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A measure-theoretic approach to the invariants of the Selberg class

by

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1. Introduction. In [6] and [7] we defined and studied the invariants of the Selberg class $\mathcal{S}$ (to be precise, of the extended Selberg class $\mathcal{S}^\sharp$). We refer to our survey papers [3], [5], [9] and [10] for the definitions and basic properties of the classes $\mathcal{S}$ and $\mathcal{S}^\sharp$. Here we recall that $\mathcal{S}^\sharp$ is the class of non-identically vanishing Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{a_n(F)}{n^s} \tag{1.1}$$

absolutely convergent for $\sigma>1$, such that $(s-1)^mF(s)$ is entire of finite order for some non-negative integer m and $F(s)$ satisfies a functional equation of the form

$$\Phi(s)=\omega\overline{\Phi}(1-s), \tag{1.2}$$

where $\overline{f}(s)=\overline{f(\overline{s})}$, $|\omega|=1$ and

$$\Phi(s)=Q^s\prod_{j=1}^{r}\Gamma(\lambda_js+\mu_j)F(s)=\gamma(s)F(s),$$

say, with $r\ge 0$, $Q>0$, $\lambda_j>0$ and $\Re\mu_j\ge 0$ ($r=0$ means that there are no Γ-factors). $\mathcal{S}$ is the subclass of the functions $F\in\mathcal{S}^\sharp$ satisfying the Ramanujan conjecture $a_n(F)\ll n^{\varepsilon}$ for every $\varepsilon>0$ and having an Euler product of type

$$\log F(s)=\sum_{n=2}^{\infty}\frac{b_n(F)}{n^s}$$

with $b_n(F)=0$ unless $n=p^m$, and $b_n(F)\ll n^{\theta}$ for some $\theta<1/2$.

We recall that the notion of invariant of $\mathcal{S}^\sharp$ arises from the fact that the data Q, λ_j, μ_j and ω of the functional equation of a function $F\in\mathcal{S}^\sharp$ are not

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uniquely determined by $F(s)$ (due, essentially, to the multiplication formula for the Γ function). Thus, an invariant is an expression defined by means of such data, but depending only on $F(s)$; invariants are denoted by I or by I_F or $I(F)$ (particularly when referred to a function $F \in \mathcal{S}^\sharp$). We refer to [6] and [7] for the meaning of several interesting invariants, such as the degree

$$d_F = 2\sum_{j=1}^r \lambda_j,$$

the conductor

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j},$$

the root number

$$\omega_F^* = \omega e^{-i\frac{\pi}{2}(\eta_F+1)}\left(\frac{q_F}{(2\pi)^{d_F}}\right)^{i\theta_F/d_F} \prod_{j=1}^r \lambda_j^{-2i\Im\mu_j}$$

and the H-invariants

$$H_F(n) = 2\sum_{j=1}^r \frac{B_n(\mu_j)}{\lambda_j^{n-1}},$$

where $B_n(z)$ denotes the nth Bernoulli polynomial; for example, $H_F(0) = d_F$. Note that the root number ω_F^* factors as

$$\omega_F^* = \left(\omega \prod_{j=1}^r \lambda_j^{-2i\Im\mu_j}\right)\left(e^{-i\frac{\pi}{2}(\eta_F+1)}\left(\frac{q_F}{(2\pi)^{d_F}}\right)^{i\theta_F/d_F}\right) = \omega'_F\omega''_F, \tag{1.3}$$

say, where ω''_F is clearly an invariant, and hence ω'_F is an invariant as well. We further recall that an invariant I is called *numerical* if $I(F) \in \mathbb{C}$ for every $F \in \mathcal{S}^\sharp$ (it is easy to construct invariants which are not numerical); in other words, a numerical invariant I is a function $I : \mathcal{S}^\sharp \to \mathbb{C}$. Note that both $\mathcal{S}$ and $\mathcal{S}^\sharp$ are multiplicative semigroups, i.e. $FG \in \mathcal{S}$ (resp. $\mathcal{S}^\sharp$) if $F, G \in \mathcal{S}$ (resp. $\mathcal{S}^\sharp$), the H-invariants are additive, i.e. $H_{FG}(n) = H_F(n) + H_G(n)$, and the conductor and ω'_F are multiplicative, i.e. $q_{FG} = q_F q_G$ and $\omega'_{FG} = \omega'_F\omega'_G$. The set of functions $F \in \mathcal{S}$ (resp. $\mathcal{S}^\sharp$) with $d_F = d$ is denoted by $\mathcal{S}_d$ (resp. $\mathcal{S}_d^\sharp$), and the order of the pole of $F(s)$ at $s = 1$ is denoted by m_F.

A fundamental problem in the theory of the Selberg class is describing the admissible values of numerical invariants, i.e. the set of values that such a numerical invariant attains at the functions of $\mathcal{S}$ and $\mathcal{S}^\sharp$. For some invariants there are nice conjectures about admissible values, for example the *degree conjecture* (asserting that $d_F \in \mathbb{N}$ for every $F \in \mathcal{S}^\sharp$) and the *conductor conjecture* (asserting that $q_F \in \mathbb{N}$ for every $F \in \mathcal{S}$). In this paper we develop a measure-theoretic approach to this problem. In order to state the results we need some definitions; we will refer to Kechris' book [8] for all the definitions and results needed from topology and measure theory.

We denote by $\mathbb{R}^+$ and $\mathbb{C}^+$ the positive real numbers and the complex numbers with non-negative real part, respectively, and by T^1 the unit circle. A numerical invariant I is called *continuous* if for every non-negative integer r there exists a continuous function

$$f_{I,r} : \mathbb{R}^+ \times (\mathbb{R}^+ \times \mathbb{C}^+)^r \times T^1 \to \mathbb{C}$$

such that

$$I(F) = f_{I,r}(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega) \tag{1.4}$$

if $F \in \mathcal{S}^\sharp$ satisfies functional equation (1.2), where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$. Examples of continuous invariants are the H-invariants, the conductor and the root numbers ω_F^*, ω_F' and ω_F''. Moreover, the real and imaginary parts of a continuous invariant are also continuous invariants.

For technical reasons, it is convenient to work with a slightly more general class than $\mathcal{S}^\sharp$, denoted by $\mathcal{S}^{\sharp\sharp}$ and consisting of the Dirichlet series (1.1), absolutely convergent for σ sufficiently large and satisfying exactly the same meromorphic continuation and functional equation axioms of $\mathcal{S}^\sharp$. Clearly, $\mathcal{S}^{\sharp\sharp}$ is a multiplicative semigroup with identity 1 and $\mathcal{S}$, $\mathcal{S}^\sharp$ are subsemigroups of $\mathcal{S}^{\sharp\sharp}$. Note that the definitions and the main properties pertaining to $\mathcal{S}^\sharp$ carry over to $\mathcal{S}^{\sharp\sharp}$. In particular, it is easy to see that Conrey–Ghosh's [1] result that the γ-factors $\gamma(s)$ of $F(s)$ are uniquely determined up to a constant factor (see also Theorem 8.1 of [5]) holds for $\mathcal{S}^{\sharp\sharp}$ as well, and the invariant theory of $\mathcal{S}^\sharp$ carries over to $\mathcal{S}^{\sharp\sharp}$.

Let $\mathcal{I} = \{I_j\}_{j\in J}$ with $J \subset \mathbb{N}$ be a countable family of continuous invariants and, for $F, G \in \mathcal{S}^{\sharp\sharp}$, write

$$\varrho_{\mathcal{I}}(F,G) = \sum_{j\in J} \frac{1}{2^j} \frac{|I_j(F) - I_j(G)|}{1 + |I_j(F) - I_j(G)|} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|a_n(F) - a_n(G)|}{1 + |a_n(F) - a_n(G)|}.$$

It is easy to check that $\varrho_{\mathcal{I}}$ is a metric on $\mathcal{S}^{\sharp\sharp}$ (recall that if $d(x,y)$ is a metric then so is $d'(x,y) = d(x,y)/(1 + d(x,y))$, and the proof for $\varrho_{\mathcal{I}}$ is similar). We define the *$\mathcal{I}$-Borel sets* to be the Borel sets of the metric space $(\mathcal{S}^{\sharp\sharp}, \varrho_{\mathcal{I}})$, and we denote by $\mathcal{B}(\mathcal{I})$ the set of the $\mathcal{I}$-Borel sets. We recall that a topological space X is *σ-compact* if

$$X = \bigcup_{n=1}^{\infty} K_n$$

with compact sets K_n satisfying $K_n \subset K_{n+1}$. Our first result is

THEOREM 1. *Let $\mathcal{I}$ be a countable family of continuous invariants. Then $(\mathcal{S}^{\sharp\sharp}, \varrho_{\mathcal{I}})$ is a σ-compact metric space and $\mathcal{S}, \mathcal{S}^\sharp \in \mathcal{B}(\mathcal{I})$.*

Theorem 1 is a basic topological result from which the following measure-theoretic consequences are deduced.

Theorem 2. *Let $\mathcal{I}$ be a countable family of continuous invariants. Then $I(B)$ is Lebesgue measurable for every $B \in \mathcal{B}(\mathcal{I})$ and every $I \in \mathcal{I}$.*

In particular, from Theorems 1 and 2 we see that *$I(\mathcal{S})$ and $I(\mathcal{S}^\sharp)$ are Lebesgue measurable for every continuous invariant I*. We remark that the measurability of $I(B)$ in Theorem 2 is obtained via Lusin's theorem (see Theorem 21.10 of [8]), and therefore *$I(B)$ is in fact measurable for every σ-finite Borel measure*, although we will only consider the Lebesgue measure in this paper.

$B = \mathcal{S}$ or $B = \mathcal{S}^\sharp$ are probably the most interesting cases of Theorem 2, and can be proved by starting directly from a single invariant I (instead of a family $\mathcal{I}$ containing I); the same remark applies to most cases where a specific set and invariant are involved. However, the definition of the metric by means of a family of invariants allows a convenient and wider choice of Borel sets, and hence a larger range of applications of our results. In fact, for example, adding a continuous invariant I to a family $\mathcal{I}$ we have $\mathcal{B}(\mathcal{I}) \subset \mathcal{B}(\mathcal{I} \cup \{I\})$. As an illustration we state the following simple corollary (examples are given later on).

Corollary 1. *Let I_0 be a continuous invariant and $B \in \mathcal{B}(I_0)$. Then $I(B)$ is Lebesgue measurable for every continuous invariant I.*

The condition that B is a Borel set in Theorem 2 can be relaxed if we assume more about the invariants of the family $\mathcal{I}$. Given $B \in \mathcal{B}(\mathcal{I})$, let G be the subsemigroup of $\mathcal{S}^{\sharp\sharp}$ generated by B; we say that G is an *$\mathcal{I}$-Borel generated* semigroup.

Theorem 3. *Let $\mathcal{I}$ be a countable family of continuous invariants such that every $I \in \mathcal{I}$ is additive or multiplicative. Then $I(G)$ is Lebesgue measurable for every $\mathcal{I}$-Borel generated semigroup G and every $I \in \mathcal{I}$.*

In analogy with Corollary 1, here is a corollary illustrating the usefulness of the family $\mathcal{I}$.

Corollary 2. *Let I_0 be an additive or multiplicative continuous invariant, $B \in \mathcal{B}(I_0)$, and G the semigroup generated by B. Then $I(G)$ is Lebesgue measurable for every additive or multiplicative continuous invariant I.*

Of course, the set B in Corollaries 1 and 2 can be intersected with $\mathcal{S}$ or $\mathcal{S}^\sharp$, and the conclusions still hold.

Of particular interest are the subsemigroups G of $\mathcal{S}^{\sharp\sharp}$ such that $I(G)$ is Lebesgue measurable for an invariant I (not necessarily continuous). In such a case, G is called an *I-measurable* semigroup. In view of Theorem 3, a first class of examples of such semigroups is given by the $\mathcal{I}$-Borel generated semigroups with all $I \in \mathcal{I}$ additive or multiplicative. Another class of examples (not disjoint from the previous one) is provided by Theorem 2 and consists of the $\mathcal{I}$-*Borel* semigroups, that is, the $\mathcal{I}$-Borel sets which are semigroups themselves. Explicit examples of measurable semigroups are as follows. First of all, by Theorem 1, $\mathcal{S}$ and $\mathcal{S}^\sharp$ are I-measurable for every continuous I. Other examples of semigroups I-measurable for every continuous I are $\mathcal{S}_0$ and $\mathcal{S}_0^\sharp$. In fact, these sets are semigroups and Corollary 1 can clearly be applied. We recall (see [1] and [4]) that $\mathcal{S}_0 = \{1\}$ and $\mathcal{S}_0^\sharp$ is a certain set of Dirichlet polynomials. Moreover, thanks to Corollary 2, the following are examples of semigroups I-measurable for every additive or multiplicative continuous I. Recalling that d denotes the degree, G^{Dir}, generated by $d^{-1}(\{1\}) \cap \mathcal{S} = \mathcal{S}_1$, is the semigroup generated by the Riemann zeta function and the shifted Dirichlet L-functions (see [4]). $G^{(1)}$, generated by $d^{-1}(\{1\}) \cap \mathcal{S}^\sharp = \mathcal{S}_1^\sharp$, can also be explicitly described (see [4]). Finally, we also mention $G^{(2)}$, generated by $d^{-1}(\{2\}) \cap \mathcal{S}^\sharp = \mathcal{S}_2^\sharp$.

In the case of I-measurable semigroups G with I additive or multiplicative we can say more about $\mu(I(G))$, where μ denotes the Lebesgue measure. Indeed, we have the following simple 0-1 *laws* for additive and multiplicative invariants.

Theorem 4. *Let G be an I-measurable semigroup. If I is additive and real-valued, then either $\mu(I(G)) = 0$ or $I(G)$ contains a half-line. If I is multiplicative and takes values in T^1 (resp. $\mathbb{R}^+$), then either $\mu(I(G)) = 0$ or $I(G) = T^1$ (resp. $I(G)$ contains a half-line).*

As is clear from the above discussion, Theorem 4 is closely related to Theorems 2 and 3. In fact, from Theorems 2–4 we easily deduce the following consequences. In view of the degree conjecture, the first part of Theorem 4 is particularly interesting in the case of the degree d, where $\mu(d(\mathcal{S}^\sharp)) = 0$ is expected. Examples of measurable semigroups G with $\mu(d(G)) = 0$ are $\mathcal{S}_0$, $\mathcal{S}_0^\sharp$, G^{Dir} and $G^{(1)}$.

The most interesting special case of the second part of Theorem 4 is the conductor q, and the conductor conjecture suggests that $\mu(q(\mathcal{S})) = 0$. For example, it follows from the characterization of the functions of degree 0 and 1 of $\mathcal{S}$ and $\mathcal{S}^\sharp$ (see [4]) that

$$\mu(q(\mathcal{S}_0)) = \mu(q(\mathcal{S}_0^\sharp)) = \mu(q(G^{\mathrm{Dir}})) = \mu(q(G^{(1)})) = 0.$$

However, probably $q(\mathcal{S}^\sharp)$ contains a half-line. In fact, in view of Hecke's theory for the groups $G(\lambda)$ (see Hecke's book [2]), already $q(G^{(2)})$ will probably contain a half-line.

Another interesting multiplicative invariant is the root number ω_F' defined by (1.3). In view of [4] we have $\mu(\omega'(\mathcal{S}_1)) = 0$, while $\omega'(G^{(1)}) = T^1$. Moreover, since the weight k in Hecke's theory with $\lambda > 2$ is arbitrary, it is very likely that $\omega'(G^{(2)}) = T^1$.

We finally remark that in all known or conjectural cases, if the set of values of a continuous invariant has 0-measure, then it is countable. We therefore state the following conjecture, clearly related to Theorem 4.

CONJECTURE. *Let I be a continuous invariant and G be an I-measurable semigroup. If I is additive or multiplicative with values in $\mathbb{R}^+$, then either $I(G)$ is countable or it contains a half-line.*

A similar conjecture can be made for multiplicative continuous invariants with values in T^1; in this case, either $I(G)$ is countable or $I(G) = T^1$.

2. Proofs. In order to prove Theorem 1 we need three lemmas.

LEMMA 1. *Let $\mathcal{I}$ be a countable family of continuous invariants. Then for every $n = 1, 2, \ldots$ and every $I \in \mathcal{I}$, the functions $F \mapsto a_n(F)$ and $F \mapsto I(F)$ are continuous with respect to the metric $\varrho_{\mathcal{I}}$.*

Proof. Given a sequence $F_m \to F_0$ in $(\mathcal{S}^{\sharp\sharp}, \varrho_{\mathcal{I}})$ we have $\varrho_{\mathcal{I}}(F_m, F_0) \to 0$, hence, in particular,

$$\frac{|a_n(F_m) - a_n(F_0)|}{1 + |a_n(F_m) - a_n(F_0)|} \to 0 \quad \text{and} \quad \frac{|I_j(F_m) - I_j(F_0)|}{1 + |I_j(F_m) - I_j(F_0)|} \to 0,$$

whence $a_n(F_m) \to a_n(F_0)$ and $I_j(F_m) \to I_j(F_0)$. ∎

For $R \geq 2$ integer, let $\mathcal{S}^{\sharp\sharp}(R)$ be the set of $F \in \mathcal{S}^{\sharp\sharp}$ such that

$$r \leq R, \quad \frac{1}{R} \leq Q, \lambda_j \leq R, \quad |\mu_j| \leq R, \quad m_F \leq R, \quad \sum_{n=1}^{\infty} \frac{|a_n(F)|}{n^R} \leq R$$

and

$$|F(s)| \leq e^{|s|^R} \quad \text{for } |s| \geq 2.$$

Clearly, $\mathcal{S}^{\sharp\sharp}(R) \subset \mathcal{S}^{\sharp\sharp}(R+1)$ and

$$\mathcal{S}^{\sharp\sharp} = \bigcup_{R=2}^{\infty} \mathcal{S}^{\sharp\sharp}(R). \tag{2.1}$$

LEMMA 2. *Let $\mathcal{I}$ be a countable set of continuous invariants. Then for $R = 2, 3, \ldots$, $\mathcal{S}^{\sharp\sharp}(R)$ is a compact subset of $(\mathcal{S}^{\sharp\sharp}, \varrho_{\mathcal{I}})$.*

Proof. Let $F_m \in \mathcal{S}^{\sharp\sharp}(R)$, $m = 1, 2, \ldots$. By the compactness of closed bounded intervals of $\mathbb{R}$, there exists a subsequence, which for ease of notation we still denote by (F_m), such that $r_m = r_0 \leq R$ and $m_{F_m} = m_0 \leq R$ for every m, and the sequences (Q_m), $(\lambda_{j,m})$, $(\mu_{j,m})$, (ω_m) and $(a_n(F_m))$ are convergent to Q_0, $\lambda_{j,0}$, $\mu_{j,0}$, ω_0 and $a_{n,0}$, respectively, all satisfying the above bounds. For $\sigma > R$ we put

$$F_0(s) = \sum_{n=1}^{\infty} \frac{a_{n,0}}{n^s},$$

which is well defined since as $m \to \infty$,

$$\sum_{n=1}^{\infty} \frac{|a_n(F_m)|}{n^R} \to \sum_{n=1}^{\infty} \frac{|a_{n,0}|}{n^R} \le R. \tag{2.2}$$

Our aim now is to prove that $F_0 \in \mathcal{S}^{\sharp\sharp}(R)$ and $F_m(s) \to F_0(s)$ as $m \to \infty$, with respect to the metric $\varrho_{\mathcal{I}}$, thus showing that $\mathcal{S}^{\sharp\sharp}(R)$ is compact.

We first prove that $F_0 \in \mathcal{S}^{\sharp\sharp}(R)$. By the definition of $\mathcal{S}^{\sharp\sharp}(R)$ and the choice of m_0 the functions

$$H_m(s) = (s-1)^{m_0} F_m(s)$$

are entire of order $\le R$. Moreover, by the functional equation, for $t \in \mathbb{R}$ we have

$$\begin{aligned} |H_m(1-R+it)| &\le (R+|t|)^R \frac{|\gamma_{F_m}(R+it)|}{|\gamma_{F_m}(1-R+it)|} |F_m(R+it)| \\ &\le c_0(R)(|t|+2)^{c_1(R)} \end{aligned}$$

for some constants $c_j(R)$, $j = 0, 1$, hence by the Phragmén–Lindelöf theorem we get

$$|H_m(\sigma+it)| \le c_0(R)(|t|+2)^{c_1(R)}, \qquad \sigma \ge 1-R.$$

Hence there exists a subsequence of $(H_m(s))$ which converges to

$$H_0(s) = (s-1)^{m_0} \widetilde{F_0}(s)$$

uniformly over compact sets in $\sigma \ge 1-R$; note that $\widetilde{F_0}(s)$ is meromorphic for $\sigma > 1-R$ with at most a pole of order $\le R$ at $s = 1$. But $(H_m(s))$ is convergent to $(s-1)^{m_0} F_0(s)$ for $\sigma > R$, thus $F_0(s) = \widetilde{F_0}(s)$ for $\sigma > R$, giving a meromorphic continuation of $F_0(s)$ to $\sigma \ge 1-R$ with at most a pole of order $\le R$ at $s = 1$. Writing

$$\gamma_0(s) = Q_0^s \prod_{j=1}^{r_0} \Gamma(\lambda_{j,0} s + \mu_{j,0}),$$

we have

$$\gamma_0(s) = \lim_{m\to\infty} \gamma_m(s), \quad \gamma_m(s) = Q_m^s \prod_{j=1}^{r_0} \Gamma(\lambda_{j,m} s + \mu_{j,m})$$

uniformly over compact sets of $\mathbb{C}$ not containing the poles of the $\gamma_m(s)$'s, and for $1-R \le \sigma \le R$ the function $F_0(s)$ satisfies the functional equation

$$\gamma_0(s) F_0(s) = \omega_0 \overline{\gamma}_0(s) \overline{F}_0(s). \tag{2.3}$$

This provides a meromorphic continuation of $F_0(s)$ to $\mathbb{C}$. Moreover, the bound

$$|F_0(s)| \le e^{|s|^R}, \qquad |s| \ge 2, \tag{2.4}$$

follows by a limiting process from the same bounds for the $F_m(s)$'s. Thus, $F_0 \in \mathcal{S}^{\sharp\sharp}(R)$ in view of (2.2)–(2.4).

Finally, since the I_j are continuous invariants and $a_n(F_m) \to a_n(F_0)$, from (1.4) we have for every positive ε

$$\begin{aligned}\varrho_{\mathcal{I}}(F_m, F_0) &= \sum_{j\in J} \frac{1}{2^j} \frac{|I_j(F_m) - I_j(F_0)|}{1 + |I_j(F_m) - I_j(F_0)|} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|a_n(F_m) - a_n(F_0)|}{1 + |a_n(F_m) - a_n(F_0)|} \\ &\le \sum_{j\in J\cap[1,N]} |f_{I_j,r_0}(Q_m, \boldsymbol{\lambda}_m, \boldsymbol{\mu}_m, \omega_m) - f_{I_j,r_0}(Q_0, \boldsymbol{\lambda}_0, \boldsymbol{\mu}_0, \omega_0)| \\ &\quad + \sum_{j=1}^{N} |a_n(F_m) - a_n(F_0)| + \frac{\varepsilon}{3} \le \varepsilon\end{aligned}$$

for N and $m \ge m_0(N)$ sufficiently large, so $\varrho_{\mathcal{I}}(F_m, F_0) \to 0$ and the lemma follows. ∎

Lemma 3. *Let $\mathcal{I}$ be a countable set of continuous invariants. Then $\mathcal{S}, \mathcal{S}^\sharp \in \mathcal{B}(\mathcal{I})$.*

Proof. From the well known formula for the abscissa of absolute convergence of Dirichlet series we see that a function $F \in \mathcal{S}^{\sharp\sharp}$ belongs to $\mathcal{S}^\sharp$ if and only if for every $\varepsilon > 0$ and $N \in \mathbb{N}$, $N > N(\varepsilon)$,

$$\sum_{n\le N} |a_n(F)| \le N^{1+\varepsilon}.$$

Therefore, for $\varepsilon > 0$ and $N \in \mathbb{N}$ we consider the function $f_{N,\varepsilon} : \mathcal{S}^{\sharp\sharp} \to \mathbb{R}$ defined by

$$f_{N,\varepsilon}(F) = \frac{1}{N^{1+\varepsilon}} \sum_{n\le N} |a_n(F)|.$$

Since $F \mapsto a_n(F)$ is continuous with respect to $\varrho_{\mathcal{I}}$, $f_{N,\varepsilon}(F)$ is also continuous with respect to $\varrho_{\mathcal{I}}$. Moreover, $\mathcal{S}^\sharp$ can be characterized as

$$\mathcal{S}^\sharp = \bigcap_{\varepsilon\in\mathbb{Q}^+} \bigcup_{M=1}^{\infty} \bigcap_{N=M}^{\infty} f_{N,\varepsilon}^{-1}([0,1]), \tag{2.5}$$

where $\mathbb{Q}^+$ denotes the positive rational numbers. Since $f_{N,\varepsilon}^{-1}([0,1])$ is a closed subset of $\mathcal{S}^{\sharp\sharp}$, (2.5) shows that $\mathcal{S}^\sharp$ is a Borel subset of $\mathcal{S}^{\sharp\sharp}$.

In order to deal with $\mathcal{S}$ we first consider

$$\mathcal{S}^\sharp(1) = \{F \in \mathcal{S}^\sharp : a_1(F) = 1\} = \mathcal{S}^\sharp \cap a_1^{-1}(\{1\}).$$

In view of the first part of the lemma, $\mathcal{S}^\sharp(1)$ is a Borel subset of $\mathcal{S}^{\sharp\sharp}$. For

$F \in \mathcal{S}^\sharp(1)$ let $\sigma_1(F) \geq 1$ be such that

$$\sum_{n=2}^{\infty} \frac{|a_n(F)|}{n^\sigma} < 1 \qquad \text{for } \sigma > \sigma_1(F).$$

Then for $\sigma > \sigma_1(F)$ the function $\log F(s)$ is well defined, and by Taylor's expansion we have

$$\log F(s) = \sum_{n=2}^{\infty} \frac{b_n(F)}{n^s}$$

with

$$b_n(F) = \sum_{m=1}^{\Omega(n)} \frac{(-1)^{m+1}}{m} \sum_{\substack{n_1 \geq 2, \ldots, n_m \geq 2 \\ n_1 \cdots n_m = n}} a_{n_1}(F) \cdots a_{n_m}(F),$$

where $\Omega(n)$ denotes the total number of prime factors of n. Thus the functions $F \mapsto b_n(F)$, $n = 2, 3, \ldots$, are continuous on $\mathcal{S}^\sharp(1)$ with respect to $\varrho_{\mathcal{I}}$. In order to deal with the Euler product axiom, for $(n,m) = 1$ we put

$$g_{n,m}(F) = a_n(F)a_m(F) - a_{nm}(F),$$

and for $\theta < 1/2$ we write

$$h_{n,\theta}(F) = n^{-\theta}|b_n(F)|;$$

note that $b_n(F) \ll n^\theta$ for some $\theta < 1/2$ is equivalent to $|b_n(F)| \leq n^\theta$ for some $0 < \theta < 1/2$ and $n \geq n(\theta)$. Moreover, in order to deal with the Ramanujan conjecture axiom, for every $\varepsilon > 0$ we define

$$l_{n,\varepsilon}(F) = n^{-\varepsilon}|a_n(F)|.$$

The three functions $g_{n,m}(F)$, $h_{n,\theta}(F)$, $l_{n,\varepsilon}(F)$ are continuous on $\mathcal{S}^\sharp(1)$ with respect to $\varrho_{\mathcal{I}}$, and $\mathcal{S}$ can be characterized as

$$\mathcal{S} = \mathcal{S}^\sharp(1) \cap \bigcap_{(n,m)=1} g_{n,m}^{-1}(\{0\}) \cap \bigcup_{\substack{0<\theta<1/2 \\ \theta \in \mathbb{Q}}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} h_{n,\theta}^{-1}([0,1])$$
$$\cap \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{K=1}^{\infty} \bigcap_{n=1}^{\infty} l_{n,\varepsilon}^{-1}([0,K]),$$

and the result follows as for $\mathcal{S}^\sharp$, thus proving the lemma. ∎

Theorem 1 follows at once from (2.1) and Lemmas 2 and 3. ∎

To prove Theorem 2, let $I \in \mathcal{I}$, $B \in \mathcal{B}(\mathcal{I})$ and $\mathcal{S}^{\sharp\sharp}(R)$ be as in (2.1). Writing

(2.6) $$B_R = B \cap \mathcal{S}^{\sharp\sharp}(R),$$

we see that B_R is a Borel set of the compact metric space $(\mathcal{S}^{\sharp\sharp}(R), \varrho_{\mathcal{I}})$. Moreover, by Proposition 4.2 of [8], $(\mathcal{S}^{\sharp\sharp}(R), \varrho_{\mathcal{I}})$ is a Polish space (see Definition 3.1 of [8]). Hence, by Theorem 13.7 (see also p. 85) of [8], B_R is analytic (in Suslin's sense, see Definition 14.1 of [8]). Therefore, by Proposition 14.4 of [8], $I(B_R)$ is analytic as well since I is continuous from $(\mathcal{S}^{\sharp\sharp}(R), \varrho_{\mathcal{I}})$ to $\mathbb{C}$ (and hence is a Borel map, see pp. 70–71 of [8]) and $\mathbb{C}$ is obviously a Polish space. Finally, by Theorem 21.10 of [8], $I(B_R)$ is Lebesgue measurable, and hence

$$I(B) = \bigcup_{R=2}^{\infty} I(B_R)$$

is Lebesgue measurable as well. ∎

The proof of Corollary 1 is very simple. Let I be a continuous invariant and $\mathcal{I} = \{I_0, I\}$. Since $B \in \mathcal{B}(I_0)$, it follows that $B \in \mathcal{B}(\mathcal{I})$, hence $I(B)$ is Lebesgue measurable by Theorem 2. ∎

We need two lemmas for the proof of Theorem 3. We recall that a *topological semigroup* $(G, \cdot)$ is a semigroup where the multiplication $\cdot$ from $G \times G$ to G is continuous.

LEMMA 4. *Let $\mathcal{I}$ be a countable family of continuous invariants and suppose that every $I \in \mathcal{I}$ is additive or multiplicative. Then $(\mathcal{S}^{\sharp\sharp}, \varrho_{\mathcal{I}})$ is a topological semigroup.*

Proof. We have to prove that the usual multiplication in $\mathcal{S}^{\sharp\sharp}$ is continuous with respect to the metric $\varrho_{\mathcal{I}}$. Let $I \in \mathcal{I}$ and write $*$ for the sum (resp. product) if I is additive (resp. multiplicative). Let $F_m \to F_0$ and $G_m \to G_0$ be two convergent sequences in $(\mathcal{S}^{\sharp\sharp}, \varrho_{\mathcal{I}})$. Since the functions in Lemma 1 are continuous, we see that as $m \to \infty$,

$$I(F_m G_m) = I(F_m) * I(G_m) \to I(F_0) * I(G_0) = I(F_0 G_0)$$

for every $I \in \mathcal{I}$, and

$$a_n(F_m G_m) = \sum_{d|n} a_d(F_m) a_{n/d}(G_m) \to \sum_{d|n} a_d(F_0) a_{n/d}(G_0) = a_n(F_0 G_0)$$

for every $n \in \mathbb{N}$. Hence $F_m G_m \to F_0 G_0$ with respect to $\varrho_{\mathcal{I}}$, and the lemma follows. ∎

Recalling that B_R is defined by (2.6), we have

LEMMA 5. *Let $\mathcal{I}$ be a countable family of continuous invariants, let $B \in \mathcal{B}(\mathcal{I})$ with $1 \in B$, and G be the semigroup generated by B. Then*

$$G = \bigcup_{R=2}^{\infty} \bigcup_{k=1}^{\infty} B_R^k.$$

Proof. The inclusion $\supset$ is obvious. To prove the opposite inclusion, given $F \in G$ we have $F(s) = \prod_{j=1}^{k} F_j(s)$ with some $k \in \mathbb{N}$ and $F_j \in B$. Then $F_j \in \mathcal{S}^{\sharp\sharp}(R_j)$ for some R_j, hence writing $R = \max\{R_1, \ldots, R_k\}$ we have $\{F_1, \ldots, F_k\} \subset B_R$. Therefore $F \in B_R^k$, and the lemma follows. ∎

In order to prove Theorem 3, we first note that clearly $\{1\} \in \mathcal{B}(\mathcal{I})$, and we may always assume that G is generated by a set $B \in \mathcal{B}(\mathcal{I})$, where $1 \in B$ (in fact, if B is an $\mathcal{I}$-Borel set then $B \cup \{1\}$ is an $\mathcal{I}$-Borel set as well and generates the same semigroup). The proof of Theorem 3 now follows the lines of the proof of Theorem 2, hence we only give a sketch. B_R is a Borel set of the Polish space $(\mathcal{S}^{\sharp\sharp}(R), \varrho_{\mathcal{I}})$, hence it is analytic in Suslin's sense. Moreover, by Lemma 4, multiplication is a continuous function, therefore B_R^k is also analytic. Since the invariant I is continuous, $I(B_R^k)$ is analytic as well, and hence Lebesgue measurable. Thus, by Lemma 5, $I(G)$ is Lebesgue measurable. ∎

The proof of Corollary 2 is similar to the proof of Corollary 1. ∎

Given a set $\mathcal{A} \subset \mathbb{R}$, $\mathcal{A} + \mathcal{A}$ denotes as usual the set of real numbers of the form $a + a'$ with $a, a' \in \mathcal{A}$. In order to prove the first part of Theorem 4 we recall that if $\mathcal{A}$ is measurable with $\mu(\mathcal{A}) > 0$, then $\mathcal{A} + \mathcal{A}$ contains an open interval; see Exercise 19 of Ch. 9 of Rudin [11]. Suppose now that $\mu(I(G)) > 0$. Since G is a semigroup and I is additive, we have

$$I(G) + I(G) \subset I(G),$$

hence there exists an interval $(a, b) \subset I(G)$. Therefore, again since G is a semigroup, for every positive integer k we have $(ka, kb) \subset I(G)$. Thus $I(G)$ contains arbitrarily long intervals. Let $F_0 \in G$ with $I(F_0) \neq 0$ and let $U_0 \subset I(G)$ be an interval of length $> |I(F_0)|$. Then

$$\bigcup_{k=1}^{\infty} (kI(F_0) + U_0) \subset I(G),$$

and such a union is a half-line, thus proving the first part of Theorem 4.

The second option of the second part of Theorem 4 follows at once from the first part. In fact, let I be multiplicative with values in $\mathbb{R}^+$. Write $\log I(G) = \{\log I(F) : F \in G\}$. The function $F \mapsto \log I(F)$ is a real-valued additive continuous invariant. Moreover, if $\mu(I(G)) > 0$ then $\mu(\log I(G)) > 0$ as well, so $\log I(G)$ contains a half-line by the first part of Theorem 4, and hence $I(G)$ contains a half-line too.

In order to prove the first option of the second part of Theorem 4, we first remark that a variant of the above mentioned exercise reads as follows. Let $\mathcal{A} \subset T^1$ and write $\mathcal{A}\mathcal{A} = \{aa' : a, a' \in \mathcal{A}\}$; if $\mathcal{A}$ is measurable and $\mu(\mathcal{A}) > 0$, then $\mathcal{A}\mathcal{A}$ contains an arc. Suppose now that $\mu(I(G)) > 0$ and argue as in the first part. Since I is multiplicative we have

$$I(G)I(G) \subset I(G),$$

thus $I(G)$ contains an arc. Hence there exists $F_0 \in G$ such that $I(F_0) = e^{2\pi i\theta_0}$ with $\theta_0 \notin \mathbb{Q}$, therefore the set $\{I(F_0^k)\}_{k\in\mathbb{N}}$ is dense in T^1. But then

$$T^1 = \bigcup_{k=1}^{\infty} I(F_0^k)I(G) \subset I(G),$$

and Theorem 4 is proved. ■

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