the fact at the beginning of the proof, this gives the lower bound.

To get the upper bound for the dimension of F we can use two covers. One is given by using the convex hulls of groups of components of F_n with $n > i$. As in the first case (taking into account the fact that $1/K < 1/(1+B-N)$), this cover gives

$$\dim_{\mathrm{H}} F \leq \frac{1}{1+B-N} + O(\varepsilon).$$

The other cover consists of the components of E_n with $n > i$. For any s the sum of the sth powers of the diameters of the components of E_n is bounded by $N_n z_n^s$, and for $s > N/K + O(\varepsilon)$ it is exponentially decreasing in n. We choose one of the two covers that gives us the smaller Hausdorff dimension. ■

The statement of Proposition 2.3 could also be written in the following way, fixing B and N and varying K:

$$S(N,B,K)=\begin{cases} 1/K, & K<1+B-N,\\ 1/(1+B-N), & 1+B-N\le K\le N(1+B-N),\\ N/K, & K>N(1+B-N).\end{cases}$$

3. Proof of Theorem 1.2. Note that the lower bound in Theorem 1.2 can be written as

$$\max\Big\{l_\varphi,\min\Big\{u_\varphi,\frac{1+u_\varphi}{1+w}\Big\}\Big\}.$$

We remind the reader that by the result of Bugeaud [Bug03] and Schmeling and Troubetzkoy [TS03], the Hausdorff dimension of E_φ is between l_φ and u_φ. So for the case of Liouville numbers, i.e., $\omega(\alpha)=\infty$, the result trivially holds and we can assume $w<\infty$. For the same reason, we only need to prove the lower bound. Moreover, we just need to show it is not smaller than $\min(u_\varphi,(1+u_\varphi)/(1+w))$ and we can assume that $l_\varphi<u_\varphi$. We shall suppose that $l_\varphi>0$; the case $l_\varphi=0$ can be obtained by a limiting argument.

For any irrational number α, $\omega(\alpha)$ can be defined alternatively by

$$\omega(\alpha)=\limsup_{n\to\infty}\frac{\log q_{n+1}(\alpha)}{\log q_n(\alpha)}.$$

Choose a sequence m_i of natural numbers such that

$$\lim_{i\to\infty}\frac{\log m_i}{-\log\varphi(m_i)}=u_\varphi.$$

Let n_i be such that $q_{n_i}<m_i\le q_{n_i+1}$. By passing to a subsequence we can assume that

- the sequence $\log q_{n_i+1}/\log q_{n_i}$ has a limit $B\in[1,w]$,
- the sequence $\log m_i/\log q_{n_i}$ has a limit $N\in[1,B]$,
- the sequence $\{n_i\}$ grows fast enough for Proposition 2.3.

Moreover, we can freely assume that $N>1$: otherwise, by the monotonicity of φ, we would have

$$\lim_{i\to\infty}\frac{\log q_{n_i}}{-\log\varphi(q_{n_i})}=u_\varphi,$$

and the assertion would follow from Theorem 1.1.

Take $K=N/u_\varphi$. By the definition of m_i, for any small $\delta>0$, we have

$$\varphi(m_i)\ge(m_i)^{-1/u_\varphi-\delta}\ge q_{n_i}^{-K}\quad\text{for all large } i.$$

Thus by monotonicity of φ,

$$\varphi(n)\ge q_{n_i}^{-K}\quad\forall n\le m_i. \tag{3.1}$$

The assumptions of Proposition 2.3 are satisfied, so we can calculate the Hausdorff dimension of the set E defined in the previous section. By (3.1), $E \subset E_\varphi$, so this gives the lower bound for the Hausdorff dimension of E_φ:

$$\dim_{\mathrm{H}} E_\varphi \geq M(N,B) := \min\left(u_\varphi, \max\left(\frac{u_\varphi}{N}, \frac{1}{1+B-N}\right)\right),$$

and we want to estimate the minimal value of M for $B \in [1,w], N \in [1,B]$.

First note that increasing B not only decreases $M(B,N)$ for a fixed N but also increases the range of possible N's. Hence, the minimum of $M(N,B)$ is achieved for $B = w$. Denote $M(N) = M(N,w)$.

We are then left with a simple optimization problem for a function of one variable. We can write

$$M(N) = \min\left(u_\varphi, \max\left(\frac{u_\varphi}{N}, \frac{1}{1+w-N}\right)\right).$$

If $wu_\varphi \leq 1$ then $u_\varphi \leq 1/(1+w-N)$ for all N, hence

$$\min_N M(N) = u_\varphi \leq \frac{1+u_\varphi}{1+w}.$$

Otherwise, as u_φ/N is a decreasing and $1/(1+w-N)$ an increasing function of N, the global minimum over N of the maximum of the two is achieved at the point N_0 where they are equal: $u_\varphi/N_0 = 1/(1+w-N_0)$, that is, for

$$N_0 = \frac{u_\varphi(1+w)}{1+u_\varphi}.$$

As $wu_\varphi > 1$ implies $1 < N_0 < wu_\varphi \leq w$, N_0 is inside the interval $[1,w]$, hence this global minimum is the local minimum we are looking for. Thus, in this case

$$\min_N M(N) = M(N_0) = \frac{1+u_\varphi}{1+w} < u_\varphi.$$

We are done.

4. Proofs of Theorems 1.5 and 1.6

Proof of Theorem 1.5. Let α be of Diophantine type $w > 1/u$. Let q_{n_i} be a sparse subsequence of denominators of convergents such that

$$w = \lim_{i\to\infty} \frac{\log q_{n_i+1}}{\log q_{n_i}}.$$

For any $0 \leq l < u \leq 1$, define

$$z = \max\left(l, \frac{1+u}{1+w}\right).$$

Note that $z < u$.

Define also a function $\varphi : \mathbb{N} \to \mathbb{R}$ as follows:

$$\varphi(n) := \max\{n^{-1/l}, k_{n_i}^{-1/u}\} \quad \text{if } k_{n_{i-1}} < n \leq k_{n_i}, \quad \text{where } k_{n_i} = q_{n_i}^{u/z}.$$

Let

$D_1 = \{y \in \mathbb{R} : \text{for infinitely many } i,\ \|n\alpha - y\| < k_{n_i}^{-1/u} \text{ for some } n \in (k_{n_{i-1}}, k_{n_i}]\}$,

$D_2 = \{y \in \mathbb{R} : \|n\alpha - y\| < n^{-1/l} \text{ for infinitely many } n\}$.

Clearly, $E_\varphi(\alpha) = D_1 \cup D_2$. The Hausdorff dimension of D_1 is given by Proposition 2.3 (with $B = w, K = 1/z, N = u/z$):

$$\dim_{\mathrm{H}} D_1 = \min\left(u, \max\left(z, \frac{z}{(1+w)z - u}\right)\right) = z$$

(the equality is valid both when $z = l$ and $z = (1+u)/(1+w)$).

By [Bug03] and [TS03] we have $\dim_{\mathrm{H}} D_2 = l$. Thus the proof is complete.

Proof of Theorem 1.6. Construct a sequence $\{n_i\}_{i \geq 1}$ by recurrence:

$$n_1 = 2, \quad n_{i+1} = 2^{n_i} \quad (i \geq 1).$$

Define a function $\varphi : \mathbb{N} \to \mathbb{R}$ as $\varphi(n) = n_i^{-1/l}$ for $n \in (n_i, n_i^{u/l})$ and $\varphi(n) = n^{-1/u}$ elsewhere.

Now we show that for this φ, $\dim_{\mathrm{H}} E_\varphi(\alpha) = u$ if α is not a Liouville number. Suppose for contradiction that $\dim_{\mathrm{H}} E_\varphi(\alpha) < u$. By Theorem 1.1, no q_m could be between n_i and $n_{i+1}^{l/u}$. Since n_i goes to infinity very fast, α cannot be of finite type. Hence, α must be a Liouville number, which is a contradiction.

Acknowledgements. M.R. was partially supported by the MNiSW (grant N201 607640, Poland). L.L. was partially supported by the ANR (grant 12R03191A, MUTADIS, France). A part of this paper was written during the visit of L.L. to the NCTS in Taiwan.

References

[BD99] V. I. Bernik and M. M. Dodson, *Metric Diophantine Approximation on Manifolds*, Cambridge Tracts in Math. 137, Cambridge Univ. Press, Cambridge, 1999.

[Bug03] Y. Bugeaud, *A note on inhomogeneous Diophantine approximation*, Glasgow Math. J. 45 (2003), 105–110.

[Cas50] J. W. S. Cassels, *Some metrical theorems in Diophantine approximation. I*, Proc. Cambridge Philos. Soc. 46 (1950), 209–218.

[Dic94] H. Dickinson, *The Hausdorff dimension of sets arising in metric Diophantine approximation*, Acta Arith. 68 (1994), 133–140.

[Dod92] M. M. Dodson, *Hausdorff dimension, lower order and Khintchine's theorem in metric Diophantine approximation*, J. Reine Angew. Math. 432 (1992), 69–76.

[F95] K. Falconer, *Fractal Geometry; Mathematical Foundations and Applications*, Wiley, Chichester, 1995.

[FW06] A.-H. Fan and J. Wu, *A note on inhomogeneous Diophantine approximation with a general error function*, Glasgow Math. J. 48 (2006), 187–191.

[Khi26] A. Khintchine, *Über eine klasse linearer Diophantischer Approximationen*, Rend. Circ. Mat. Palermo 50 (1926), 170–195.

[Kim12] D. H. Kim, *A note on metric inhomogeneous Diophantine approximation*, arXiv: 1201.4568v1, 2012.

[Kur55] J. Kurzweil, *On the metric theory of inhomogeneous Diophantine approximations*, Studia Math. 15 (1955), 84–112.

[LN12] M. Laurent and A. Nogueira, *Inhomogeneous approximation with coprime integers and lattice orbits*, Acta Arith. 154 (2012), 413–427.

[Min57] H. Minkowski, *Diophantische Approximationen: Eine Einführung in die Zahlentheorie*, Chelsea Publ., New York, 1957.

[TS03] S. Troubetzkoy and J. Schmeling, *Inhomogeneous Diophantine approximations and angular recurrence for billiards in polygons*, Mat. Sb. 194 (2003), 129–144.

[Xu10] J. Xu, *Inhomogeneous Diophantine approximation and Hausdorff dimension*, preprint, 2010.

Lingmin Liao
LAMA UMR 8050, CNRS
Université Paris-Est Créteil
61 Avenue du Général de Gaulle
94010 Créteil Cedex, France
E-mail: lingmin.liao@u-pec.fr

Michał Rams
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: M.Rams@impan.gov.pl

Received on 13.9.2012
and in revised form on 24.1.2013 (7192)