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Torsion points in families of Drinfeld modules

by

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1. Introduction. Lang [22] proved that if a curve $C \subset \mathbb{A}^2$ contains infinitely many points whose coordinates are both roots of unity, then C is the zero set of an equation of the form $X^m Y^n = \alpha$ for some $m, n \in \mathbb{Z}$ (not both equal to 0) and some root of unity α. In this case we note that if C projects dominantly on both axes, then for *each* point $(x, y) \in C$ we know that x is a root of unity *if and only if* y is a root of unity. In particular, the following result is a corollary of Lang's Theorem.

Corollary 1.1. *Let $F_1, F_2 \in \mathbb{C}(z)$ be nonconstant rational maps such that there exist infinitely many $\lambda \in \mathbb{C}$ such that both $F_1(\lambda)$ and $F_2(\lambda)$ are roots of unity. Then F_1 and F_2 are multiplicatively dependent, and therefore for each $\lambda \in \mathbb{C}$, $F_1(\lambda)$ is a root of unity if and only if $F_2(\lambda)$ is a root of unity.*

Lang's result is a special case of the Manin–Mumford Conjecture (proven by Raynaud [28, 29] for abelian varieties, and by Hindry [20] for semiabelian varieties). The Manin–Mumford Conjecture (in its most general form asked by Lang) predicts that the set of torsion points of a semiabelian variety G defined over $\mathbb{C}$ is not Zariski dense in a subvariety V of G, unless V is a translate of an algebraic subgroup of G by a torsion point. Pink and Zilber have suggested extending the Manin–Mumford Conjecture to a more general question regarding unlikely intersections between a subvariety V of a semiabelian scheme G and algebraic subgroups of the fibers of G having codimension greater than the dimension of V (see [6, 19, 24, 25, 26]).

Masser and Zannier [24, 25] studied the Pink–Zilber Conjecture when G is the square of the Legendre family of elliptic curves. We state below a special case of their theorem.

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Theorem 1.2 (Masser–Zannier). *For each $\lambda \in \mathbb{C} \setminus \{0,1\}$, let E_λ be the elliptic curve given by the equation $Y^2 = X(X-1)(X-\lambda)$. Let P_λ and Q_λ be two given families of points on E_λ depending algebraically on the parameter λ. Suppose there exist infinitely many $\lambda \in \mathbb{C}$ such that both P_λ and Q_λ are torsion points for E_λ. Then the points P_λ and Q_λ are linearly dependent over $\mathbb{Z}$ on the generic fiber of the elliptic surface E_λ.*

Both Lang's result and Masser–Zannier's result are special cases of the following problem. Let $\{G_\lambda\}$ be an algebraic family of algebraic groups, and let $\{P_\lambda\}$ and $\{Q_\lambda\}$ be two algebraic families of points on G_λ. If there exist infinitely many λ such that both P_λ and Q_λ are torsion points, then at least one of the following three properties holds:

(1) P_λ is torsion for all λ.
(2) Q_λ is torsion for all λ.
(3) For each λ, P_λ is a torsion point if and only if Q_λ is a torsion point.

In the arithmetic theory of function fields of positive characteristic, *Drinfeld modules* play a role similar to elliptic curves over number fields. Hence, it is natural to ask if there exist results for the family of Drinfeld modules that are parallel to those obtained by Masser and Zannier.

In this paper we study the first instance of the above problem in characteristic p, where $G_\lambda := (\mathbb{G}_{\mathrm{a}}, \Phi^\lambda)$ is the constant family of additive group schemes endowed with the action of a Drinfeld module Φ^λ (which belongs to an algebraic family of such Drinfeld modules). More precisely, let K be a field extension of $\mathbb{F}_q(t)$ and let $K(z)$ be the rational function field in variable z. Let $\Phi : \mathbb{F}_q[t] \to \mathrm{End}_{K(z)}(\mathbb{G}_{\mathrm{a}})$ be a Drinfeld module defined over $K(z)$ (see Section 2 for details). Equivalently, Φ is also regarded as a family of Drinfeld modules by specialization. That is, by letting $z = \lambda$ for λ in the separable closure K^{sep} of K inside $\overline{K}$, we obtain an algebraic family of Drinfeld modules, denoted by Φ^λ, which are defined over K^{sep}. Then we are able to prove that the above trichotomy must hold.

The following result is a special case of our more general result (Theorem 2.6).

Theorem 1.3. *Let $r \geq 2$ be an integer, let q be a power of a prime number p, let K be a finitely generated extension of $\mathbb{F}_q(t)$, and let $g_1, \ldots, g_{r-1} \in K[z]$. We let $\Phi : \mathbb{F}_q[t] \to \mathrm{End}_{K(z)}(\mathbb{G}_{\mathrm{a}})$ be the family of Drinfeld modules defined by*

$$\Phi_t(x) = tx + \sum_{i=1}^{r-1} g_i(z) x^{q^i} + x^{q^r}.$$

Let $\mathbf{a}, \mathbf{b} \in K[z]$ and assume there exist infinitely many $\lambda \in K^{\mathrm{sep}}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are torsion points for Φ^λ. Then at least one of the following properties holds:

(1) $\mathbf{a}$ *is a torsion point for* Φ.
(2) $\mathbf{b}$ *is a torsion point for* Φ.
(3) *For each* $\lambda \in \overline{K}$, $\mathbf{a}(\lambda)$ *is a torsion point for* Φ^λ *if and only if* $\mathbf{b}(\lambda)$ *is a torsion point for* Φ^λ.

For certain families of Drinfeld modules and for constant starting points $\mathbf{a}, \mathbf{b}$ (i.e., with no dependence on z) we can give an explicit condition for there to exist infinitely many $\lambda \in K^{\mathrm{sep}}$ such that both $\mathbf{a}$ and $\mathbf{b}$ are torsion points for Φ^λ.

THEOREM 1.4. *Let* $r \geq 2$ *be an integer and let* K *be a finitely generated extension of* $\mathbb{F}_q(t)$ *such that* $\mathbb{F}_q$ *is algebraically closed in* K. *Let* $\mathbf{a}, \mathbf{b} \in K$ *and let* $\Phi : \mathbb{F}_q[t] \to \mathrm{End}_{K(z)}(\mathbb{G}_\mathrm{a})$ *be the family of Drinfeld modules given by*

$$\Phi_t(x) = tx + zx^q + x^{q^r}.$$

If there exist infinitely many $\lambda \in K^{\mathrm{sep}}$ *such that both* $\mathbf{a}$ *and* $\mathbf{b}$ *are torsion points for* Φ^λ, *then* $\mathbf{a}$ *and* $\mathbf{b}$ *are linearly dependent over* $\mathbb{F}_q$. *Moreover, for* $\lambda \in \overline{K}$, $\mathbf{a}$ *is torsion for* Φ^λ *if and only if* $\mathbf{b}$ *is torsion for* Φ^λ.

Going in the opposite direction, when K is a constant field extension of $\mathbb{F}_q(t)$ we can also give a more precise relation between $\mathbf{a}$ and $\mathbf{b}$.

THEOREM 1.5. *Let* $r \geq 2$ *be an integer, let* q *be a power of a prime number* p, *let* $\mathbf{a}, \mathbf{b} \in K = \overline{\mathbb{F}_q}(t)$ *and let* $\Phi : \mathbb{F}_q[t] \to \mathrm{End}(\mathbb{G}_\mathrm{a})$ *be the family of Drinfeld modules given by*

$$\Phi_t(x) = tx + zx^q + x^{q^r}.$$

If there exist infinitely many $\lambda \in K^{\mathrm{sep}}$ *such that both* $\mathbf{a}$ *and* $\mathbf{b}$ *are torsion points for* Φ^λ, *then* $\mathbf{a}$ *and* $\mathbf{b}$ *are linearly dependent over* $\mathbb{F}_q$. *Moreover, for* $\lambda \in \overline{K}$, $\mathbf{a}$ *is torsion for* Φ^λ *if and only if* $\mathbf{b}$ *is torsion for* Φ^λ.

Both Theorems 1.4 and 1.5 should be true without any extra condition on the (constant) starting points $\mathbf{a}$ and $\mathbf{b}$. However, our methods cannot be extended to this general case. In order to prove Theorems 1.4 and 1.5 we employ a finer analysis of the valuation of torsion points for Drinfeld modules which fails for arbitrary extension fields K of $\mathbb{F}_q(t)$. Also, for the family of Drinfeld modules from Theorems 1.4 and 1.5, we prove in Proposition 6.2 that for almost all $\mathbf{c} \in \mathbb{F}_q[t]$ there are indeed infinitely many $\lambda \in \mathbb{F}_q(t)^{\mathrm{sep}}$ such that $\mathbf{c}$ is torsion for Φ^λ, thus justifying that the conclusion in both Theorems 1.4 and 1.5 is not vacuous.

A question analogous to the problem studied by Masser and Zannier in the setting of arithmetic dynamics is the following. Given complex numbers a and b and an integer $d \geq 2$, when do there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic for the action of $f_\lambda(x) := x^d + \lambda$ on $\mathbb{C}$? This question was first raised by Zannier at an American Institute of Mathematics workshop in 2008 and further studied by Baker and DeMarco in [1]. They show that this happens if and only if $a^d = b^d$. In [15], the present authors together with Tucker extended Baker–DeMarco's result to general families of polynomials.

On the other hand, a Drinfeld module can be viewed as a collection of (additive) polynomials acting on the affine line—so we can apply the techniques and results that have been developed in arithmetic dynamics into the study of Drinfeld modules, especially to Diophantine problems arising from Drinfeld modules; this point of view will prove useful in our paper.

The problem studied in this paper naturally can be viewed as an analogue of the problems from [1] and [15] in the setting of Drinfeld modules. Hence, our method for studying the question raised above follows closely the methods used in [1]. We will apply the techniques used in [15] to our situation. The reader will find the material in Section 3 and results in Section 4 parallel to [15, Sections 4–6].

However, as we work in the world of positive characteristic, we cannot expect that all the techniques developed in [1] and [15] can be applied to our situation. This paper reveals the differences and difficulties in the study of the same questions in the setting of Drinfeld modules. In particular, we cannot use complex analysis in order to derive the explicit relations between **a** and **b** as in Theorems 1.4 and 1.5; instead we use arguments from valuation theory which are amenable to Drinfeld modules. Also, in all of our results we need to restrict to specializations $\lambda \in K^{\mathrm{sep}}$ rather than $\lambda \in \overline{K}$ since the powerful equidistribution results on Berkovich spaces that we will be using are also restricted to separable points.

The plan of our paper is as follows. In Section 2 we set up our notation and state our main result (Theorem 2.6), and then we describe the method of our proof. In Section 3 we give a brief overview of Berkovich spaces. Then, in Section 4 we compute the capacities of the generalized v-adic Mandelbrot sets associated to a generic point **c** for a family of Drinfeld modules. We proceed with the proof of Theorem 2.6 in Section 5. Then, in Section 6 we conclude the paper by proving Theorems 1.3–1.5.

2. Preliminaries and statement of the main results. In this section, we give a brief review of the theory of Drinfeld modules and height functions that is relevant to the discussion below. Throughout the paper, we let q be a power of a prime p and let $\mathbb{F}_q$ be a finite field of q elements.

2.1. Drinfeld modules. We first define Drinfeld modules (of generic characteristic)—for more details, see [18]. Let L be a field extension of the rational function field $\mathbb{F}_q(t)$. A *Drinfeld* $\mathbb{F}_q[t]$*-module* defined over L is an $\mathbb{F}_q$-algebra homomorphism $\Phi : \mathbb{F}_q[t] \to \mathrm{End}_L(\mathbb{G}_{\mathrm{a}})$ such that

$$\Phi_t(x) := tx + \sum_{i=1}^{r-1} a_i x^{q^i} + x^{q^r} \qquad \text{with } a_i \in L \text{ for all } i = 1, \ldots, r-1,$$

where Φ_a denotes the image of $a \in \mathbb{F}_q[t]$ under Φ, and $r \geq 1$ is called the *rank* of Φ. Note that as an $\mathbb{F}_q$-algebra homomorphism, Φ is uniquely determined by the action of Φ_t on $\mathbb{G}_{\mathrm{a}}$. In general, $\Phi_t(x)$ is not required to be monic in x. However, at the expense of replacing Φ by a Drinfeld module which is conjugate to it, i.e.

$$\Psi_t(x) = \gamma^{-1}\Phi_t(\gamma x), \tag{2.1}$$

for a suitable $\gamma \in \overline{L}$ we can reduce ourselves to the case that Ψ_t is monic. Note that Ψ is defined over $L(\gamma)$ which is a finite extension of L.

A point $x \in \overline{L}$ is called *torsion* for Φ if there exists a nonzero $a \in \mathbb{F}_q[t]$ such that x is in the kernel of Φ_a. Since each Φ_a is a separable polynomial, we conclude that actually x lives in the separable closure L^{sep} of L inside $\overline{L}$. We denote by Φ_{tor} the set of all torsion points for Φ. It is immediate to see that x is torsion if and only if its orbit under the action of Φ_t is finite, i.e. x is preperiodic for the map Φ_t. Note that if Ψ is a Drinfeld module conjugate to Φ as in (2.1), then $x \in \Phi_{\mathrm{tor}}$ if and only if $\gamma^{-1}x \in \Psi_{\mathrm{tor}}$.

2.2. Canonical heights. Let K be a finitely generated transcendental extension of $\mathbb{F}_q$. At the expense of replacing K by a finite extension and replacing $\mathbb{F}_q$ with its algebraic closure in this finite extension, we may assume there exists a smooth projective, geometrically irreducible variety $\mathcal{V}$ defined over $\mathbb{F}_q$ whose function field is K (see [10]). We let then $\Omega := \Omega_K$ be the set of places of K corresponding to the codimension one irreducible subvarieties of $\mathcal{V}$. Furthermore, we normalize the absolute values $|\cdot|_v$ so that for each $\alpha \in K^*$ we have

$$\prod_{v \in \Omega} |\alpha|_v = 1. \tag{2.2}$$

For more details see [23, §2.3] or [5, §1.4.6]. We note that if $x \in K$ is a v-adic unit for all $v \in \Omega_K$, then $x \in \mathbb{F}_q$.

Let $\mathbb{C}_v$ be a fixed completion of the algebraic closure of a completion K_v of $(K, |\cdot|_v)$. Note that there is a unique extension of $|\cdot|_v$ to an absolute value on $\mathbb{C}_v$. By abuse of notation, we still denote this extension by $|\cdot|_v$. Let Φ be a Drinfeld module of rank r defined over $\mathbb{C}_v$. Following Poonen [27] and Wang [30], for each $x \in \mathbb{C}_v$, the *local canonical height* of x is defined as

$$\widehat{\mathrm{h}}_{\Phi,v}(x) := \lim_{n\to\infty} \frac{\log^+ |\Phi_{t^n}(x)|_v}{q^{rn}}, \tag{2.3}$$

where $\log^+ z$ always denotes $\log\max\{z, 1\}$ (for any real number z). It is immediate that $\widehat{\mathrm{h}}_{\Phi,v}(\Phi_{t^i}(x)) = q^{ir}\,\widehat{\mathrm{h}}_{\Phi,v}(x)$, and thus $\widehat{\mathrm{h}}_{\Phi,v}(x) = 0$ whenever $x \in \Phi_{\mathrm{tor}}$.

Now, if $f(x) = \sum_{i=0}^{d} a_i x^i$ is any polynomial defined over $\mathbb{C}_v$, then $|f(x)|_v = |a_d x^d| > |x|_v$ when $|x|_v > r_v$, where

$$r_v = r_v(f) := \max\{1, \max\{|a_i/a_d|^{1/(d-i)}\}_{0 \le i < d}\}. \tag{2.4}$$

Moreover, for a Drinfeld module Φ, if $|x|_v > r_v(\Phi_t)$ then $\widehat{\mathrm{h}}_{\Phi,v}(x) = \log |x|_v > 0$. For more details see [17] and [21].

We fix an algebraic closure $\overline{K}$ of K, and for each $v \in \Omega_K$ we fix an embedding $\overline{K} \hookrightarrow \mathbb{C}_v$. The *global canonical height* $\widehat{\mathrm{h}}_{\Phi}(x)$ associated to the Drinfeld module Φ was first introduced by Denis [11] (Denis defined the global canonical heights for general T-modules which are higher dimensional analogue of Drinfeld modules). For each $x \in \overline{K}$, the *global canonical height* is defined as

$$\widehat{\mathrm{h}}_{\Phi}(x) = \lim_{n\to\infty} \frac{h\left(\Phi_{t^n}(x)\right)}{q^{rn}},$$

where h is the usual (logarithmic) Weil height on $\overline{K}$. As shown in [27] and [30], the global canonical height decomposes into a sum of the corresponding local canonical heights. Furthermore, $\widehat{\mathrm{h}}_{\Phi}(x) = 0$ if and only if $x \in \Phi_{\mathrm{tor}}$ (see [30, Proposition 3(v)]).

Remark 2.5. (1) We note that the theory of canonical height associated to a Drinfeld module is a special case of the canonical heights associated to morphisms on algebraic varieties developed by Call and Silverman (see [8] for details).

(2) The definition for the canonical height functions given above seems to depend on the particular choice of the map Φ_t. On the other hand, one can define the canonical heights $\widehat{\mathrm{h}}_{\Phi}$ as in [11] by letting

$$\widehat{\mathrm{h}}_{\Phi}(x) = \lim_{\deg(R)\to\infty} \frac{h\left(\Phi_R(x)\right)}{q^{r\deg(R)}},$$

and a similar formula (see (2.3)) for canonical local heights $\widehat{\mathrm{h}}_{\Phi,v}(x)$ where R runs through all nonconstant polynomials in $\mathbb{F}_q[t]$. In [27] and [30] it is proven that both definitions yield the same height function.

Let $\Phi : \mathbb{F}_q[t] \to \mathrm{End}_L(\mathbb{G}_{\mathrm{a}})$ be the Drinfeld module as in Theorem 1.3 with $L = K(z)$. Then our main result is the following.

Theorem 2.6. *Let K be a finitely generated extension of $\mathbb{F}_q(t)$, let $r \ge 2$ be an integer, and let $g_1, \dots, g_{r-1} \in K[z]$. We let $\Phi : \mathbb{F}_q[t] \to \mathrm{End}_{K(z)}(\mathbb{G}_{\mathrm{a}})$ be the family of Drinfeld modules defined by*

$$\Phi_t(x) = tx + \sum_{i=1}^{r-1} g_i(z) x^{q^i} + x^{q^r}.$$

Let $\mathbf{a}, \mathbf{b} \in K[z]$ *and assume the following inequality holds:*

$$\min\{\deg(\mathbf{a}), \deg(\mathbf{b})\} > \max\left\{\frac{\deg(g_1)}{q^r - q}, \dots, \frac{\deg(g_{r-1})}{q^r - q^{r-1}}\right\}. \tag{2.7}$$

If there exist infinitely many $\lambda \in K^{\mathrm{sep}}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are torsion points for* Φ^λ*, then for each* $\lambda \in \overline{K}$*,* $\mathbf{a}(\lambda)$ *is torsion for* Φ^λ *if and only if* $\mathbf{b}(\lambda)$ *is torsion for* Φ^λ*. Furthermore, in this case, if* $C_{\mathbf{a}}$ *and* $C_{\mathbf{b}}$ *are the leading coefficients of* $\mathbf{a}$*, respectively of* $\mathbf{b}$*, then* $C_{\mathbf{a}}^{\deg(\mathbf{b})}/C_{\mathbf{b}}^{\deg(\mathbf{a})} \in \overline{\mathbb{F}_q}$*.*

Note that inequality (2.7) prevents $\mathbf{a}$ and $\mathbf{b}$ from being torsion points for *all* λ (see Proposition 4.2).

Remark 2.8. One can consider a more general Drinfeld module such that the additive group $\mathbb{G}_{\mathrm{a}}$ is equipped with an action by a ring A consisting of all regular functions on a smooth projective curve defined over $\mathbb{F}_q$ with a fixed point removed. In our definition above, the curve is $\mathbb{P}^1$ with the distinguished point at infinity. However, we lose no generality by restricting ourselves to the case $A = \mathbb{F}_q[t]$ since $\mathbb{F}_q[t]$ always embeds into such a ring A, and moreover, A is a finite integral extension of $\mathbb{F}_q[t]$. Indeed, for any two Drinfeld modules $\Phi : \mathbb{F}_q[t] \to \mathrm{End}(\mathbb{G}_{\mathrm{a}})$ and $\Phi' : A \to \mathrm{End}(\mathbb{G}_{\mathrm{a}})$ with $\Phi'|_{\mathbb{F}_q[t]} = \Phi$, $x \in \Phi_{\mathrm{tor}}$ if and only if $x \in \Phi'_{\mathrm{tor}}$ (provided A is a finite integral extension of $\mathbb{F}_q[t]$). One direction is obvious, while if $x \in \Phi'[a]$ for some $a \in A$, we let $c_0, c_1, \dots, c_{m-1} \in \mathbb{F}_q[t]$ (with $c_0 \neq 0$) be such that

$$a^m + c_{m-1}a^{m-1} + \dots + c_1 a + c_0 = 0,$$

and then $x \in \Phi[c_0]$, as desired.

Theorem 1.4 is an easy consequence of the following more general result.

Theorem 2.9. *Let* K *be a finitely generated extension of* $\mathbb{F}_q(t)$ *such that* $\mathbb{F}_q$ *is algebraically closed in* K*, let* $r \geq 2$ *be an integer, let* $\mathbf{a}, \mathbf{b} \in K$*, and let* $g_1, \dots, g_{r-1} \in K[z]$*. We let* $\Phi : \mathbb{F}_q[t] \to \mathrm{End}_{K(z)}(\mathbb{G}_{\mathrm{a}})$ *be the family of Drinfeld modules defined by*

$$\Phi_t(x) = tx + \sum_{i=1}^{r-1} g_i(z) x^{q^i} + x^{q^r}.$$

Assume there exists $i_0 \in \{1, \dots, r-1\}$ *such that*

$$\deg(g_{i_0}) > \deg(g_i) \quad \text{for each } i \neq i_0. \tag{2.10}$$

If there exist infinitely many $\lambda \in K^{\mathrm{sep}}$ *such that both* $\mathbf{a}$ *and* $\mathbf{b}$ *are torsion points for* Φ^λ*, then* $\mathbf{a}$ *and* $\mathbf{b}$ *are* $\mathbb{F}_q$*-linearly dependent. Moreover,* $\mathbf{a}$ *is torsion for* Φ^λ *if and only if* $\mathbf{b}$ *is torsion for* Φ^λ*.*

We note that the “moreover” clause follows immediately since we have $\Phi_{f(t)}(c \cdot \mathbf{a}) = c \cdot \Phi_{f(t)}(\mathbf{a})$ for each $c \in \mathbb{F}_q$, and for each $f(t) \in \mathbb{F}_q[t]$.

Our results and proofs are inspired by the results of [1] so that the strategy for the proof of Theorem 2.6 essentially follows the ideas in the paper [1]. However there are different technical details in our proofs. Also, Drinfeld modules are a better vehicle for the methods of [1] (see also [15, 16]).

Following [1], the strategy to prove Theorem 2.6 is the following. For the family of Drinfeld modules Φ, we define the v-adic generalized Mandelbrot sets $M_{\mathbf{c},v}$ (inside the Berkovich affine line) associated to any $\mathbf{c} \in K[z]$ similar to that introduced in [1]. We then use the equidistribution result discovered independently by Baker–Rumely [2], Chambert-Loir [9] and Favre–Rivera-Letelier [12, 13] to deduce that for any given points $\mathbf{a}$ and $\mathbf{b}$ the v-adic Mandelbrot sets $M_{\mathbf{a},v}$ and $M_{\mathbf{b},v}$ are equal for each $v \in \Omega_K$. Here we use the version obtained by Baker and Rumely [2], which connects the equidistribution result to arithmetic capacities; we apply it in the final step of the proof to show that the leading coefficients of $\mathbf{a}$ and $\mathbf{b}$ have the desired relation.

In order to apply the equidistribution results and capacity theory over nonarchimedean fields mentioned above, we need to introduce the Berkovich space associated to the affine line, which we overview in the next section.

3. Equidistribution theorem and Berkovich spaces. As mentioned above, we will need to apply the arithmetic equidistribution discovered independently by Baker–Rumely, Chambert-Loir and Favre–Rivera-Letelier. When the base field is a nonarchimedean field, the equidistribution theorem is best stated over the Berkovich space associated to the underlying variety in question. For the convenience of the reader, we review the Berkovich spaces in this section.

We use the version of the equidistribution theorem obtained by Baker and Rumely which connects the equidistribution result to the theory of arithmetic capacities. Hence, the material presented in this section is mainly from the book [3] by Baker and Rumely. The main body of this section is taken from [15, Section 4] which is written according to the summary in [1]. For more details on arithmetic equidistribution as well as a detailed introduction to the Berkovich line we refer the reader to [3].

Let K be a field of characteristic p endowed with a product formula, and let Ω_K be the set of its inequivalent absolute values. For each $v \in \Omega := \Omega_K$, we let $\mathbb{C}_v$ be the completion of an algebraic closure of the completion of K at v. Let $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ denote the Berkovich affine line over $\mathbb{C}_v$ (see [4] or [3, §2.1] for details). Then $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ is a locally compact, Hausdorff, path-connected space containing $\mathbb{C}_v$ as a dense subspace (with the topology induced by the v-adic absolute value). As a topological space, $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ is the set consisting of all multiplicative seminorms, denoted by $[\cdot]_x$, on $\mathbb{C}_v[T]$ extending the absolute value $|\cdot|_v$ on $\mathbb{C}_v$ endowed with the weakest topology such that the map $[\cdot]_z \mapsto [f]_z$ is continuous for all $f \in \mathbb{C}_v[T]$.

The set of seminorms can be described as follows. If $\{D(c_i, r_i)\}_i$ is any decreasing nested sequence of closed disks centered at $c_i \in \mathbb{C}_v$ of radius $r_i \geq 0$, then the map $f \mapsto \lim_{i\to\infty} [f]_{D(c_i,r_i)}$ defines a multiplicative seminorm on $\mathbb{C}_v[T]$ where $[f]_{D(c_i,r_i)}$ is the sup-norm of f over the closed disk $D(c_i, r_i)$.

Berkovich's classification theorem says that there are exactly four types of points, Type I, II, III, and IV. The first three types of points can be described in terms of closed disks $\zeta = D(c, r) = \bigcap D(c_i, r_i)$ where $c \in \mathbb{C}_v$ and $r \geq 0$. The corresponding multiplicative seminorm is just $f \mapsto [f]_{D(c,r)}$ for $f \in \mathbb{C}_v[T]$. Then ζ is of Type I, II or III if and only if $r = 0, r \in |\mathbb{C}_v^*|_v$ or $r \notin |\mathbb{C}_v^*|_v$, respectively. As for Type IV points, they correspond to sequences of decreasing nested disks $D(c_i, r_i)$ such that $\bigcap D(c_i, r_i) = \emptyset$ and the multiplicative seminorm is $f \mapsto \lim_{i\to\infty}[f]_{D(c_i,r_i)}$ as described above. For details, see [4] or [3]. For $\zeta \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, we sometimes write $|\zeta|_v$ instead of $[T]_\zeta$.

In order to apply the main equidistribution result from [3, Theorem 7.52], we recall the potential theory on the affine line over $\mathbb{C}_v$. The right setting for nonarchimedean potential theory is the potential theory on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ developed in [3]. We quote part of a nice summary of the theory from [1, §2.2 and §2.3] without going into details. We refer the reader to [1, 3] for all the details and proofs.

Let E be a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. Then analogous to the complex case, the logarithmic capacity $\gamma(E) = e^{-V(E)}$ and the Green's function G_E of E relative to ∞ can be defined, where $V(E)$ is the infimum of the *energy integral* with respect to all possible probability measures supported on E. More precisely,

$$V(E) = \inf_{\mu} \iint_{E\times E} -\log \delta(x, y)\, d\mu(x)\, d\mu(y),$$

where the infimum is taken with respect to all probability measures μ supported on E, while for $x, y \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, the function $\delta(x, y)$ is the *Hsia kernel* (see [3, Proposition 4.1]):

$$\delta(x, y) := \limsup_{\substack{z,w\in\mathbb{C}_v \\ z\to x,\, w\to y}} |z - w|_v.$$

The following are the basic properties of the logarithmic capacity of E:

- If E_1, E_2 are two compact subsets of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ such that $E_1 \subset E_2$, then $\gamma(E_1) \leq \gamma(E_2)$.
- If $E = \{\zeta\}$ where ζ is a Type II or III point corresponding to a closed disk $D(c, r)$, then $\gamma(E) = r > 0$ [3, Example 6.3]. (This can be viewed

as an analogue of the fact that a closed disk $D(c,r)$ of positive radius r in $\mathbb{C}_v$ has logarithmic capacity $\gamma(D(c,r)) = r$.)

If $\gamma(E) > 0$, then there exists a unique probability measure μ_E, also called the *equilibrum measure on* E, attaining the infimum of the energy integral. Furthermore, the support of μ_E is contained in the boundary of the unbounded component of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \setminus E$.

The Green's function $G_E(z)$ of E relative to infinity is a well-defined nonnegative real-valued subharmonic function on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ which is harmonic on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \setminus E$ (in the sense of [3, Chapter 8]). If $\gamma(E) = 0$, then the Green's function associated to the set E does not exist. Indeed, as shown in [3, Proposition 7.17, p. 151], if $\gamma(\partial E) = 0$ then there exists no nonconstant harmonic function on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \setminus E$ which is bounded below (this is the Strong Maximum Principle for harmonic functions defined on Berkovich spaces). The following result is [1, Lemmas 2.2 and 2.5], and it gives a characterization of the Green's function of the set E.

LEMMA 3.1. *Let E be a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, and let U be the unbounded component of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \setminus E$.*

(1) *If $\gamma(E) > 0$ (i.e. $V(E) < \infty$), then $G_E(z) = V(E) + \log|z|_v$ for all $z \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ such that $|z|_v$ is sufficiently large.*
(2) *If $G_E(z) = 0$ for all $z \in E$, then G_E is continuous on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, $\mathrm{supp}(\mu_E) = \partial U$, and $G_E(z) > 0$ if and only if $z \in U$.*
(3) *If $G : \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \to \mathbb{R}$ is a continuous subharmonic function which is harmonic on U, identically zero on E, and such that $G(z) - \log^+|z|_v$ is bounded, then $G = G_E$. Furthermore, if $G(z) = \log|z|_v + V + o(1)$ (as $|z|_v \to \infty$) for some $V < \infty$, then $V(E) = V$, and so $\gamma(E) = e^{-V}$.*

To state the equidistribution result from [3], we consider the compact *Berkovich adelic sets* which are of the form

$$\mathbb{E} := \prod_{v\in\Omega} E_v$$

where E_v is a nonempty compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ for each $v \in \Omega$ and where E_v is the closed unit disk $\mathcal{D}(0,1)$ in $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ for all but finitely many $v \in \Omega$. The *logarithmic capacity* $\gamma(\mathbb{E})$ of $\mathbb{E}$ is defined as

$$\gamma(\mathbb{E}) = \prod_{v\in\Omega} \gamma(E_v). \tag{3.2}$$

Note that in (3.2) we have a finite product, as for all but finitely many $v \in \Omega$, $\gamma(E_v) = \gamma(\mathcal{D}(0,1)) = 1$. Let $G_v = G_{E_v}$ be the Green's function of E_v relative to ∞ for each $v \in \Omega$. For every $v \in \Omega$, we fix an embedding of the separable closure K^{sep} of K into $\mathbb{C}_v$. Let $S \subset K^{\mathrm{sep}}$ be any finite subset that is invariant under the action of the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. We define the *height* $h_{\mathbb{E}}(S)$ of S relative to $\mathbb{E}$ by

$$h_{\mathbb{E}}(S) = \sum_{v \in \Omega} \bigg(\frac{1}{|S|} \sum_{z \in S} G_v(z) \bigg). \tag{3.3}$$

Note that this definition is independent of any particular embedding $K^{\mathrm{sep}} \hookrightarrow \mathbb{C}_v$ that we choose at $v \in \Omega$. For each v, we let μ_v be the equilibrium measure on E_v. The following is a special case of the equidistribution result [3, Theorem 7.52] that we need for our application.

THEOREM 3.4. *Let $\mathbb{E} = \prod_{v \in \Omega} E_v$ be a compact Berkovich adelic set with $\gamma(\mathbb{E}) = 1$. Suppose that S_n is a sequence of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant finite subsets of K^{sep} with $|S_n| \to \infty$ and $h_{\mathbb{E}}(S_n) \to 0$ as $n \to \infty$. For each $v \in \Omega$ and for each n let δ_n be the discrete probability measure supported equally on the elements of S_n. Then the sequence $\{\delta_n\}$ of measures converges weakly to μ_v, the equilibrium measure on E_v.*

4. The dynamics of the Drinfeld module family Φ. Let K be a finitely generated field extension of $\mathbb{F}_q(t)$. We work with a family of Drinfeld modules Φ as given in Theorem 2.6, i.e.

$$\Phi_t(x) = tx + \sum_{i=1}^{r-1} g_i(z) x^{q^i} + x^{q^r},$$

with $g_i(z) \in K[z]$ for $i = 1, \ldots, r-1$. For convenience, we let $g_0(z) := t$ be the corresponding constant polynomial. Let $\mathbf{c} \in K[z]$ be given. We define $f_{\mathbf{c},n}(z) := \Phi_{t^n}(\mathbf{c})$ for each $n \in \mathbb{N}$. Note that $f_{\mathbf{c},n}$ is a polynomial in z with coefficients in K. Assume $m := \deg(\mathbf{c})$ satisfies the inequality

$$m = \deg(\mathbf{c}) > \max_{i=0}^{r-1} \frac{\deg(g_i)}{q^r - q^i}. \tag{4.1}$$

We let C_m be the leading coefficient of $\mathbf{c}$. In the next lemma we compute the degrees of all polynomials $f_{\mathbf{c},n}$ for all positive integers n.

LEMMA 4.2. *With the above hypothesis, the polynomial $f_{\mathbf{c},n}(z)$ has degree $m \cdot q^{rn}$ and leading coefficient $C_m^{q^{rn}}$ for each $n \in \mathbb{N}$.*

Proof. The assertion follows easily by induction on n, using (4.1), since the term of $f_{\mathbf{c},n}(z)$ of highest degree in z is $\mathbf{c}^{q^{rn}}$. ∎

We immediately obtain as a corollary of Lemma 4.2 the fact that $\mathbf{c}$ is not preperiodic for $\mathbf{f}$. Furthermore we find that if $\mathbf{c}(\lambda) \in \Phi^{\lambda}_{\mathrm{tor}}$, then $\lambda \in \overline{K}$.

Fix a place $v \in \Omega_K$. Our first task is to define the *generalized Mandelbrot set* $M_{\mathbf{c},v}$ associated to $\mathbf{c}$ and establish that $M_{\mathbf{c},v}$ is a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. Roughly speaking, $M_{\mathbf{c},v}$ is the subset of $\mathbb{C}_v$ consisting of all $\lambda \in \mathbb{C}_v$ such that $\mathbf{c}(\lambda)$ is v-adic bounded under the action of $\mathbb{F}_q[t]$ with respect to the Drinfeld module structure Φ^λ. A simple observation is that the orbit of $\mathbf{c}(\lambda)$ under the action of $\mathbb{F}_q[t]$ is v-adic bounded if and only if under the action of Φ^λ_t the orbit $\{\Phi^\lambda_{t^n}(\mathbf{c}(\lambda)) \mid n = 0, 1, 2, \ldots\}$ is v-adic bounded. Hence, in our definition for the generalized Mandelbrot set $M_{\mathbf{c},v}$ below we shall only consider the orbit of $\mathbf{c}(\lambda)$ under the action of Φ^λ_t. Following [1, 15], we put

$$M_{\mathbf{c},v} := \Big\{\lambda \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} : \sup_n [f_{\mathbf{c},n}(T)]_\lambda < \infty\Big\}.$$

Let $\lambda \in \mathbb{C}_v$ and recall the local canonical height $\widehat{\mathrm{h}}_{\Phi^\lambda,v}(x)$ of $x \in \mathbb{C}_v$, given by

$$\widehat{\mathrm{h}}_{\Phi^\lambda,v}(x) = \lim_{n\to\infty} \frac{\log^+ |\Phi^\lambda_{t^n}(x)|_v}{q^{rn}}.$$

Notice that $\widehat{\mathrm{h}}_{\Phi^\lambda,v}(x)$ is a continuous function of both λ and x. As $\mathbb{C}_v$ is a dense subspace of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, continuity in λ implies that the canonical local height function $\widehat{\mathrm{h}}_{\Phi^\lambda,v}(\mathbf{c}(\lambda))$ has a natural extension on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ (note that the topology on $\mathbb{C}_v$ is the restriction of the weak topology on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, so any continuous function on $\mathbb{C}_v$ will automatically have a unique extension to $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$). In the following, we will extend $\widehat{\mathrm{h}}_{\Phi^\lambda,v}(\mathbf{c}(\lambda))$ to a function of λ on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ and view it as a continuous function on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. It follows from the definition of $M_{\mathbf{c},v}$ that $\lambda \in M_{\mathbf{c},v}$ if and only if $\widehat{\mathrm{h}}_{\Phi^\lambda,v}(\mathbf{c}(\lambda)) = 0$. Thus, $M_{\mathbf{c},v}$ is a closed subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. In fact, the following is true.

PROPOSITION 4.3. *$M_{\mathbf{c},v}$ is a compact subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$.*

We already showed that $M_{\mathbf{c},v}$ is a closed subset of the locally compact space $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, and thus in order to prove Proposition 4.3 we only need to show that $M_{\mathbf{c},v}$ is a bounded subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$.

LEMMA 4.4. *$M_{\mathbf{c},v}$ is a bounded subset of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$.*

Proof. For each $i = 0, \ldots, r-1$, we let D_i be the leading coefficient of g_i (recall that $g_0 = t$); also, let $d_i := \deg(g_i)$. Using an argument identical to that in deriving (2.4), we see that there exists $M > 1$ such that if $|\lambda|_v > M$, then

$$|g_i(\lambda)|_v = |D_i|_v \cdot |\lambda|_v^{d_i} \quad \text{for each } i = 0, \ldots, r-1,$$
$$|\mathbf{c}(\lambda)|_v = |C_m|_v \cdot |\lambda|_v^m > |\lambda|_v > 1.$$

At the expense of replacing M with a larger number we may assume that

$$M^{m(q^r - q^i) - d_i} \geq |D_i|_v \cdot |C_m|_v^{q^i - q^r} \quad \text{for each } i = 0, \ldots, r-1.$$

Note that we may achieve the above inequality for large M since (by our assumption (4.1)) $m > d_i/(q^r - q^i)$ for each $i = 0, \ldots, r-1$. Therefore, if

$|\lambda|_v > M$, then

$$|\mathbf{c}(\lambda)|_v^{q^r} > |g_i(\lambda) \cdot \mathbf{c}(\lambda)^{q^i}|_v \quad \text{for each } i = 0, \dots, r-1.$$

Hence,

$$|\Phi_t^\lambda(\mathbf{c}(\lambda))|_v = |\mathbf{c}(\lambda)|_v^{q^r} > |\mathbf{c}(\lambda)|_v > |\lambda|_v > 1.$$

This allows us to conclude that if $|\lambda|_v > M$, then $|\Phi_{t^n}^\lambda(\mathbf{c}(\lambda))|_v \to \infty$ as $n \to \infty$. Thus $\lambda \notin M_{\mathbf{c},v}$ if $|\lambda|_v > M$. ∎

Next our goal is to compute the logarithmic capacities of the v-adic generalized Mandelbrot sets $M_{\mathbf{c},v}$ associated to $\mathbf{c}$ for the given family Φ of Drinfeld modules.

THEOREM 4.5. *The logarithmic capacity of $M_{\mathbf{c},v}$ is $\gamma(M_{\mathbf{c},v}) = |C_m|_v^{-1/m}$.*

The strategy for the proof of Theorem 4.5 is to construct a continuous subharmonic function $G_{c,v} : \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} \to \mathbb{R}$ satisfying the conditions of Lemma 3.1(3). We let

$$G_{\mathbf{c},v}(\lambda) := \lim_{n\to\infty} \frac{1}{\deg(f_{\mathbf{c},n})} \log^+[f_{\mathbf{c},n}(T)]_\lambda. \tag{4.6}$$

By a similar reasoning to the proof of [1, Prop. 3.7], it can be shown that the limit exists for all $\lambda \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. In fact, by the definition of canonical local height, for $\lambda \in \mathbb{C}_v$ we have

$$\begin{aligned} G_{\mathbf{c},v}(\lambda) &= \lim_{n\to\infty} \frac{1}{mq^{rn}} \log^+ |f_\lambda^n(\mathbf{c}(\lambda))|_v \quad \text{as } \deg(f_{\mathbf{c},n}) = mq^{rn} \text{ by Lemma 4.2,} \\ &= \frac{1}{m} \cdot \widehat{\mathrm{h}}_{\Phi^\lambda,v}(\mathbf{c}(\lambda)) \quad \text{by the definition of canonical local height.} \end{aligned}$$

Note that $G_{\mathbf{c},v}(\lambda) \geq 0$ for all $\lambda \in \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. Moreover, by definition we see that $\lambda \in M_{\mathbf{c},v}$ if and only if $G_{\mathbf{c},v}(\lambda) = 0$.

LEMMA 4.7. *$G_{\mathbf{c},v}$ is the Green's function for $M_{\mathbf{c},v}$ relative to ∞.*

The proof is essentially the same as the proof of [1, Prop. 3.7]; we just give a sketch of the idea.

Proof of Lemma 4.7. By the same argument as in the proof of [7, Prop. 1.2], we observe that as a function of λ, the function $\log^+[f_{\mathbf{c},n}(T)]_\lambda/\deg(f_{\mathbf{c},n})$ converges uniformly on compact subsets of $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. So, it is a continuous subharmonic function on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$, which converges to $G_{\mathbf{c},v}$ uniformly; hence it follows from [3, Prop. 8.26(c)] that $G_{\mathbf{c},v}$ is continuous and subharmonic on $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$. Furthermore, as remarked above, $G_{\mathbf{c},v}$ is zero on $M_{\mathbf{c},v}$.

Arguing as in the proof of Lemma 4.4, if $|\lambda|_v$ is sufficiently large, then for $n \geq 1$ we have

$$|f_{\mathbf{c},n}(\lambda)|_v = |\Phi_{t^n}^\lambda(\mathbf{c}(\lambda))|_v = |C_m\lambda^m|_v^{q^{rn}}.$$

Hence, for $|\lambda|_v$ sufficiently large we have

$$G_{\mathbf{c},v}(\lambda) = \lim_{n\to\infty} \frac{1}{mq^{rn}} \log|f_{\mathbf{c},n}(\lambda)|_v = \log|\lambda|_v + \frac{\log|C_m|_v}{m}.$$

It follows from Lemma 3.1(3) that $G_{\mathbf{c},v}$ is indeed the Green's function of $M_{\mathbf{c},v}$. ■

Now we are ready to prove Theorem 4.5.

Proof of Theorem 4.5. As in the proof of Lemma 4.7, we have

$$G_{\mathbf{c},v}(\lambda) = \log|\lambda|_v + \frac{\log|C_m|_v}{m}$$

for $|\lambda|_v$ sufficiently large. By Lemma 3.1(3), we find that $V(M_{\mathbf{c},v}) = \frac{\log|C_m|_v}{m}$. Hence, the logarithmic capacity of $M_{\mathbf{c},v}$ is

$$\gamma(M_{\mathbf{c},v}) = e^{-V(M_{\mathbf{c},v})} = \frac{1}{|C_m|_v^{1/m}}$$

as desired. ■

Let us call $\mathbb{M}_{\mathbf{c}} = \prod_{v\in\Omega} M_{\mathbf{c},v}$ the *generalized adelic Mandelbrot set* associated to $\mathbf{c}$. As a corollary to Theorem 4.5 we see that $\mathbb{M}_{\mathbf{c}}$ satisfies the hypothesis of Theorem 3.4.

Corollary 4.8. *For all but finitely many nonarchimedean places v, $M_{\mathbf{c},v}$ is the closed unit disk $\mathcal{D}(0,1)$ in $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$; furthermore $\gamma(\mathbb{M}_{\mathbf{c}}) = 1$.*

Proof. For each place v where all coefficients of $g_i(z), i = 0,\dots,r-1$, and of $\mathbf{c}(z)$ are v-adic integral, and moreover $|C_m|_v = 1$, we have $M_{\mathbf{c},v} = \mathcal{D}(0,1)$. Indeed, $\mathcal{D}(0,1) \subset M_{\mathbf{c},v}$ since $\Phi^\lambda_{t^n}(\mathbf{c}(\lambda))$ is then always a v-adic integer. For the converse inclusion we note that each coefficient of $f_{\mathbf{c},n}(z)$ is a v-adic integer, while the leading coefficient is a v-adic unit for all $n \geq 1$; thus $|f_{\mathbf{c},n}(\lambda)|_v = |\lambda|_v^{mq^{rn}} \to \infty$ if $|\lambda|_v > 1$. Note that $C_m \neq 0$, and so the second assertion in the corollary follows immediately by the product formula in K. ■

Using the decomposition of the global canonical height as a sum of local canonical heights we obtain the following result.

Corollary 4.9. *Let $\lambda \in K^{\mathrm{sep}}$, let S be the set of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$*-conjugates of λ, and let $h_{\mathbb{M}_{\mathbf{c}}}$ be defined as in* (3.3). *Then* $\deg(\mathbf{c}) \cdot h_{\mathbb{M}_{\mathbf{c}}}(S) = \widehat{\mathrm{h}}_{\Phi^\lambda}(\mathbf{c}(\lambda))$.

Remark 4.10. Corollary 4.9 is a result on specialization of height functions relating the canonical heights of $\mathbf{c}(\lambda)$ in the family and the height of the parameter λ. As a consequence, we find that $\mathbf{c}(\lambda)$ is a torsion point for Φ^λ if and only if $h_{\mathbb{M}_{\mathbf{c}}}(S) = 0$.

By abuse of notation we will write $h_{\mathbb{M}_{\mathbf{c}}}(\lambda) := h_{\mathbb{M}_{\mathbf{c}}}(S)$, where S is the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-orbit of λ.

5. Proof of Theorem 2.6. We work under the hypothesis of Theorem 2.6, and we keep the notation from the previous sections.

Recall that $\Phi_t(x) = tx + \sum_{i=0}^{r-1} g_i(z)x^{q^i} + x^{q^r}$ where $g_i \in K[z]$ for $i = 1, \dots, r-1$. Let $\mathbf{a}, \mathbf{b} \in K[z]$ satisfy the hypothesis (2.7) of Theorem 2.6. In particular, both $\mathbf{a}$ and $\mathbf{b}$ have positive degrees. Let Ω_K be the set of all inequivalent absolute values on K constructed as in Subsection 2.2.

Next, assume there exist infinitely many $\lambda \in K^{\mathrm{sep}}$ such that $\mathbf{a}(\lambda), \mathbf{b}(\lambda) \in \Phi^{\lambda}_{\mathrm{tor}}$.

Let $h_{\mathbb{M}_{\mathbf{a}}}(z)$ (respectively, $h_{\mathbb{M}_{\mathbf{b}}}(z)$) be the height of $z \in K^{\mathrm{sep}}$ relative to the adelic generalized Mandelbrot set $\mathbb{M}_{\mathbf{a}} := \prod_{v \in \Omega_K} M_{\mathbf{a},v}$ (respectively, $\mathbb{M}_{\mathbf{b}}$) defined as in Section 4 (see also Remark 4.10). Note that if $\lambda \in K^{\mathrm{sep}}$ is a parameter such that $\mathbf{a}(\lambda)$ (and $\mathbf{b}(\lambda)$) is torsion for Φ^{λ}, then $h_{\mathbb{M}_{\mathbf{a}}}(\lambda) = 0$ by Corollary 4.9. So, we will apply the equidistribution result from Theorem 3.4 to conclude that $M_{\mathbf{a},v} = M_{\mathbf{b},v}$ for each place $v \in \Omega_K$.

Indeed, we know that there exists an infinite sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of distinct numbers $\lambda \in K^{\mathrm{sep}}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are torsion points for Φ^{λ}. So, for each $n \in \mathbb{N}$, we may define S_n to be the union of the sets of Galois conjugates for λ_m for all $1 \le m \le n$. Clearly $\#S_n \to \infty$ as $n \to \infty$, and also each S_n is $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant. Finally, $h_{\mathbb{M}_{\mathbf{a}}}(S_n) = h_{\mathbb{M}_{\mathbf{b}}}(S_n) = 0$ for all $n \in \mathbb{N}$, and thus Theorem 3.4 applies in this case. We obtain $\mu_{M_{\mathbf{a},v}} = \mu_{M_{\mathbf{b},v}}$ for each $v \in \Omega_K$ and since they are both supported on $M_{\mathbf{a},v}$ (resp. $M_{\mathbf{b},v}$), we also get $M_{\mathbf{a},v} = M_{\mathbf{b},v}$.

It follows that the two Green's functions $G_{\mathbf{a},v}$ and $G_{\mathbf{b},v}$ for $M_{\mathbf{a},v}$ and $M_{\mathbf{b},v}$ are the same. By the definitions of $G_{\mathbf{a},v}$ ($G_{\mathbf{b},v}$ respectively) (see (4.6)), for each $v \in \Omega_K$ we have

$$\frac{1}{\deg(\mathbf{a})}\,\widehat{\mathrm{h}}_{\Phi^{\lambda},v}(\mathbf{a}(\lambda)) = G_{\mathbf{a},v}(\lambda) = G_{\mathbf{b},v}(\lambda) = \frac{1}{\deg(\mathbf{b})}\,\widehat{\mathrm{h}}_{\Phi^{\lambda}}(\mathbf{a}(\lambda)) \quad \text{for } \textit{all } \lambda \in \mathbb{C}_v.$$

Using the decomposition of the global canonical height as a sum of local canonical heights, we conclude that

$$\frac{1}{\deg(\mathbf{a})}\,\widehat{\mathrm{h}}_{\Phi^{\lambda}}(\mathbf{a}(\lambda)) = \frac{1}{\deg(\mathbf{b})}\,\widehat{\mathrm{h}}_{\Phi^{\lambda}}(\mathbf{b}(\lambda)) \qquad \text{for all } \lambda \in \overline{K}.$$

Thus, for each $\lambda \in \overline{K}$ we see that $\widehat{\mathrm{h}}_{\Phi^{\lambda}}(\mathbf{a}(\lambda)) = 0$ if and only if $\widehat{\mathrm{h}}_{\Phi^{\lambda}}(\mathbf{b}(\lambda)) = 0$. Hence $\mathbf{a}(\lambda) \in \Phi^{\lambda}_{\mathrm{tor}}$ if and only if $\mathbf{b}(\lambda) \in \Phi^{\lambda}_{\mathrm{tor}}$.

Finally, knowing that $\mathbb{M}_{\mathbf{a}} = \mathbb{M}_{\mathbf{b}}$ we deduce that the capacities of the corresponding generalized Mandelbrot sets are equal to each other, i.e.

$$|C_{\mathbf{a}}|_v^{1/\deg(\mathbf{a})} = |C_{\mathbf{b}}|_v^{1/\deg(\mathbf{b})} \qquad \text{for each place } v.$$

We let $U := C_{\mathbf{a}}^{\deg(\mathbf{b})}/C_{\mathbf{b}}^{\deg(\mathbf{a})}$. Then for each $v \in \Omega_K$ we know that $|U|_v = 1$. Since the only elements of K which are units for all places $v \in \Omega_K$ are the ones living in $\overline{\mathbb{F}_q}$, we conclude that $U \in \overline{\mathbb{F}_q}$, as desired.

6. Proof of our main results. Theorem 1.3 follows from Theorem 2.6.

Proof of Theorem 1.3. If either $\mathbf{a}$ or $\mathbf{b}$ is a torsion point for Φ, then we are done. So, from now on, assume that neither $\mathbf{a}$ nor $\mathbf{b}$ is torsion for Φ.

We now consider the given Drinfeld module $\Phi : \mathbb{F}_q[t] \to \mathrm{End}_L(\mathbb{G}_a)$ as a Drinfeld module over $L = K(z)$, the rational function field with constant field K. We let $\Omega_{L/K}$ be the set of all places of the function field L/K, and let $\widehat{\mathrm{h}}_\Phi$ be the canonical height for the Drinfeld module Φ over L. We first show that there exists a place $w \in \Omega_{L/K}$ such that $\widehat{\mathrm{h}}_{\Phi,w}(\mathbf{a}) > 0$ and similarly $\widehat{\mathrm{h}}_{\Phi,w'}(\mathbf{b}) > 0$ possibly for a different place $w' \in \Omega_{L/K}$. There are two cases to consider.

CASE 1: *Each polynomial g_i is constant.* If for infinitely many $\lambda \in K^{\mathrm{sep}}$, $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are torsion points but neither $\mathbf{a}$ nor $\mathbf{b}$ is torsion, we would infer in particular that $\mathbf{a}$ is nonconstant, and thus for some $w \in \Omega_{L/K}$ we have $\widehat{\mathrm{h}}_{\Phi,w}(\mathbf{a}) > 0$ by [14, Theorem 4.15(a)] (a similar statement holds for $\mathbf{b}$).

CASE 2: *There exists at least one polynomial g_i which is not constant.* In this case, there exists at least one place $v \in \Omega_{L/K}$ such that for some $i = 1, \ldots, r-1$, we have $|g_i|_v > 1$ and thus [14, Theorem 4.15(b)] implies that there exists some place $w \in \Omega_{L/K}$ such that $\widehat{\mathrm{h}}_{\Phi,w}(\mathbf{a}) > 0$. A similar statement holds for $\mathbf{b}$ since we know that neither $\mathbf{a}$ nor $\mathbf{b}$ is torsion for Φ.

Therefore in all cases we know that there exists some place $w \in \Omega_{L/K}$ such that $\widehat{\mathrm{h}}_{\Phi,w}(\mathbf{a}) > 0$, i.e.

$$|\Phi_{t^n}(\mathbf{a})|_w \to \infty \quad \text{as } n \to \infty. \tag{6.1}$$

However, since $\mathbf{a} \in K[z]$ and each $g_i \in K[z]$ and also $t \in K$, we deduce that for each place $v \in \Omega_{L/K}$ *except* the place at infinity, the iterates $\Phi_{t^n}(\mathbf{a})$ are all v-integral. Hence it must be that for the place w at infinity of the function field $K(z)$ the convergence (6.1) holds, i.e.

$$\deg_z(\Phi_{t^n}(\mathbf{a})) \to \infty \quad \text{as } n \to \infty.$$

A similar argument shows that also $\deg_z(\Phi_{t^n}(\mathbf{b})) \to \infty$ as $n \to \infty$. Hence, at the expense of replacing both $\mathbf{a}$ and $\mathbf{b}$ by $\Phi_{t^n}(\mathbf{a})$ respectively $\Phi_{t^n}(\mathbf{b})$ (for a sufficiently large integer n), we may achieve that inequality (2.7) is satisfied. Moreover, note that for each λ, $\mathbf{a}(\lambda)$ (or $\mathbf{b}(\lambda)$) is a torsion point for Φ^λ if and only if $\Phi^\lambda_{t^n}(\mathbf{a}(\lambda))$ (respectively $\Phi^\lambda_{t^n}(\mathbf{b}(\lambda))$) is a torsion point for Φ^λ. Theorem 2.6 shows that for each $\lambda \in \overline{K}$, $\Phi^\lambda_{t^n}(\mathbf{a}(\lambda))$ (and thus $\mathbf{a}(\lambda)$) is a torsion point for Φ^λ if and only if $\Phi^\lambda_{t^n}(\mathbf{b}(\lambda))$ (and thus $\mathbf{b}(\lambda)$) is a torsion point for Φ^λ. ∎

Next we prove that for the family of Drinfeld modules from Theorems 1.4 and 1.5, and for almost all $\mathbf{c} \in \mathbb{F}_q[t]$, there exist indeed infinitely many

$\lambda \in K^{\mathrm{sep}}$ such that $\mathbf{c}$ is torsion for Φ^λ; thus the conclusion in those results is not vacuous.

PROPOSITION 6.2. *Assume not both q and r equal 2. Then for the Drinfeld module family $\Phi^\lambda : \mathbb{F}_q[t] \to \mathrm{End}(\mathbb{G}_{\mathrm{a}})$ given by $\Phi^\lambda_t(x) = tx + \lambda x^q + x^{q^r}$ (for some $r \geq 2$), and for all $\mathbf{c} \in \mathbb{F}_q[t]$ of degree larger than 1, there exist infinitely many $\lambda \in \mathbb{F}_q(t)^{\mathrm{sep}}$ such that $\mathbf{c}$ is torsion for Φ^λ.*

Proof. Indeed, we let as before

$$f_{\mathbf{c},n}(\lambda) = \Phi^\lambda_{t^n}(\mathbf{c}).$$

An easy induction shows that for each $n \geq 2$ we have

$$f_{\mathbf{c},n}(\lambda) = P_n + \lambda Q_n(\lambda^q) + R_n(\lambda^q),$$

where $P_n \in \mathbb{F}_q[t]$, while $\deg(Q_n) \leq q^{r(n-2)}$ and $\deg(R_n) = q^{r(n-1)-1}$ (where Q_n and R_n are polynomials with coefficients in $\mathbb{F}_q[t]$).

CLAIM 6.3. *For each $n \geq 1$, we have $Q_n \neq 0$ provided $\mathbf{c} \in \mathbb{F}_q[t]$ has degree larger than 1.*

Proof. It suffices to prove that the constant coefficient of Q_n is nonzero, for which in turn it suffices to show that the coefficient b_n of λ in $f_{\mathbf{c},n}$ is nonzero. We observe the recurrence relations

$$b_{n+1} = tb_n + P_n^q \quad \text{and} \quad P_{n+1} = tP_n + P_n^{q^r}.$$

We start with $b_1 = \mathbf{c}^q$ and $P_1 = t\mathbf{c} + \mathbf{c}^{q^r}$. Hence, if $\mathbf{c} \in \mathbb{F}_q[t]$ has degree $D > 1$, then both b_n and P_n are in $\mathbb{F}_q[t]$ of degrees $q^{r(n-1)+1}D$ respectively $q^{rn}D$ for $n \geq 1$. ∎

We let $\lambda_1, \ldots, \lambda_s$ be the roots of $f_{\mathbf{c},n}$ with multiplicities $d_1, \ldots, d_s \geq 1$. Then $f'_{\mathbf{c},n}$ will have the roots $\lambda_1, \ldots, \lambda_s$ with multiplicities $d_1 - 1, \ldots, d_s - 1$ (at least). If some d_i is less than p, then the corresponding root λ_i is separable over $\mathbb{F}_q(t)$.

Now, $f'_{c,n}(\lambda) = Q_n(\lambda^q)$, which has degree at most $q^{r(n-2)+1}$ in λ. But note that $Q_n \neq 0$ for all $\mathbf{c} \in \mathbb{F}_q[t]$ of degree larger than 1. So, assume $\lambda_1, \ldots, \lambda_u$ have multiplicities less than p, while $\lambda_{u+1}, \ldots, \lambda_s$ have multiplicities at least p. Then

$$(d_{u+1} - 1) + \cdots + (d_s - 1) \leq q^{r(n-2)+1}$$

and since $d_i \geq p$ for $i = u+1, \ldots, s$, we have

$$d_{u+1} + \cdots + d_s \leq \frac{p}{p-1} \cdot q^{r(n-2)+1}.$$

Because

$$d_1 + \cdots + d_s = \deg_\lambda(R_n(\lambda^q)) = q^{r(n-1)}$$

and $d_i \le p-1$ for $i = 1, \dots, u$, we conclude that

$$u \ge \frac{q^{r(n-1)} - \frac{p}{p-1} \cdot q^{r(n-2)+1}}{p-1},$$

which goes to infinity as n goes to infinity (as long as not both q and r are equal to 2). ∎

In the following proof, for a nonzero element $f \in \mathbb{F}_q[t]$ and a Drinfeld module Φ we denote the submodule of f-torsion by $\Phi[f]$ as usual.

Proof of Theorem 2.9. If either $\mathbf{a}$ or $\mathbf{b}$ equals 0, then the conclusion is immediate. So, assume now that both $\mathbf{a}$ and $\mathbf{b}$ are nonzero.

We note that

$$\mathbf{a}_1(z) := \Phi_t(\mathbf{a}) = \mathbf{a}^{q^{i_0}} \cdot g_{i_0}(z) + O(z^{d_0-1})$$

and

$$\mathbf{b}_1(z) := \Phi_t(\mathbf{b}) = \mathbf{b}^{q^{i_0}} \cdot g_{i_0}(z) + O(z^{d_0-1})$$

are both polynomials in z of the same degree d_0. Furthermore their leading coefficients are $c_{d_0} \cdot \mathbf{a}^{q^{i_0}}$, respectively $c_{d_0} \cdot \mathbf{b}^{q^{i_0}}$, where c_{d_0} is the leading coefficient of g_{i_0}. Using (2.10), we find that inequality (2.7) from Theorem 2.6 is satisfied and thus we conclude that $\mathbf{a}/\mathbf{b} \in \overline{\mathbb{F}_q}$. On the other hand, we know that $\mathbf{a}, \mathbf{b} \in K$ and $\mathbb{F}_q$ is algebraically closed in K; hence $\mathbf{a}/\mathbf{b} \in \mathbb{F}_q$ as desired. ∎

We omit the proof of Theorem 1.4 since it is an immediate corollary to Theorem 2.9. Similarly to the proof of Theorem 2.9, we can prove Theorem 1.5.

Proof of Theorem 1.5. Let s be a positive integer such that $\mathbf{a}, \mathbf{b} \in \mathbb{F}_{q^s}(t)$ and let $\Omega_s := \Omega_{\mathbb{F}_{q^s}(t)}$. Assume there exist infinitely many $\lambda \in \mathbb{F}_q(t)^{\mathrm{sep}}$ such that $\mathbf{a}, \mathbf{b} \in \Phi^{\lambda}_{\mathrm{tor}}$. In addition, we may assume both $\mathbf{a}$ and $\mathbf{b}$ are nonzero. The proof of Theorem 1.4 shows that $\mathbf{a}/\mathbf{b} \in \overline{\mathbb{F}_q}$. In addition, Theorem 2.6 implies that for each $\lambda \in \mathbb{F}_q(t)^{\mathrm{sep}}$, we have $\mathbf{a} \in \Phi^{\lambda}_{\mathrm{tor}}$ if and only if $\mathbf{b} \in \Phi^{\lambda}_{\mathrm{tor}}$. In order to finish the proof of Theorem 1.5 we will use both consequences of Theorem 2.6 stated above.

We let $\gamma := \mathbf{b}/\mathbf{a} \in \mathbb{F}_{q^s}$, and assume $\gamma \notin \mathbb{F}_q$. Let λ_0 be such that $\Phi^{\lambda_0}_t(\mathbf{a}) = 0$. Then we have

$$\lambda_0 \mathbf{a}^q = -t\mathbf{a} - \mathbf{a}^{q^r}. \tag{6.4}$$

Since $\mathbf{a} \in \mathbb{F}_{q^s}(t)$, we conclude that also $\lambda_0 \in \mathbb{F}_{q^s}(t) \subset \mathbb{F}_q(t)^{\mathrm{sep}}$. We will show that $\mathbf{b} \notin \Phi^{\lambda_0}_{\mathrm{tor}}$, which yields a contradiction to the conclusion of Theorem 2.6. Before we proceed, we note that

$$\begin{aligned} \Phi^{\lambda_0}_t(\mathbf{b}) &= \gamma t\mathbf{a} + \gamma^q \lambda_0 \mathbf{a}^q + \gamma^{q^r} \mathbf{a}^{q^r} \\ &= (\gamma - \gamma^q)t\mathbf{a} + (\gamma^{q^r} - \gamma^q)\mathbf{a}^{q^r} \quad \text{by (6.4)}. \end{aligned} \tag{6.5}$$

In the following, we denote by $|\cdot|_\infty$ the absolute value corresponding to the unique place of $\mathbb{F}_{q^s}(t)$ where t is not integral. We split our analysis into two cases:

CASE 1: $\mathbf{a} \in \mathbb{F}_{q^s}$. By (6.4) we have $|\lambda_0|_\infty = |t|_\infty$ and it follows from (6.5) that

$$\begin{aligned}|\Phi_t^{\lambda_0}(\mathbf{b})|_\infty &= |(\gamma - \gamma^q)t\mathbf{a} + (\gamma^{q^r} - \gamma^q)\mathbf{a}^{q^r}|_\infty = |t|_\infty \\ &> \max\{1, |t|_\infty^{1/(q^r-1)}, |\lambda_0|_\infty^{1/(q^r-q)}\}.\end{aligned}$$

Using (2.4), we conclude that $|\Phi_{t^n}^{\lambda_0}(\mathbf{b})|_\infty \to \infty$ as $n \to \infty$ and hence $\mathbf{b} \notin \Phi_{\mathrm{tor}}^{\lambda_0}$ as desired.

CASE 2: $\mathbf{a} \notin \mathbb{F}_{q^s}$. In this case there exists a place $v \in \Omega_s$ such that $|\mathbf{a}|_v > 1$. Note that it is not possible to have $|t\mathbf{a}|_v \geq |\mathbf{a}|^{q^r}$ for then also $|t|_v > 1$. This implies that $|\cdot|_v = |\cdot|_\infty$ and $|t|_\infty \geq |\mathbf{a}|_\infty^{q^r-1}$. But this is impossible in $\mathbb{F}_{q^s}(t)$. Hence, $|t\mathbf{a}|_v < |\mathbf{a}|_v^{q^r}$, and so $|\lambda_0|_v = |\mathbf{a}|_v^{q^r-q}$.

There are two possibilities now.

CASE 2a: $\gamma^{q^r} - \gamma^q \neq 0$. Then

$$|\Phi_t^{\lambda_0}(\mathbf{b})|_v = \max\{|t\mathbf{a}|_v, |\mathbf{a}|_v^{q^r}\} = |\mathbf{a}|_v^{q^r}.$$

It follows that

- $|t|_v^{1/(q^r-1)} \leq |\mathbf{a}|_v$,
- $|\lambda_0|_v^{1/(q^r-q)} = |\mathbf{a}|_v$, and
- $|\Phi_t^{\lambda_0}(\mathbf{b})|_v = |\mathbf{a}|_v^{q^r} > |\mathbf{a}|_v = \max\{1, |t|_v^{1/(q^r-1)}, |\lambda_0|_v^{1/(q^r-q)}\}$.

Using (2.4) again, we conclude that $|\Phi_{t^n}^{\lambda_0}(\mathbf{b})|_v \to \infty$ as $n \to \infty$ and $\mathbf{b} \notin \Phi_{\mathrm{tor}}^{\lambda_0}$.

CASE 2b: $\gamma^{q^r} = \gamma^q$. Since we assumed that $\gamma \notin \mathbb{F}_q$, we deduce that $r \geq 3$. By (6.5) and the assumption in Case 2b we have

$$\Phi_t^{\lambda_0}(\mathbf{b}) = (\gamma - \gamma^q)t\mathbf{a}.$$

If $|t|_v > 1$ (i.e. v is the place ∞) then $|\Phi_t^{\lambda_0}(\mathbf{b})|_v > |\mathbf{a}|_v$, and so again (2.4) can be used to infer that $|\Phi_{t^n}^{\lambda_0}(\mathbf{b})|_v \to \infty$ since

$$|\Phi_t^{\lambda_0}(\mathbf{b})|_v > |\mathbf{a}|_v = \max\{|t|_v^{1/(q^r-1)}, |\lambda_0|^{1/(q^r-q)}\}.$$

Assume now that $|t|_v \leq 1$. Using the assumption of Case 2b we get

$$\begin{aligned}\Phi_{t^2}^{\lambda_0}(\mathbf{b}) &= (\gamma - \gamma^q)t^2\mathbf{a} + (\gamma - \gamma^q)^q t^q \mathbf{a}^q \lambda_0 + (\gamma - \gamma^q)^{q^r} t^{q^r} \mathbf{a}^{q^r} \\ &= (\gamma - \gamma^q)t^2\mathbf{a} + (\gamma^q - \gamma^{q^2})t^q\mathbf{a}^q\lambda_0 + (\gamma^q - \gamma^{q^2})t^{q^r}\mathbf{a}^{q^r} \\ &= (\gamma - \gamma^q)t^2\mathbf{a} + (\gamma^q - \gamma^{q^2}) \cdot (t^q\mathbf{a}^q\lambda_0 + t^{q^r}\mathbf{a}^{q^r}) \\ &= (\gamma - \gamma^q)t^2\mathbf{a} + (\gamma^q - \gamma^{q^2}) \cdot (-t^{q+1}\mathbf{a} - t^q\mathbf{a}^{q^r} + t^{q^r}\mathbf{a}^{q^r}) \qquad \text{using (6.4)} \\ &= (\gamma - \gamma^q)t^2\mathbf{a} + (\gamma^q - \gamma^{q^2}) \cdot (-t^{q+1}\mathbf{a} + \mathbf{a}^{q^r} \cdot (t^{q^r} - t^q)).\end{aligned}$$

Since $|\mathbf{a}|_v > 1$ and $v \in \Omega_s$ we conclude that

$$|t^{q^r} - t^q|_v = |t^{q^{r-1}} - t|_v^q \geq |\mathbf{a}|_v^{-q} \tag{6.6}$$

since $t^{q^{r-1}} - t$ is a separable polynomial and thus it is either a v-adic unit or has the v-adic absolute value equal to that of a uniformizer of v in $\mathbb{F}_{q^s}(t)$ (note the assumption that $|t|_v \leq 1$). So, using (6.6) we get

$$\begin{aligned} |\mathbf{a}^{q^r} \cdot (t^{q^r} - t^q)|_v &\geq |\mathbf{a}|_v^{q^r-q} \\ &\geq |\mathbf{a}|_v^{q^3-q} && \text{because } r \geq 3 \text{ in Case 2b} \\ &> |\mathbf{a}|_v && \text{because } q \geq 2 \\ &\geq |t^{q+1}\mathbf{a}|_v && \text{because } |t|_v \leq 1 \text{ by assumption.} \end{aligned}$$

Therefore (using also the assumption that $\gamma \notin \mathbb{F}_q$ and thus $\gamma^q - \gamma^{q^2} \neq 0$)

$$|(\gamma^q - \gamma^{q^2}) \cdot (-t^{q+1}\mathbf{a} + \mathbf{a}^{q^r} \cdot (t^{q^r} - t^q))|_v \geq |\mathbf{a}|_v^{q^r-q}. \tag{6.7}$$

On the other hand

$$|(\gamma - \gamma^2)t^2\mathbf{a}|_v \leq |\mathbf{a}|_v \tag{6.8}$$

because $|t|_v \leq 1$. Using (6.7) and (6.8) coupled with the fact that

$$q^r - q > 1 \qquad \text{since } r \geq 3 \text{ and } q \geq 2,$$

we conclude that

$$|\Phi_{t^2}^{\lambda_0}(\mathbf{b})|_v \geq |\mathbf{a}|_v^{q^r-q} > |\mathbf{a}|_v = \max\{|t|_v^{1/(q^r-1)}, |\lambda_0|^{1/(q^r-q)}\}.$$

Again using (2.4) yields $|\Phi_{t^n}^{\lambda_0}(\mathbf{b})|_v \to \infty$ as $n \to \infty$ and thus $\mathbf{b} \notin \Phi_{\mathrm{tor}}^{\lambda_0}$.

In conclusion, assuming that $\gamma \notin \mathbb{F}_q$ yields in each case a contradiction; this finishes the proof of Theorem 1.5. ∎

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