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The binary Goldbach conjecture with primes in arithmetic progressions with large modulus

by

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1. Introduction. The binary Goldbach conjecture states that every even integer larger than 2 can be written as the sum of two prime numbers. In 1975, Montgomery and Vaughan [10] considered the corresponding exceptional set $E(X)$ defined as

$$E(X) := \#\{n \le X : 2 \mid n,\ n \ne p_1 + p_2 \text{ for any primes } p_1,\ p_2\}.$$

They could show that

$$E(X) < X^{1-\delta}$$

for a small positive number $\delta > 0$. It was later shown in [9] that δ can be chosen as large as $\delta = 0.121$. Lavrik [5] investigated a special case of the binary Goldbach conjecture requiring that the two prime summands belong to a given arithmetic progression. In particular, he considered the following exceptional set:

$$E_{k,b_1,b_2}(X) := \#\{n \le X : n \equiv b_1 + b_2 \pmod{k},\ n \ne p_1 + p_2 \text{ for any primes } p_i \equiv b_i \pmod{k},\ i = 1, 2\},$$

$$E_k(X) := \max_{\substack{1 \le b_1, b_2 \le k \\ (b_1 b_2, k) = 1}} E_{k,b_1,b_2}(X).$$

He could show that for all integers $k \le (\log X)^c$, where c is a positive integer, and any $D > 0$,

$$E_k(X) \ll X(\log X)^{-D}\phi(k)^{-1}. \tag{1.1}$$

Later, Liu and Zhan [8] showed that the following estimate holds for *all* $k \le X^{\delta}$ for small $\delta, \delta_1 > 0$:

$$E_k(X) \ll X^{1-\delta_1}\phi^{-1}(k). \tag{1.2}$$

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In this paper, we show that $E_{k,b_1,b_2}(X)$ can be estimated for all but some exceptional prime numbers k in the range $k \le X^{1/4}(\log X)^{-W}$. For each non-exceptional k, our result is not valid for all pairs of integers b_1 and b_2. Instead for each fixed b_2, it is only valid for *almost all* b_1. We prove:

THEOREM 1. *Set* $R = X^{1/4}(\log X)^{-W}$, $W > 0$. *For any* $D, U, E > 0$, *all but* $\ll RL^{-E}$ *prime numbers* $k \le R$, *any fixed* $b_2 \bmod k$, $(k, b_2) = 1$, *and all but* $\ll k(\log k)^{-U}$ *integers* $b_1 \bmod k$, $(b_1, k) = 1$,

$$E_{k,b_1,b_2}(X) \ll X(\log X)^{-D}k^{-1}.$$

The ternary Goldbach conjecture with primes in arithmetic progressions to a large modulus has been investigated in [4], [14]. The methods applied in these publications cannot be simply applied to prove Theorem 1. They rely on estimates for Dirichlet polynomials which are used to estimate the error term induced by the integral over the major arcs. However these estimates, which are discussed in depth in [6], have so far only been successfully applied to problems in additive prime number theorem involving at least three summands.

Therefore, when considering the problem of two prime summands in fixed arithmetic progressions, we need to develop a new approach to calculate the integral over the major arcs. We divide the major arcs into two sets $M_1(k)$ and $M_2(k)$:

$M_1(k)$: The major arcs defined around a fraction a/q where $k \le q \le k(\log X)^B$, $k \mid q$.

$M_2(k)$: The major arcs defined around a fraction a/q where $q \le k(\log X)^B$, $(q, k) = 1$.

We show that the major term is derived from $M_1(k)$ and that the contribution of the integral over $M_2(k)$ is a permissible error term. We note that the major arcs $M_1(k)$ are different from the major arcs for the general Goldbach conjecture without restrictions on the primes. It has been known since the papers of Hardy and Littlewood that for the general Goldbach conjecture the major arcs can be defined around fractions a/q with $q \le (\log X)^B$. A similar observation was made in [13] for the ternary Goldbach problem with primes in progressions.

Modifying an idea from [12], we use the Bombieri–Vinogradov theorem to calculate the integral over $M_1(k)$. In order to calculate the contribution of the integral over $M_2(k)$, $M_2(k)$ is further divided into subsets defined by the size of the denominator q. For $q \le N^{\delta_1}$, where δ_1 is a very small positive constant, we divide the corresponding major arcs into *inner* and *outer* major arcs. The integral over the inner major arcs can be trivially estimated. For the outer major arcs, we require a new estimate for exponential sums over primes in progressions defined as

$$S(N,\alpha,k,b)=\sum_{\substack{M<m\le N\\ m\equiv b\,(\mathrm{mod}\,k)}}\Lambda(m)e(m\alpha),\tag{1.3}$$

where $M=N(\log N)^{-3D}$ for some $D>0$ defined later. For the subset of $M_2(k)$ with $q\ge N^{\delta_1}$, the same estimate is applied.

To describe the new estimate, we first recall that using Dirichlet's theorem on rational approximation, we can always write

$$\alpha=a/q+\Lambda,\qquad |\Lambda|\le 1/qQ,\tag{1.4}$$

where $(a,q)=1$, $q\le Q$, and Q is a real number satisfying $N^{3/4}\le Q\le N$. We define $L=\log N$ and $d(N)$ is the number of divisors of N. For any two positive integers q and k we define

$$\begin{aligned}&h=(k,q),\ k=\prod_{i=1}^{g}p_i^{\alpha_i}k_0,\ q=\prod_{i=1}^{g}p_i^{\beta_i}q_0,\ \Big(q_0k_0,\prod_{i=1}^{g}p_i\Big)=1,\ \alpha_i,\beta_i\ge 1,\\ &\gamma_i=\min(\alpha_i,\beta_i),\ \delta_i=\begin{cases}\beta_i & \text{if } \gamma_i=\beta_i,\\ 0 & \text{else},\end{cases}\quad h_2=\prod_{i=1}^{g}p_i^{\delta_i},\ h_1=hh_2^{-1}.\end{aligned}\tag{1.5}$$

At each occurrence of h, its dependence on specific values of q and k will be obvious. Thus we do not need to index h as $h_{k,q}$. Using this notation, we will prove the following two results:

THEOREM 2. *For* $(k,b)=1$,

$$\begin{aligned}S(N,\alpha,k,b)&\ll d^{1/2}([q,k])L^c\frac{(q/h_2)^{1/2}}{[k,q]}\\ &\quad\times\Big([q,k]N^{1/2}\sqrt{1+|\Lambda|N}+[q,k]^{1/2}N^{4/5}+\frac{N}{\sqrt{1+|\Lambda|N}}\Big)+O(qL).\end{aligned}$$

THEOREM 3. *For* $(k,b)=1$,

$$\begin{aligned}&S(N,\alpha,k,b)\\ &\quad\ll L^cd([q,k])\frac{(q/h_2)^{1/2}}{[k,q]}\Big([q,k]N^{11/20}\sqrt{1+|\Lambda|N}+\frac{N}{\sqrt{1+|\Lambda|N}}\Big)+O(qL).\end{aligned}$$

In what follows, we first prove Theorems 2 and 3. Then we prove Theorem 1 using Theorem 2. We note that for $k=1$, Theorem 2 is [11, Theorem 1.1]. Theorem 3 will not be applied for the proof of Theorem 1, but it is proved here as an independent result whose proof is similar to the proof of Theorem 2.

2. Proof of Theorem 2. This proof uses an approach introduced in [11]. In particular, we will make use of [11, Lemma 3.1]:

LEMMA 2.1. *For any integer $u \ge 1$, $2 \le q \le x$, $q|\Lambda| \le x^{1-u}$,*

$$\sum_{\chi \bmod q} \Big| \sum_{M<m\le N} \Lambda(m)\chi(m)e(\Lambda m^u)\Big| \ll d^{1/2}(q)L^c\bigg(qN^{1/2}\sqrt{1+|\Lambda|N^u} + q^{1/2}N^{4/5} + \frac{N}{\sqrt{1+|\Lambda|N^u}}\bigg).$$

For any Dirichlet character χ, we introduce the following definition:

$$C_\chi(q,a,d,b) = \sum_{\substack{m=1\\ (m,q)=1\\ m\equiv b \,(\mathrm{mod}\, d)}}^{q} \chi(a)e(ma/q).$$

We will use [7, Lemma 2]:

LEMMA 2.2.

$$S(N, a/q+\Lambda, k, b) = \frac{1}{\phi(k/h_1)\phi(q/h_2)} \times \sum_{\psi \bmod k/h_1} \overline{\psi}(b) \sum_{\eta \bmod q/h_2} C_{\overline{\eta}}(q,a,h,b) \sum_{M<m\le N} \Lambda(m)\psi(m)\eta(m)e(m\Lambda) + O(L^2),$$

where h_1, h_2 are as defined in (1.5).

REMARKS. 1. Lemma 2.1 is shown in [11] with the summation range $N/2 < m \le N$ instead of $M < m \le N$. Dividing the interval $]M,N]$ into $\ll \log\log N$ intervals $]D,2D]$, Lemma 2.1 with $M < m \le N$ follows from [11]. The same applies to Lemma 2.2 which is stated with the summation range $N/2 < m \le N$ in [6]. Without further mentioning it in later parts of the paper, we will use the same argument to derive Lemmata 3.1 and 5.1 from [5] and [4], respectively.

2. Lemma 2.2 is identical to [7, Lemma 2] with one modification. In [7], the following definitions are used:

$$h_1 = \prod_{i=1}^{g} p_i^{\gamma_i}, \qquad h_2 = hh_1^{-1}. \tag{2.1}$$

One sees that in (1.5) we interchange the definitions of h_1 and h_2 given in (2.1). The proof in [7] uses the fact that $(k/h_1, q/h_2) = 1$. As this is also true with h_1 and h_2 as defined in (1.5), the proof of Lemma 2.2 is literally identical to the proof [7, Lemma 2].

We will also apply [7, Lemma 3]:

LEMMA 2.3. *Let $d\,|\,q$, $g\,|\,q$ and $(ab,q)=1$. Let χ* mod g *be induced by the primitive character χ^** mod g^*, *$g^*\,|\,g$. Then*

$$|C_\chi(q,a,d,b)| \le (g^*)^{1/2}.$$

We see from Lemmata 2.2 and 2.3 that

$$
\begin{aligned}
(2.2)\qquad & |S(N, a/q + \varLambda, r, b)| \\
&\le \frac{1}{\frac{k}{h_1}\frac{q}{h_2}} \sum_{\psi \bmod k/h_1} \sum_{\eta \bmod q/h_2} |C_{\overline{\eta}}(q, a, h, b)| \\
&\quad \times \Big| \sum_{M<m\le N} \varLambda(m)\psi(m)\eta(m)e(m\varLambda) \Big| + O(L^2) \\
&\ll \frac{L(q/h_2)^{1/2}}{[k,q]} \sum_{\psi \bmod k/h_1} \sum_{\eta \bmod q/h_2} \Big| \sum_{M<m\le N} \varLambda(m)\psi(m)\eta(m)e(m\varLambda) \Big| + O(L^2) \\
&\le \frac{L(q/h_2)^{1/2}}{[k,q]} \sum_{\chi \bmod [k,q]} \Big| \sum_{M<m\le N} \varLambda(m)\chi(m)e(m\varLambda) \Big| + O(L^2).
\end{aligned}
$$

Applying Lemma 2.1 with $u = 1$ to (2.2), we find

$$
\begin{aligned}
& S(N, \alpha, k, b) \\
&\quad \ll d^{1/2}([q,k]) L^c \frac{(q/h_2)^{1/2}}{[k,q]} \\
&\qquad \times \Big([q,k] N^{1/2} \sqrt{1 + |\varLambda| N} + [q,k]^{1/2} N^{4/5} + \frac{N}{\sqrt{1 + |\varLambda| N}} \Big) + O(L^2).
\end{aligned}
$$

3. Proof of Theorem 3. We will make use of [2, Lemma 4.1]:

LEMMA 3.1. *Let $s, r \ge 1$, and $Q \ge r$. Consider a set $H(s, r, Q)$ of characters $\chi = \xi\psi \bmod sq$, where ξ is a character modulo s and ψ is a primitive character modulo q, with $r \le q \le Q$, $r \mid q$, and $(q, s) = 1$. Then,*

$$
\sum_{\chi \in H(s,r,Q)} \Big| \sum_{M<m\le N} \varLambda(m)\chi(m)e(\varLambda m^u) \Big| \ll L^c \Big(\frac{Q^2 s}{r} x^{11/20} \sqrt{1 + |\varLambda| x} + \frac{x}{\sqrt{1 + |\varLambda| x}} \Big).
$$

From Lemma 3.1, we derive:

LEMMA 3.2. *For any integer $u \ge 1$, and any $\varLambda$ satisfying $|\varLambda| \le N^{-1/2}$,*

$$
(3.1)\qquad \sum_{\chi \bmod q} \Big| \sum_{M<m\le N} \varLambda(m)\chi(m)e(\varLambda m^u) \Big| \ll L^c d(q) \Big(q x^{11/20} \sqrt{1 + |\varLambda| x} + \frac{x}{\sqrt{1 + |\varLambda| x}} \Big).
$$

Proof. The characters χ mod q are induced by primitive characters χ_1^* mod $q_1, \ldots, \chi_w^*$ mod q_w to moduli $q_i \mid q$. Thus, using the notation from

Lemma 3.1, the left-hand side of (3.1) can be estimated as

$$(3.2)\qquad \begin{aligned} &\ll \sum_{q_i|q} \sum_{\chi\in H(q/q_i,q_i,q_i)} \Big| \sum_{M<m\le N} \Lambda(m)\chi(m)e(\Lambda m)\Big| \\ &\le d(q) \max_{q_i|q} \sum_{\chi\in H(q/q_i,q_i,q_i)} \Big| \sum_{M<m\le N} \Lambda(m)\chi(m)e(\Lambda m)\Big|. \end{aligned}$$

Applying Lemma 3.2 to (2.2), we find

$$\begin{aligned} &S(N,\alpha,k,b) \\ &\quad \ll d([q,k])L^c \frac{(q/h_2)^{1/2}}{[k,q]} \left([q,k]x^{11/20}\sqrt{1+|\Lambda|x} + \frac{x}{\sqrt{1+|\Lambda|x}} \right) + O(L^2). \end{aligned}$$

4. Outline of the proof of Theorem 1. In what follows, $[a_1,\ldots,a_n]$ and $(a_1,\ldots,a_n)$ denote the least common multiple and the greatest common divisor of the integers $a_1,\ldots,a_n$, respectively; c is an effective positive constant that may take different values at different occasions. For example, we may write $L^cL^c \ll L^c$. We use the familiar notations

$$r\sim R \;\Leftrightarrow\; R/2<r\le R, \qquad \sum_{1\le a\le q}{}^{*} := \sum_{\substack{1\le a\le q\\ (a,q)=1}}.$$

We write $e(\alpha)=e^{2\pi i\alpha}$ and p_i always denotes a prime number. We set

$$(4.1)\qquad C(q,a,k,b) = \sum_{\substack{m=1\\ (m,q)=1\\ m\equiv b\,(\mathrm{mod}\,(k,q))}}^{q} e(ma/q), \quad M(\alpha) = \sum_{M<m\le N} e(\alpha m),$$

$$(q,k,b_1,b_2) = \sum_{a=1}^{q}{}^{*} C(q,a,k,b_1)C(q,a,k,b_2)e(-an/q),$$

$$A(q,k,b_1,b_2) = B(q,k,b_1,b_2)/\phi^2([k,q]).$$

Finally, for $B>0$ we define

$$(4.2)\qquad P=N^{\delta_1},\quad P_1=kL^B,\quad Q=Nk^{-1}L^{-3B},\quad Q_1=N^{1-2\delta_1}.$$

where $0<\delta_1<1/10000$ is a very small positive constant. For a sufficiently small ϑ, the case $k<N^{\vartheta}$ is treated in [8]. Throughout this paper, we therefore limit our investigations to the case

$$(4.3)\qquad k>N^{\vartheta}$$

for a sufficiently small ϑ, and we further assume $\vartheta>100\delta_1>0$. We see from (4.2) that for $q\le P$, we have $Q_1>Qq$.

The proof uses the circle method. We divide the unit interval into the set of primary major arcs $M_1(k)$, the set of secondary major arcs $M_2(k)$, and the minor arcs $m(k)$, as follows:

$$M_1(k) = \bigcup_{\substack{k\le q\le P_1\\ k|q}} \bigcup_{(a,q)=1} \left[\frac{a}{q}-\frac{1}{qQ}, \frac{a}{q}+\frac{1}{qQ}\right],$$

$$M_2(k) = \bigcup_{j=1}^{3} M_{2j}(k),$$

$$M_{21}(k) = \bigcup_{\substack{q\le P\\ (k,q)=1}} \bigcup_{(a,q)=1} \left[\frac{a}{q}-\frac{1}{Q_1}, \frac{a}{q}+\frac{1}{Q_1}\right],$$

$$M_{22}(k) = \bigcup_{\substack{q\le P\\ (k,q)=1}} \bigcup_{(a,q)=1} \left[\frac{a}{q}-\frac{1}{qQ}, \frac{a}{q}-\frac{1}{Q_1}\right] \cup \left[\frac{a}{q}+\frac{1}{Q_1}, \frac{a}{q}+\frac{1}{qQ}\right],$$

$$M_{23}(k) = \bigcup_{\substack{P<q\le P_1\\ (k,q)=1}} \bigcup_{(a,q)=1} \left[\frac{a}{q}-\frac{1}{qQ}, \frac{a}{q}+\frac{1}{qQ}\right],$$

$$m(k) = \left[-\frac{1}{Q}, 1-\frac{1}{Q}\right] \setminus \bigcup_{1\le i\le 2} M_i(k).$$

Accordingly, we split the unit integral as follows:

$$(4.4)\qquad \int_{-1/Q}^{1-1/Q} \prod_{i=1}^{2} S(N,\alpha,k,b_i)e(-\alpha n)\,d\alpha$$

$$= \int_{M_1(k)} \prod_{i=1}^{2} S(N,\alpha,k,b_i)e(-n\alpha)\,d\alpha + \sum_{j=1}^{3} \int_{M_{2j}(k)} \prod_{i=1}^{2} S(N,\alpha,k,b_i)e(-n\alpha)\,d\alpha$$

$$+ \int_{m(k)} \prod_{i=1}^{2} S(N,\alpha,k,b_i)e(-n\alpha)\,d\alpha$$

$$=: R_{M_1}(n,k,b_1,b_2) + \sum_{j=1}^{3} R_{M_{2j}}(n,k,b_1,b_2) + R_m(n,k,b_1,b_2).$$

Theorem 1 can be derived from the following three results which we will show in the remainder of this paper. Let A, E, U, W, D be large integers that can take any positive value. In the following three results, the implied $\ll$-constants depend on some of these constants.

1. Set $T = N^s L^{-W}$, $\vartheta \le s \le 1/2$. For all but $\ll TL^{-E}$ prime numbers $k \le T$, all pairs of numbers b_1, b_2 satisfying $(b_1 b_2, k) = 1$, and all but $\ll Nk^{-1}L^{-D}$, integers n satisfying $N/2 < n \le N$, $n \equiv b_1 + b_2 \pmod{k}$, we have

$$R_{M_1}(n,k,b_1,b_2) = n\sigma(n,k) + O(Nk^{-1}L^{-A}), \tag{4.5}$$

where $\sigma(n,k) \gg 1/k$ is defined in (7.1).

2. For any prime number $k \le N^{1/3-5\delta_1}$, where δ_1 is as defined in (4.2), any integers b_1, b_2, and all but $\ll nk^{-1}L^{-D}$ integers $N/2 < n \le N$, $n \equiv b_1 + b_2 \pmod{k}$, we have

$$\max_{1\le j\le 3} |R_{M_{2j}}(n,k,b_1,b_2)| \ll Nk^{-1}L^{-A}. \tag{4.6}$$

3. For all $k \le N^{1/4}L^{-W}$ and any fixed, positive integer b_2 with $(b_2,k)=1$,

$$|R_m(n,k,b_1,b_2)| \ll Nk^{-1}L^{-A} \tag{4.7}$$

for all but $\ll kL^{-U}$ integers b_1, b_2 with $1 \le b_1 \le k$, $(b_1,k) = 1$, and all but $\ll Nk^{-1}L^{-D}$ integers n, $N/2 < n \le N$, satisfying $n \equiv b_1 + b_2 \pmod{k}$.

5. The integral over the minor arcs. We will use [4, Lemma 4.1]:

LEMMA 5.1. *For all $F > 0$, $V > 2F + 1$, $W > V/2$, let $Z \le N^{1/4}L^{-W}$ and $\alpha \in \mathbb{R}$ with $\|\alpha - u/v\| \le v^{-2}$ for some integers u, v with $(u,v) = 1$, and $ZL^V \le v \le NZ^{-1}L^{-V}$. Then for $Z \le d \le 2Z$,*

$$\frac{1}{\phi(d)} \sum_{\substack{b=1\\(b,d)=1}}^{d} |S(N,\alpha,d,b)|^2 \le N^2Z^{-2}L^{-F}.$$

We use Lemma 5.1 to show the following:

LEMMA 5.2. *For any $H > 0$ and any b_2,*

$$\sum_{\substack{b_1=1\\(b_1,k)=1}}^{k} \int_{m(k)} |S(N,\alpha,k,b_1)S(N,\alpha,k,b_2)|^2\,d\alpha \ll N^3k^{-2}L^{-H}.$$

Proof. Using Lemma 5.1 with $d = k$, $Z = k$, and $F \ge H + 2$, we see

$$\begin{aligned}
&\sum_{\substack{b_1=1\\(b_1,k)=1}}^{k} \int_{m(k)} |S(N,\alpha,k,b_1)S(N,\alpha,k,b_2)|^2\,d\alpha \\
&\qquad = \int_{m(k)} |S(N,\alpha,k,b_2)|^2 \sum_{\substack{b_1=1\\(b_1,k)=1}}^{k} |S(N,\alpha,k,b_1)|^2\,d\alpha \\
&\qquad \le \max_{\alpha\in m(k)} \sum_{\substack{1\le b_1\le k\\(b_1,k)=1}} |S(N,\alpha,k,b_1)|^2 \int_0^1 |S(N,\alpha,k,b_2)|^2\,d\alpha \\
&\qquad \ll N^2k^{-1}L^{-F}NL^2k^{-1} \le N^3k^{-2}L^{-H}.
\end{aligned}$$

We now prove (4.7). Let Y denote the number of integers b_1, $(b_1,k)=1$, $1\le b_1\le k$, not satisfying

$$\int\limits_{m(k)} |S(N,\alpha,k,b_1)S(N,\alpha,k,b_2)|^2\,d\alpha \ll N^3k^{-3}L^{-H/2}. \tag{5.1}$$

Then we see from Lemma 5.2 that

$$YN^3k^{-3}L^{-H/2} \ll N^3k^{-2}L^{-H} \;\Leftrightarrow\; Y\ll kL^{-H/2}. \tag{5.2}$$

Assume that b_1 satisfies (5.1). Using Parseval's identity and (5.1), we find

$$\begin{aligned}\sum_{\substack{N/2<n\le N\\ n\equiv b_1+b_2 \,(\mathrm{mod}\, k)}} |R_m(n,k,b_1,b_2)|^2 &\le \sum_{N/2<n\le N} |R_m(n,k,b_1,b_2)|^2\\ &\le \int\limits_{m(k)} |S(N,\alpha,k,b_1)S(N,\alpha,k,b_2)|^2\,d\alpha\\ &\ll N^3k^{-3}L^{-H/2}\ll N^3k^{-3}L^{-2A-D}\end{aligned} \tag{5.3}$$

for $H>H(A,D)$. We now derive (4.7) from (5.2) (for $H>H(U)$) and (5.3) in a standard way.

6. Upper bounds for the integral over $M_2(k)$. Using (4.3), (4.6) can be shown for $j=1$ by using a trivial upper bound:

$$\begin{aligned}R_{M_{21}}(n,k,b_1,b_2) &\ll \sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q}\frac{1}{Q_1}\left(\frac{NL}{k}\right)^2 \ll P^2Q_1^{-1}\left(\frac{NL}{k}\right)^2\\ &= N^{1+4\delta_1}L^2k^{-2}\ll Nk^{-1}L^{-A}.\end{aligned} \tag{6.1}$$

We now apply Theorem 2 to estimate the contribution of the integral over $M_{22}(k)$ and $M_{23}(k)$. We note that $d([q,k])\ll N^{\delta_1/100}$. Thus, noting that $|\Lambda|\ge Q_1^{-1}$ for $\alpha\in M_{22}(k)$, we have

$$\begin{aligned}&\max_{\alpha\in M_{22}(k)}|S(N,\alpha,k,b)|\\ &\quad\ll L^cN^{\delta_1/100}\max_{q\le P}\left(N^{1/2}q^{1/2}(1+k^{1/2}L^{3B/2}q^{-1/2})+\frac{N^{4/5}}{k^{1/2}}+\frac{N}{kN^{\delta_1}}\right)\\ &\qquad+O(qL)\\ &\quad\ll Nk^{-1}L^{-(A+D+1)/2}\end{aligned} \tag{6.2}$$

for $k\le N^{1/3-5\delta_1}$. For $\alpha\in M_{23}(k)$, we note that $(q/h_2)^{1/2}/[k,q]=q^{1/2}/kq=q^{-1/2}k^{-1}$, and obtain

$$(6.3)\qquad \max_{\alpha\in M_{23}(k)} |S(N,\alpha,k,b)|$$
$$\ll \max_{P<q\le P_1} L^c N^{\delta_1/100}\Big(N^{1/2}q^{1/2}(1+k^{1/2}q^{-1/2}L^{3B/2}) + \frac{N^{4/5}}{k^{1/2}} + \frac{N}{kP^{1/2}}\Big) + O(qL)$$
$$\ll Nk^{-1}L^{-(A+D+1)/2}$$

for $k\le N^{1/3-5\delta_1}$.

Using Parseval's identity, (6.2), and (6.3), we find

$$(6.4)\qquad \max_{j=2,3} \sum_{\substack{N/2<n\le N\\ n\equiv b_1+b_2 \,(\mathrm{mod}\, k)}} |R_{M_{2j}}(n,k,b_1,b_2)|^2$$
$$\le \max_{j=2,3} \sum_{N/2<n\le N} |R_{M_{2j}}(n,k,b_1,b_2)|^2$$
$$= \max_{j=2,3} \int_{M_{2j}} |S(N,\alpha,k,b_1)S(N,\alpha,k,b_2)|^2\,d\alpha$$
$$\le \max_{j=2,3}\max_{\alpha\in M_{2j}} |S(N,\alpha,k,b_1)|^2 \int_{-1/Q}^{1-1/Q} |S(N,\alpha,k,b_2)|^2\,d\alpha$$
$$\le \frac{N^2}{k^2}L^{-A-D-1}\frac{NL}{k} = N^3k^{-3}L^{-A-D}.$$

Now, for $j=2,3$, (4.6) follows from (6.4) in a standard way.

7. Lemmas for the integral over $M_1(k)$

LEMMA 7.1 (Bombieri–Vinogradov Theorem; see [3]). *Define*

$$E(x,q,a) = \sum_{\substack{p\le x\\ p\equiv a\,(\mathrm{mod}\, q)}} \log p - \frac{x}{\phi(q)},$$
$$E(x,q) = \max_{a,\,(a,q)=1} |E(x,q,a)|,\qquad E^*(x,q) = \max_{y\le x} E(y,q).$$

Let $W>0$ be fixed. Then, for $x^{1/2}L^{-W}\le Q\le x^{1/2}$,

$$\sum_{q\le Q} E^*(x,q) \ll x^{1/2}QL^5.$$

LEMMA 7.2 (see [1]). *Let k be an integer and set $k_q=(k,q)$. Then*

$$C(q,a,k,b) = \begin{cases} \mu(q/k_q)e(tba/k_q) & \textit{if } (q/k_q,k_q)=1,\ tq/k_q\equiv 1 \ (\mathrm{mod}\ k_q),\\ 0 & \textit{otherwise.}\end{cases}$$

LEMMA 7.3. *For $b_1 + b_2 \equiv n \pmod{k}$, any integer k, and k_q as defined in Lemma 7.2, we have:*

(a)
$$B(p^u, k, b_1, b_2) = \begin{cases} -1, & u = 1,\ (p, kn) = 1, \\ p - 1, & u = 1,\ (p, k) = 1,\ p \mid n, \\ p^u - p^{u-1}, & p \mid k,\ p^u = k_{p^u}, \\ 0, & \textit{else}. \end{cases}$$

(b) *$B(q, k, b_1, b_2)$ is multiplicative in q.*

Proof. Part (a) follows from Lemma 7.2. Part (b) is shown in a standard way.

LEMMA 7.4.

(a) *For any prime k, and any integers q, b_1, b_2,*

$$\sum_{\substack{q \geq U \\ (q,k)=1}} |A(q, k, b_1, b_2)| \ll k^{-2} L d(n) U^{-1}.$$

(b) *For any prime k and $U \geq k$,*

$$\sum_{\substack{q \geq U \\ k \mid q}} |A(q, k, b_1, b_2)| \ll L d(n) U^{-1}.$$

(c) *For any prime k,*

$$\sum_{q \geq 1} A(q, k, b_1, b_2) = \sigma(n, k),$$

where

$$\sigma(n, k) = \frac{kn}{\phi(k)\phi(kn)} \prod_{\substack{p \geq 2 \\ (p,kn)=1}} \left(1 - \frac{1}{(p-1)^2}\right). \tag{7.1}$$

Proof. For (a), we see from Lemma 7.3 that $|B(p^u, k, b_1, b_2)| \leq \phi((p^u, n))$ for $(p, k) = 1$. Therefore,

$$\begin{aligned} \sum_{\substack{q \geq U \\ (q,k)=1}} |A(q, k, b_1, b_2)| &\leq \frac{1}{\phi^2(k)} \sum_{q \geq U} \frac{1}{\phi^2(q)} \phi((q, n)) \\ &\leq \phi(k)^{-2} \sum_{d \mid n} \phi^{-1}(d) \sum_{q \geq U/d} \phi^{-2}(q) \\ &\ll \phi(k)^{-2} L^{1/2} U^{-1} d(n). \end{aligned} \tag{7.2}$$

For (b), we use Lemma 7.3 and (7.2) with U/d instead of U:

$$\sum_{\substack{q\ge U\\ k|q}} |A(q,k,b_1,b_2)| \le \sum_{d|k}\phi(d) \sum_{\substack{q\ge U/d\\ (q,k)=1}} |A(q,k,b_1,b_2)| \tag{7.3}$$
$$\ll \phi^{-2}(k)L^{1/2}U^{-1}d(n)\sum_{d|k}\phi(d)d \ll d(n)LU^{-1}.$$

(c) We see from (a) and (b) that the left-hand side of (7.1) is absolutely convergent. Thus, it is equal to its Euler product. Applying Lemma 7.3, we see that

$$\sum_{q\ge 1} A(q,k,b_1,b_2)$$
$$= \frac{1}{\phi^2(k)} \prod_{\substack{p\ge 2\\ (p,kn)=1}} \Big(1-\frac{1}{(p-1)^2}\Big) \prod_{\substack{(p,k)=1\\ p|n}} \Big(1+\frac{1}{(p-1)}\Big) \prod_{p|k}\Big(1+\sum_{\substack{b\ge 1\\ p^b|k}}(p^b-p^{b-1})\Big)$$
$$= \frac{kn}{\phi(k)\phi(kn)} \prod_{\substack{p\ge 2\\ (p,kn)=1}} \Big(1-\frac{1}{(p-1)^2}\Big).$$

LEMMA 7.5. *For any two coprime integers g and k, and any $D>0$,*

$$d(n) \ll L^{D+1}$$

for all but $Nk^{-1}L^{-D}$ integers $n\le N$ satisfying $n\equiv g \pmod{k}$.

Proof. We have the following estimate:

$$\sum_{\substack{n\le N\\ n\equiv g\,(\mathrm{mod}\,k)}} d(n) \ll \sum_{a\le N} \sum_{\substack{b\le N/a\\ ab\equiv g\,(\mathrm{mod}\,k)}} 1 \ll N/k \sum_{a\le N} 1/a \ll NL/k.$$

This implies the lemma.

8. The integral over $M_1(k)$. We first use an idea from [12] to modify the exponential sum $S(N,\alpha,k,b)$:

$$S(N,\alpha,k,b) = \sum_{\substack{c=1\\ c\equiv b\,(\mathrm{mod}\,(q,k))\\ (c,q)=1}}^{q} e(ac/q)T_{q,k,b,c}(\Lambda) + O(qL), \tag{8.1}$$

where

$$T_{q,k,b,c}(\Lambda) = \sum_{\substack{M<m\le N\\ m\equiv b\,(\mathrm{mod}\,k)\\ m\equiv c\,(\mathrm{mod}\,q)}} \Lambda(m)e(\Lambda m). \tag{8.2}$$

This implies

$$(8.3)\qquad |S(N,\alpha,k,b)| \le \sum_{\substack{c=1\\ c\equiv b\,(\mathrm{mod}\,(q,k))\\ (c,q)=1}}^{q} |T_{q,k,b,c}(\varLambda)| + O(qL).$$

Applying the theory of congruences, we find that for integers k,b,c,q that satisfy the conditions $(k,b)=(c,q)=1$ and $c\equiv b\ (\mathrm{mod}\,(k,q))$, there exists an integer $f=f(k,b,c,q)$ with $(f,[k,q])=1$ such that

$$n\equiv f\ (\mathrm{mod}\,[k,q]) \ \Leftrightarrow\ n\equiv b\ (\mathrm{mod}\ k),\ n\equiv c\ (\mathrm{mod}\ q).$$

Consequently,

$$\begin{aligned}(8.4)\quad T_{q,k,b,c}(\varLambda) &= \sum_{\substack{M<m\le N\\ m\equiv f\,(\mathrm{mod}\,[k,q])}} \Lambda(m)e(\varLambda m)\\ &= \sum_{\substack{m\le N\\ m\equiv f\,(\mathrm{mod}\,[k,q])}} \Lambda(m)e(\varLambda m) - \sum_{\substack{m\le M\\ m\equiv f\,(\mathrm{mod}\,[k,q])}} \Lambda(m)e(\varLambda m)\\ &=: T_1(\varLambda)-T_2(\varLambda).\end{aligned}$$

We follow the argument of [12] and approximate $T_1(\varLambda)$ as follows:

$$\begin{aligned}(8.5)\qquad T_1(\varLambda) &= -\int_0^N \Big(\sum_{\substack{m\le y\\ n\equiv f([k,q])}} \Lambda(m)\Big)\frac{d}{dy}e(\varLambda y)\,dy + \Big(\sum_{\substack{m\le N\\ n\equiv f([k,q])}} \Lambda(m)\Big)e(\varLambda N)\\ &= -\int_0^N \Big(\frac{y}{\phi([k,q])}+O(E^*(N,[k,q]))\Big)\frac{d}{dy}e(\varLambda y)\,dy\\ &\quad + \Big(\frac{N}{\phi([k,q])}+O(E^*(N,[k,q]))\Big)e(\varLambda N)+O(N^{1/2}L^c)\\ &= \frac{1}{\phi([k,q])}\Big(-\int_0^N y\Big(\frac{d}{dy}e(\varLambda y)\Big)\,dy + Ne(\varLambda N)\Big)\\ &\quad + O((1+|\varLambda|N)E^*(N,[k,q]))+O(N^{1/2}L^c)\\ &= \frac{1}{\phi([k,q])}\Big(-\int_0^N y\Big(\frac{d}{dy}e(\varLambda y)\Big)\,dy + Ne(\varLambda N)\Big)\\ &\quad + O(L^{3B}E^*(N,[k,q]))+O(N^{1/2}L^c)\end{aligned}$$

because $|\varLambda|N\le N/qQ\le N/kQ=L^{3B}$ as $k\le q\le P_1$ for $M_1(k)$. Here, the error term $O(N^{1/2}L^c)$ derives from a trivial estimate of the powers of primes with exponent larger than 1 which are not included in the Bombieri–Vinogradov terms $E^*(N,[k,q])$. $T_2(\varLambda)$ can be expressed in the same way with the variable M instead of N. Using that

$$\int_M^N e(\Lambda y)\,dy = M(\Lambda) + O(1),$$

where $M(\Lambda)$ is defined in (4.1), and applying partial integration, we derive from (8.4) and (8.5) that

$$T_{q,k,b,c}(\Lambda) = \frac{M(\Lambda)}{\phi([k,q])} + O(L^{3B}E^*(N,[k,q])) + O(N^{1/2}L^c). \tag{8.6}$$

We recall that for $M_1(k)$, we have $k \parallel q$. We will use this fact repeatedly during the remainder of this section. Combining (8.1) and (8.6), we obtain

$$\begin{aligned} S(N,\alpha,k,b_i) = {} & \frac{C(q,a,k,b_i)}{\phi([k,q])} M(\Lambda) + O(P_1k^{-1}L^{3B}E^*(N,[k,q])) \\ & + O(P_1k^{-1}qL + P_1k^{-1}N^{1/2}L^c). \end{aligned} \tag{8.7}$$

As $P_1k^{-1} = L^B$, we can introduce the following abbreviation:

$$\begin{aligned} A_i &= B_i + C + D := S(N,\alpha,k,b_i) \\ &= \frac{C(q,a,k,b_i)}{\phi([k,q])} M(\Lambda) + O(L^{4B}E^*(N,[k,q])) + O(qL^{B+1} + N^{1/2}L^{B+c}). \end{aligned} \tag{8.8}$$

We note the elementary identity

$$A_1A_2 = B_1B_2 + O\Big(C\Big(\max_{1\le i\le 2}|B_i| + C\Big)\Big) + O\Big(\max_{1\le i\le 2}|B_i|D + D^2 + DC\Big). \tag{8.9}$$

From (8.8), we see that

$$\max_{1\le i\le 2}|B_i| + C \ll \frac{NL^{4B+1}}{q}, \qquad D \ll N^{1/2}L^{B+c} \tag{8.10}$$

for $q \le P_1 = kL^B$ and $k \le N^{1/2}$. For the estimation of B_i we have used Lemma 7.2 and for the estimate of C we have used the definition of $E^*(x,q)$ in Lemma 7.1. Using (8.10), we see that the second error term in (8.9) is $O\big(\frac{NL^{4B+1}}{q}N^{1/2}L^{B+c}\big) = O(N^{3/2}L^{5B+c+1}/q)$. Using (8.9)–(8.10) and $M(\Lambda) \ll 1/|\Lambda|$, we find

$$\begin{aligned} R_{M_1}(n,k,b_1,b_2) = {} & \sum_{\substack{q\le P_1\\ k|q}} A(q,k,b_1,b_2) \int_{|\Lambda|\le 1/qQ} M^2(\Lambda)e(-n\Lambda)\,d\Lambda \\ & + O\Big(\sum_{\substack{q\le P_1\\ k|q}} \sideset{}{^*}\sum_{a=1}^{q} \Big(\frac{L^{3B}}{N}\,\frac{NL^{4B+1}}{q} L^{4B}E^*(N,[k,q]) + \frac{N^{3/2}L^{8B+c+1}}{qN}\Big)\Big) \end{aligned} \tag{8.11}$$

$$
\begin{aligned}
&= n \sum_{\substack{q \le P_1 \\ k \mid q}} A(q,k,b_1,b_2) + O\Big(NL^{-3D} \sum_{\substack{q \le P_1 \\ k \mid q}} |A(q,k,b_1,b_2)|\Big) \\
&\quad + O\Big(\sum_{\substack{q \le P_1 \\ k \mid q}} |A(q,k,b_1,b_2)| \int\limits_{1/qQ}^{1/2} \Lambda^{-2}\, d\Lambda\Big) \\
&\quad + O\Big(\sum_{\substack{q \le P_1 \\ k \mid q}} L^{11B+1} E^*(N,[k,q])\Big) + O(N^{1/2} L^{9B+c+1}) \\
&=: G_1(k) + G_2(k) + G_3(k) + G_4(k) + G_5(k).
\end{aligned}
$$

Using Lemma 7.4(b) with $U = k$, and Lemma 7.5, we see that for $D > D(A,E)$ and all but $\ll Nk^{-1}L^{-D}$ integers n satisfying $n \sim N$, $n \equiv b_1 + b_2 \pmod{k}$,

$$
G_2(k) \ll NL^{-3D} d(n) L k^{-1} \ll Nk^{-1} L^{-A-E-1}. \tag{8.12}
$$

Using once more Lemma 7.4(b) with $U = k$, and Lemma 7.5, we see that for $B > B(A,E,D)$ and for all but $\ll nk^{-1}L^{-D}$ integers n satisfying $n \sim N$, $n \equiv b_1 + b_2 \pmod{k}$,

$$
\begin{aligned}
G_3(k) &\ll d(n) L k^{-1} P_1 Q = d(n) L k^{-1} k L^B N k^{-1} L^{-3B} \\
&\ll Nk^{-1} L^{-A-E-1}.
\end{aligned} \tag{8.13}
$$

Obviously, for $k \le N^{1/4}$,

$$
G_5(k) \ll Nk^{-1} L^{-A-E-1}. \tag{8.14}
$$

Using Lemma, 7.4(a)–(c), we transform the main term

$$
\begin{aligned}
G_1(k) &= \sigma(n,k) n + O\Big(N \sum_{\substack{q \ge 1 \\ (q,k)=1}} |A(q,k,b_1,b_2)|\Big) \\
&\qquad + O\Big(N \sum_{\substack{q > P_1 \\ k \mid q}} |A(q,k,b_1,b_2)|\Big) \\
&= \sigma(n,k) n + O\big(N(d(N) L k^{-2} + d(N) L P_1^{-1})\big) \\
&=: \sigma(n,k) n + G_{11}(k).
\end{aligned} \tag{8.15}
$$

Using Lemma 7.5, we see that for all but $\ll Nk^{-1}L^{-D}$ integers n satisfying $n \sim N$, $n \equiv b_1 + b_2 \pmod{k}$, $A > 0$, $B > B(E,A,D)$,

$$
G_{11}(k) \ll Nk^{-1} L^{-A-E-1}. \tag{8.16}
$$

From (8.11)–(8.16) we obtain for all but $\ll Nk^{-1}L^{-D}$ integers n satisfying $n \sim N$, $n \equiv b_1 + b_2 \pmod{k}$, $A > 0$, $B > B(E, A, D)$,

$$\begin{aligned}(8.17)\qquad \sum_{\substack{k\le R\\ k\text{ prime}}} &|R_{M_1}(n,k,b_1,b_2) - n\sigma(n,k)| \\ &\le \sum_{\substack{k\le R\\ k\text{ prime}}} \big(|G_{11}(k)| + |G_2(k)| + |G_3(k)| + |G_4(k)| + |G_5(k)|\big) \\ &\ll L^{11B+1} \sum_{\substack{k\le R\\ k\text{ prime}}} \sum_{\substack{q\le P_1\\ k\mid q}} E^*(N,q) + \sum_{k\le R} Nk^{-1}L^{-A-E-1} \\ &\ll L^{11B+1} \sum_{\substack{k\le R\\ k\text{ prime}}} \sum_{\substack{q\le P_1\\ k\mid q}} E^*(N,q) + NL^{-A-E}\end{aligned}$$

for $B > B(A)$. Recalling $P_1 = kL^B$, we estimate the double summation in (8.17) as

$$\begin{aligned}(8.18)\qquad \sum_{\substack{k\le R\\ k\text{ prime}}} \sum_{\substack{q\le kL^B\\ k\mid q}} E^*(N,q) &\ll \sum_{q\le RL^B} E^*(N,q) \sum_{\substack{k\mid q\\ k\text{ prime}}} 1 \\ &\ll L \sum_{q\le RL^B} E^*(N,q).\end{aligned}$$

Inserting (8.18) into (8.17), we obtain, for $B > B(A)$,

$$\begin{aligned}(8.19)\qquad \sum_{\substack{k\le R\\ k\text{ prime}}} &|R_{M_1}(n,k,b_1,b_2) - n\sigma(n,k)| \\ &\ll L^{11B+2} \sum_{q\le RL^B} E^*(N,q) + NL^{-A-E}.\end{aligned}$$

We now assume that $R = N^{1/2}L^{-W}$ where $W = A + 12B + E + 7$ such that $RL^B \le N^{1/2}L^{-11B-A-E-7}$. Applying Lemma 7.1 to (8.19), we obtain

$$\begin{aligned}\sum_{\substack{k\le R\\ k\text{ prime}}} &|R_{M_1}(n,k,b_1,b_2) - n\sigma(n,k)| \\ &\ll L^{11B+2}NL^{-11B-A-E-7}L^5 + NL^{-A-E} \ll NL^{-A-E},\end{aligned}$$

for all but $\ll Nk^{-1}L^{-D}$, $n \sim N$, $n \equiv b_1 + b_2 \pmod{k}$, $A > 0$, $B > B(E, A, D)$. This implies (4.5).

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