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Cesàro means related to the square of the divisor function

by

Gintautas Bareikis and Algirdas Mačiulis (Vilnius)

1. Introduction. In what follows, p stands for a prime number. We use the notations $f = \mathrm{O}(g)$ and $f \ll g$ in their usual meaning. When implied constants depend upon some parameters, we sometimes indicate that by a subscript.

Let $\tau(n,u)$ be the number of natural divisors of n which do not exceed $u \geq 0$, and $\tau(n) := \tau(n,n)$.

Following J.-M. Deshouillers, F. Dress and G. Tenenbaum [3] for each $n \geq 2$ we define a random variable ξ_n by

$$P\left(\xi_n = \frac{\ln d}{\ln n}\right) = \frac{1}{\tau(n)},$$

as d runs through the set of all $\tau(n)$ divisors of n. In addition we may assume that $P(\xi_1 = 0) = 1$. Then the distribution function of the variable ξ_n is

$$F_n(t) := P(\xi_n \leq t) = \frac{\tau(n,n^t)}{\tau(n)}, \quad 0 \leq t \leq 1,\ n \in \mathbb{N}.$$

The sequence $\{F_n\}$ does not converge pointwise on $[0,1]$. However the averages

$$\frac{1}{x}\sum_{n\leq x} F_n(t)$$

uniformly converge for $t \in [0,1]$ as $x \to \infty$. Namely in [3] (see also [8, Section II.6.2]) the following result was obtained.

Theorem DDT. *Uniformly in* $t \in [0,1]$,

$$\frac{1}{x}\sum_{n\leq x} F_n(t) = \frac{2}{\pi}\arcsin\sqrt{t} + \mathrm{O}\left(\frac{1}{\sqrt{\ln x}}\right), \quad x \to \infty.$$

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Some generalizations of the DDT theorem were studied in [1, 2]. In the present paper we deal with the means of the squares

$$S_x(t) := \frac{1}{x}\sum_{n\le x} F_n^2(t),$$

and give for $S_x(t)$ an asymptotic formula similar to the DDT theorem.

2. Results. For $0 \le t \le 1/2$, define

$$I(t) := \frac{2}{\Gamma^4(1/4)}\int_0^t \frac{dw}{w^{3/4}}\int_0^{t-w}\frac{dv}{v^{3/4}}\int_0^w \frac{du}{(u(1-u-v-w))^{3/4}}.$$

This integral may be evaluated using the representation

$$I(t) = \frac{8}{\sqrt{2\pi}\,\Gamma^2(1/4)}\sum_{k=0}^{\infty}\frac{(2k)!}{(k!)^2 4^k(4k+1)}J_k(t), \tag{2.1}$$

where

$$J_k(t) := \int_0^t u^{k-1/4}(1-u)^{-k-3/4}\,du.$$

Moreover, integrating $J_k(t)$ by parts yields a simple iteration formula

$$J_k(t) = \frac{4}{4k-1}\left(\frac{t}{1-t}\right)^{k-1/4} - J_{k-1}(t), \qquad k\in\mathbb{N},$$

with the expression of $J_0(t)$ in terms of the distribution function of the beta law, $J_0(t) = \sqrt{2}\,\pi B(t; 3/4, 1/4)$.

The main result of this paper is

THEOREM 2.1. *Uniformly in $0 \le t \le 1$, we have*

$$S_x(t) = Q(t) + \mathrm{O}\left(\frac{1}{\sqrt[4]{\ln x}}\right) \tag{2.2}$$

as $x \to \infty$. Here

$$Q(t) := \begin{cases} I(t) & \text{if } t\in[0,1/2],\\ I(1-t) + \frac{4}{\pi}\arcsin\sqrt{t} - 1 & \text{if } t \in (1/2, 1].\end{cases}$$

Note that the DDT theorem and (2.2) remain true if $F_n(t)$ is replaced by $X_x(n,t) := \tau(n, x^t)/\tau(n)$. Following E. Manstavičius, N. M. Timofeev and G. Tenenbaum [5, 6, 7, 9], $X_x(n,t)$ can be considered as the sequence of arithmetical stochastic processes, provided a positive integer $n \le x$ is taken with probability $\nu_x(\{n\}) = 1/[x]$.

G. Tenenbaum [9] partly described the limit arithmetical process showing that the traces of this process are continuous. In addition he proved a general formula for the moments of $X_x(n,t)$ which approach the corresponding moments of the limit process. However the expression of these limit moments in simple form is an interesting but still open problem.

The limits in the DDT theorem and (2.2) can be understood as the asymptotic first and second moments of $X_x(n,t)$ respectively and give a partial solution to this problem.

3. Lemmas and preliminaries. As usual, let $\zeta(s)$, $s=\sigma+iu\in\mathbb{C}$, be the Riemann zeta function. The following lemma is a version of Theorem 3 in [8, Section II.5.3].

LEMMA 3.1 ([2]). *Let $z\in\mathbb{C}$, $M>0$, $0\le\rho<1$, and*

$$F(s):=\sum_{n=1}^{\infty}\frac{a_n}{n^s},\quad a_n\ge 0,$$

be such that the function $G_z(s):\ =F(s)\zeta^{-z}(s)$ can be continued as a holomorphic function for $\sigma\ge 1-c/\log(|u|+2)$ and in this domain satisfies the bound

$$|G_z(s)|\le M(1+|u|)^{\rho}.$$

Then for $A>0$, $|z|\le A$ and $x\ge 2$, we have

$$\sum_{n\le x}a_n=\frac{x}{\log^{1-z}x}\biggl(\frac{G_z(1)}{\Gamma(z)}+\mathrm{O}\biggl(\frac{M}{\log x}\biggr)\biggr),$$

where the implied constant depends at most on c, ρ, and A.

To estimate the mean values of positive multiplicative functions we will use the following classical result.

LEMMA 3.2 ([4]). *Let $g(m)$ be a multiplicative function with $0\le g(p^l)\le C$ for all prime numbers p and $l\in\mathbb{N}$ with some $C\ge 0$. Then*

$$\sum_{m\le x}g(m)\ll_C\frac{x}{\log x}\exp\biggl\{\sum_{p\le x}\frac{g(p)}{p}\biggr\},$$

$$\sum_{m\le x}\frac{g(m)}{m}\ll_C\exp\biggl\{\sum_{p\le x}\frac{g(p)}{p}\biggr\}.$$

The following result will be useful:

LEMMA 3.3. *Let $g:\mathbb{N}\to\mathbb{R}$ be a multiplicative function with $0\le g(m)\le 1$ for all $m\in\mathbb{N}$ and let $\sigma_0:=\log_2(9/5)$, $\varkappa>0$. Set*

$$L:=\prod_{p}\biggl(\sum_{m=0}^{\infty}\frac{g(p^m)}{p^m}\biggr)\biggl(1-\frac{1}{p}\biggr)^{\varkappa}.$$

If $0 \le g(2^k) \le 3/4$ *for all* $k \in \mathbb{N}$, *and*

$$\sum_p \frac{|g(p) - \varkappa|}{p^{\sigma_0}} \ll 1, \tag{3.1}$$

then for all $x \ge 1$ *and* $d \in \mathbb{N}$,

$$M(x,d) := \sum_{n \le x} g(nd) = \frac{x}{\ln^{1-\varkappa}(ex)}\left(\frac{L\tilde{g}(d)}{\Gamma(\varkappa)} + \mathrm{O}\left(\frac{\hat{g}(d)}{\ln(ex)}\right)\right). \tag{3.2}$$

Here $\tilde{g}$ *and* $\hat{g}$ *are the multiplicative functions defined by*

$$\tilde{g}(p^k) = \left(\sum_{m=0}^{\infty} \frac{g(p^m)}{p^m}\right)^{-1} \sum_{m=0}^{\infty} \frac{g(p^{m+k})}{p^m},$$

$$\hat{g}(p^k) = \left(1 - \sum_{m=1}^{\infty} \frac{g(p^m)}{p^{m\sigma_0}}\right)^{-1} \sum_{m=0}^{\infty} \frac{g(p^{m+k})}{p^{m\sigma_0}}, \qquad k \in \mathbb{N}.$$

Moreover, the estimates

$$\tilde{g}(p^k) = g(p^k) + \mathrm{O}\left(\frac{1}{p}\right), \qquad \hat{g}(p^k) = g(p^k) + \mathrm{O}\left(\frac{1}{p^{\sigma_0}}\right) \tag{3.3}$$

hold uniformly for any prime p *and integer* $k \ge 0$.

Proof. For $\operatorname{Re} s = \sigma > 1$ define

$$G_\varkappa(s) := \zeta^{-\varkappa}(s) \sum_{n=1}^{\infty} \frac{g(nd)}{n^s} = g(d,s)L(s),$$

where

$$g(d,s) := \prod_{p^k \| d} \left(\sum_{m=0}^{\infty} \frac{g(p^{m+k})}{p^{ms}}\right) \cdot \left(\sum_{m=0}^{\infty} \frac{g(p^m)}{p^{ms}}\right)^{-1},$$

$$L(s) := \prod_p \left(\sum_{m=0}^{\infty} \frac{g(p^m)}{p^{ms}}\right)\left(1 - \frac{1}{p^s}\right)^{\varkappa}.$$

Taking exponent and logarithm and then expanding the logarithms we get

$$L(s) = \exp\left\{\sum_p \frac{g(p) - \varkappa}{p^s} + \mathrm{O}\left(\sum_p \frac{1}{p^{2\sigma}}\right)\right\}.$$

Assume that $\sigma \ge \sigma_0$. Then $|g(d,s)| \le |\hat{g}(d)|$ and according to (3.1) we have $|L(s)| \ll 1$. Thus $G_\varkappa(s)$ has an analytic continuation into the region $\sigma \ge \sigma_0$ and in this domain satisfies the bound

$$|G_\varkappa(s)| \ll \hat{g}(d).$$

On the other hand it is easily verified that

$$\frac{x}{\ln^{1-\varkappa} x} = \frac{x}{\ln^{1-\varkappa} ex}\left(1 + \mathrm{O}\left(\frac{1}{\ln x}\right)\right), \qquad x \ge 2. \tag{3.4}$$

Since $g(d,1)=\tilde{g}(d)$ and $L(1)=L$, we see that Lemma 3.1 implies (3.2) for $x\geq 2$. If $1\leq x<2$, then

$$M(x,d)=g(d)\leq \hat{g}(d).$$

This completes the proof of (3.2) for all $x\geq 1$.

The relations (3.3) follow directly from the definitions, having in mind that

$$\sum_{m=1}^{\infty}\frac{g(p^m)}{p^{m\sigma_0}}\leq\frac{15}{16}$$

for any prime p. ∎

Set $g_0(m):=\tau^{-2}(m)$ and define the multiplicative functions

$$g_i(m):=\tilde{g}_{i-1}(m),\qquad h_i(m):=\hat{g}_{i-1}(m),\qquad i=1,2.$$

We note that

$$g_1(p^k)=\frac{\Phi(1/p,2,k+1)}{\Phi(1/p,2,1)},$$
$$g_2(p^k)=\left(\sum_{m=0}^{\infty}\frac{\Phi(1/p,2,m+1)}{p^m}\right)^{-1}\sum_{m=0}^{\infty}\frac{\Phi(1/p,2,k+m+1)}{p^m},$$

where

$$\Phi(x,l,a):=\sum_{i=0}^{\infty}\frac{x^i}{(i+a)^l}$$

is the function usually referred to as *Lerch transcendent*. Lemma 3.3 when used with the functions $g_0(m)$ and $g_1(m)$ implies the following corollary.

Corollary 3.4. *For $x\geq 1$, $d\in\mathbb{N}$ and $i\in\{0,1\}$ we have*

$$M_i(x,d):=\sum_{n\leq x}g_i(nd)=\frac{x}{\ln^{3/4}(ex)}\left(\frac{L_i g_{i+1}(d)}{\Gamma(1/4)}+\mathrm{O}\left(\frac{h_{i+1}(d)}{\ln(ex)}\right)\right),\tag{3.5}$$

where

$$L_i:=\prod_p\left(\sum_{m=0}^{\infty}\frac{g_i(p^m)}{p^m}\right)\left(1-\frac{1}{p}\right)^{1/4}.$$

The estimates

$$g_{i+1}(p^k)=\frac{1}{(k+1)^2}+\mathrm{O}\left(\frac{1}{p}\right),\qquad h_{i+1}(p^k)=\frac{1}{(k+1)^2}+\mathrm{O}\left(\frac{1}{p^{\sigma_0}}\right)\tag{3.6}$$

hold uniformly for any prime p and integer $k\geq 0$.

Proof. We have $g_0(p^k)=1/(k+1)^2$ for any prime p and $k\in\mathbb{N}$. Therefore $g_0(m)$ satisfies the conditions of Lemma 3.3 with $\varkappa=1/4$, and consequently (3.5) and (3.6) hold with $i=0$.

Further, since

$$h_1(p^k) < \left(\frac{1}{4} + \frac{g_0(p^2)}{2^{\sigma_0}} + \frac{g_0(p^3)}{2^{\sigma_0}(2^{\sigma_0}-1)}\right) \cdot \left(1 - \frac{g_0(p)}{2^{\sigma_0}} - \frac{g_0(p^2)}{2^{\sigma_0}(2^{\sigma_0}-1)}\right)^{-1},$$

it can be easily checked that

$$g_1(p^k) \le h_1(p^k) < \frac{1841}{4064}.$$

Hence we can apply Lemma 3.3 to the multiplicative function $g_1(m)$ and obtain the desired estimates with $i = 1$. ∎

Let us define the strongly multiplicative functions

$$g_3(n) := \prod_{p|n} \left(\sum_{m=0}^{\infty} \frac{g_2(p^m)}{p^m}\right)^{-1}, \quad h_3(n) := \prod_{p|n} \left(1 + \frac{1}{p^{\sigma_0}}\right)^{1/4}.$$

Lemma 3.5. *Let $\chi_0(n,l)$ be the principal character modulo l. Then for each $x \ge 1$ we have*

$$M_2(x,l) := \sum_{n \le x} g_2(n)\chi_0(n,l) = \frac{x}{\ln^{3/4}(ex)} \left(\frac{g_3(l)L_2}{\Gamma(1/4)} + \mathrm{O}\left(\frac{h_3(l)}{\ln(ex)}\right)\right), \tag{3.7}$$

where

$$L_2 := \prod_{p} \left(\sum_{m=0}^{\infty} \frac{g_2(p^m)}{p^m}\right) \left(1 - \frac{1}{p}\right)^{1/4}.$$

Proof. The argument goes along the same lines as in the proof of Lemma 3.3. For $\operatorname{Re} s = \sigma > 1$ the corresponding generating function can be written in the form

$$\begin{aligned} G(l,s) &= \zeta^{-1/4}(s) \sum_{m=1}^{\infty} \frac{g_2(m)\chi_0(m,l)}{m^s} \\ &= \prod_{(p,l)=1} \left(\sum_{m=0}^{\infty} \frac{g_2(p^m)}{p^{sm}}\right) \left(1 - \frac{1}{p^s}\right)^{1/4} \cdot \prod_{p|l} \left(1 - \frac{1}{p^s}\right)^{1/4}. \end{aligned}$$

From (3.6) it follows that $G(l,s)$ can be continued as a holomorphic function for $\sigma \ge \sigma_0$ and in this domain satisfies the bound $|G(l,s)| \ll h_3(l)$. Lemma 3.1 now yields

$$M_2(x,l) = \frac{x}{\ln^{3/4} x} \left(\frac{G(l,1)}{\Gamma(1/4)} + \mathrm{O}\left(\frac{h_3(l)}{\ln x}\right)\right)$$

for $x \ge 2$. Since $G(l,1) = g_3(l)L_2$, from this and (3.4) we obtain (3.7). ∎

We will need some estimates of the integral

$$I(T,y,\alpha,\beta) := \int_0^T \frac{dv}{(1+v)^{\alpha}(y-v)^{\beta}}. \tag{3.8}$$

Lemma 3.6. *Assume that α, β are positive numbers and $\alpha \neq 1$, $\beta \neq 1$. If $y \geq T+1 \geq 1$, then*

$$I(T,y,\alpha,\beta) \ll_{\alpha,\beta} y^{-\beta}|(T+1)^{1-\alpha}-1| + y^{-\alpha}|(y-T)^{1-\beta}-y^{1-\beta}|. \tag{3.9}$$

Proof. In case $T \leq y/2$ we have

$$I(T,y,\alpha,\beta) \leq \left(\frac{y}{2}\right)^{-\beta} \int_0^T \frac{dv}{(1+v)^\alpha}.$$

If $T > y/2$, then

$$\begin{aligned} I(T,y,\alpha,\beta) &= I(y/2,y,\alpha,\beta) + (T,y,\alpha,\beta) - I(y/2,y,\alpha,\beta)) \\ &\leq (y/2)^{-\beta} \int_0^T \frac{dv}{(1+v)^\alpha} + (y/2)^{-\alpha} \int_0^T \frac{dv}{(y-v)^\beta}, \end{aligned}$$

and (3.9) follows. ∎

For $K \geq eN \geq e$ and $\beta > 0$ we set

$$S := \sum_{1 \leq m \leq N} \frac{a_m}{m \ln^\beta(K/m)}, \qquad a_m \geq 0.$$

This sum may be evaluated in terms of the integral (3.8) provided some information about the behaviour of the sum

$$M(u) := \sum_{m \leq u} a_m$$

is given. Let us denote $I(\alpha,\beta) := I(\ln N, \ln K, \alpha, \beta)$ for short.

Lemma 3.7. *Assume that*

$$\left| M(u) - \frac{Au}{\ln^\alpha(eu)} \right| \leq \frac{Bu}{\ln^{\alpha+1}(eu)} \tag{3.10}$$

for some $\alpha > 0$ and $A, B \geq 0$. Then

$$\begin{aligned} |S - A \cdot I(\alpha,\beta)| &\leq A(1+\ln N)^{-\alpha} \ln^{-\beta} K \\ &\quad + B\frac{(\ln K - \varepsilon_\beta \ln N)^{-\beta}}{(1+\ln N)^{\alpha+1}} + (\alpha A + B\max(1,\,\beta))I(\alpha+1,\beta). \end{aligned}$$

Here $\varepsilon_\beta = 1$ if $\beta > 1$, and $\varepsilon_\beta = 0$ otherwise.

Proof. With $y := \ln K$ integration by parts yields

$$\begin{aligned} S &= \sum_{1 \leq m \leq N} \frac{a_m}{m(y - \ln m)^\beta} = \int_{0-}^{\ln N} \frac{e^{-v}}{(y-v)^\beta}\, dM(e^v) \\ &= \frac{M(N)}{N}(y - \ln N)^{-\beta} + \int_0^{\ln N} \frac{e^{-v}M(e^v)}{(y-v)^\beta}\left(1 - \frac{\beta}{y-v}\right) dv. \end{aligned}$$

In view of (3.10) this implies

$$|S - A\cdot I(\alpha,\beta)| \le A|\ln^{-\alpha}(eN)(y-\ln N)^{-\beta} - \beta I(\alpha,\beta+1)| + \frac{B(y-\ln N)^{-\beta}}{\ln^{\alpha+1}(eN)} + B\int_0^{\ln N}\left|1-\frac{\beta}{y-v}\right|\frac{dv}{(1+v)^{\alpha+1}(y-v)^{\beta}}. \tag{3.11}$$

We note that $y-v\ge 1$ and

$$I(\alpha,\beta+1)\ge(1+\ln N)^{-\alpha}\int_0^{\ln N}\frac{dv}{(y-v)^{\beta+1}} = \frac{(y-\ln N)^{-\beta}-y^{-\beta}}{\beta(1+\ln N)^{\alpha}}.$$

Now the desired inequality follows from (3.11) by considering the cases $\beta\le 1$ and $\beta>1$. ■

When (3.10) holds with $A=0$, the inequality (3.11) implies the following, estimate.

Corollary 3.8. *If for some $\alpha, B>0$,*

$$M(u)\le\frac{Bu}{\ln^{\alpha}(eu)},$$

then

$$S\le B\biggl(\frac{(\ln K-\varepsilon_\beta\ln N)^{-\beta}}{(1+\ln N)^{\alpha}}+\max(1,\beta)I(\alpha,\beta)\biggr).$$

Let us denote, for $\alpha\ge 0$ and $t\in(0,1/2]$,

$$\psi(\alpha,t):=\int_0^t\frac{dw}{(\alpha+w)^{3/4}(\alpha+t-w)^{3/4}}\int_0^w\frac{du}{(\alpha+u)^{3/4}(\alpha+1-t-u)^{3/4}}.$$

The next lemma yields the estimate of $\psi(\alpha,t)$ for small α.

Lemma 3.9. *Uniformly for $\alpha\ge0$ and $t\in(0,1/2]$,*

$$\psi(\alpha,t)=\psi(0,t)+\mathrm{O}(t^{-1/2}\alpha^{1/4}).$$

Moreover

$$\psi(0,t)=\frac{4\Gamma^2(1/4)}{\sqrt{2\pi}}(1-t)^{-3/4}t^{-1/4}\sum_{k=0}^{\infty}b_k\left(\frac{t}{1-t}\right)^k, \tag{3.12}$$

where

$$b_k=\frac{(2k)!}{(k!)^2 4^k(4k+1)},\qquad k=0,1,\dots.$$

Proof. To evaluate the inner integral in ψ we will use the hypergeometric function. For $|z|\le 1$ we have

$$\int_0^1 x^{-3/4}(1-xz)^{-3/4}\,dx=\sum_{k=0}^{\infty}c_k z^k,$$

where

$$c_k := \frac{\Gamma(3/4+k)\Gamma(1/4+k)}{\Gamma(3/4)\Gamma(5/4+k)k!} = \frac{4\Gamma(3/4+k)}{\Gamma(3/4)(4k+1)k!}.$$

Standard transformations yield

$$\int_0^w \frac{du}{(\alpha+u)^{3/4}(\alpha+1-t-u)^{3/4}} = \Big(\int_0^{w+\alpha} - \int_0^{\alpha}\Big)\frac{du}{u^{3/4}(2\alpha+1-t-u)^{3/4}}$$
$$= (2\alpha+1-t)^{-3/4}\sum_{k=0}^{\infty} \frac{c_k}{(2\alpha+1-t)^k}((w+\alpha)^{k+1/4} - \alpha^{k+1/4}).$$

Thus

$$(2\alpha+1-t)^{3/4}\psi(\alpha,t) = \sum_{k=0}^{\infty}\frac{c_k\psi_k(\alpha,t)}{(2\alpha+1-t)^k} - \alpha^{1/4}\int_0^t \frac{dw}{(\alpha+w)^{3/4}(\alpha+t-w)^{3/4}}\cdot\sum_{k=0}^{\infty} c_k\Big(\frac{\alpha}{2\alpha+1-t}\Big)^k, \tag{3.13}$$

where

$$\psi_k(\alpha,t) := \int_0^t (\alpha+w)^{k-1/2}(\alpha+t-w)^{-3/4}\,dw, \qquad k = 0,1,\ldots.$$

The integral in (3.13) can be estimated using the beta function:

$$\int_0^t \frac{dw}{(\alpha+w)^{3/4}(\alpha+t-w)^{3/4}} \le \int_0^t \frac{dw}{w^{3/4}(t-w)^{3/4}} = t^{-1/2}B(1/4,1/4).$$

Moreover, for $k \ge 1$,

$$c_k = \frac{4}{4k+1}\prod_{m=1}^{k}\Big(1-\frac{1}{4m}\Big) \le \frac{4k^{-1/4}}{4k+1}.$$

Therefore (3.13) becomes

$$\psi(\alpha,t) = (2\alpha+1-t)^{-3/4}\sum_{k=0}^{\infty}\frac{c_k\psi_k(\alpha,t)}{(2\alpha+1-t)^k} + \mathrm{O}(\alpha^{1/4}t^{-1/2}). \tag{3.14}$$

Let us estimate the distance $\psi_k(\alpha,t) - \psi_k(0,t)$. We have

$$\psi_k(0,t) = \int_0^t w^{k-1/2}(t-w)^{-3/4}\,dw = t^{k-1/4}B(k+1/2,1/4). \tag{3.15}$$

If $t \geq 2\alpha$, then

$$\begin{aligned}\psi_k(\alpha,t)-\psi_k(0,t) &= \int\limits_0^t (\alpha+w)^{k-1/2}((\alpha+t-w)^{-3/4}-(t-w)^{-3/4})\,dw \\ &\quad+\int\limits_0^t (t-w)^{-3/4}((\alpha+w)^{k-1/2}-w^{k-1/2})\,dw \\ &=: \psi_{k1}(\alpha,t)+\psi_{k2}(\alpha,t).\end{aligned}$$

Having in mind that $t \geq 2\alpha$, we can estimate

$$\begin{aligned}\psi_{k1}(\alpha,t) &\ll \alpha \int\limits_0^{t-\alpha} (\alpha+w)^{k-1/2}(t-w)^{-7/4}\,dw + \int\limits_{t-\alpha}^{t} (w+\alpha)^{k-1/2}(t-w)^{-3/4}\,dw \\ &\ll (\alpha+t)^{k-1/2}\alpha^{1/4}.\end{aligned}$$

Similarly we obtain, for $k \geq 1$,

$$\psi_{k2}(\alpha,t) \ll (\alpha+t)^{k-1/2}\alpha^{1/4}.$$

If $k=0$, then

$$\psi_{02}(\alpha,t) \ll \int\limits_0^\alpha w^{-1/2}(t-w)^{-3/4}\,dw + \alpha\int\limits_\alpha^t w^{-3/2}(t-w)^{-3/4}\,dw \ll t^{-1/2}\alpha^{1/4}.$$

Thus for $t \geq 2\alpha$ and $k \geq 0$ we have

$$\psi_k(\alpha,t)=\psi_k(0,t)+\mathrm{O}((\alpha+t)^{k-1/2}\alpha^{1/4}).$$

If $t \leq 2\alpha$, then it follows from (3.15) that

$$|\psi_k(\alpha,t)-\psi_k(0,t)| \leq \psi_k(\alpha,t)+\psi_k(0,t) \ll (\alpha+t)^{k-1/4}+t^{k-1/4},$$

since $\psi_0(\alpha,t) \leq \psi_0(0,t)$.

Substituting these estimates of $\psi_k(\alpha,t)$ into (3.14) we get

$$\psi(\alpha,t)=(2\alpha+1-t)^{-3/4}\sum_{k=0}^{\infty}\frac{c_k\psi_k(0,t)}{(2\alpha+1-t)^k}+\mathrm{O}(t^{-1/2}\alpha^{1/4}). \tag{3.16}$$

According to definition of c_k, it follows from (3.15) that

$$c_k\psi_k(0,t)=\frac{4\Gamma(k+1/2)\Gamma(1/4)}{\Gamma(3/4)(4k+1)k!}t^{k-1/4}=\frac{4\Gamma^2(1/4)}{\sqrt{2\pi}}b_k t^{k-1/4}.$$

Hence, taking $\alpha=0$ in (3.16), we obtain (3.12).

Finally, since

$$b_k \leq \frac{1}{(4k+1)\sqrt{k+1}},$$

the estimate (3.16) yields, for $\alpha > 0$,

$$|\psi(\alpha,t)-\psi(0,t)| \ll \sum_{k=0}^{\infty}\bigg(\frac{b_k t^{k-1/4}}{(1-t)^{k+3/4}}-\frac{b_k t^{k-1/4}}{(2\alpha+1-t)^{k+3/4}}\bigg)+t^{-1/2}\alpha^{1/4}$$
$$\ll \alpha t^{-1/4}(1-t)^{-7/4}\sum_{k\le 1/\alpha}(k+1)b_k\Big(\frac{t}{1-t}\Big)^k$$
$$+t^{-1/4}(1-t)^{-3/4}\sum_{k>1/\alpha}b_k\Big(\frac{t}{1-t}\Big)^k+t^{-1/2}\alpha^{1/4}$$
$$\ll t^{-1/4}\min(1,\sqrt{\alpha})+t^{-1/2}\alpha^{1/4}\ll t^{-1/2}\alpha^{1/4}.$$

This completes the proof of Lemma 3.9. ∎

4. Proof of Theorem 2.1. Our proof starts with the observation that

$$F_n^2(t)=(1-F_n(1-t))^2+\mathrm{O}\Big(\frac{1}{\tau(n)}\Big),$$

for $0\le t\le 1$. Hence Lemma 3.2 implies

$$S_x(t)=1-\frac{2}{x}\sum_{n\le x}F_n(1-t)+S_x(1-t)+\mathrm{O}\Big(\frac{1}{\sqrt{\ln x}}\Big).$$

Having in mind that $\arcsin\sqrt{1-t}=\pi/2-\arcsin\sqrt{t}$, by the DDT theorem we have

$$S_x(t)=S_x(1-t)+\frac{4}{\pi}\arcsin\sqrt{t}-1+\mathrm{O}\Big(\frac{1}{\sqrt{\ln x}}\Big), \tag{4.1}$$

uniformly for $t\in[0,1]$.

Therefore from now on we may consider $S_x(t)$ with $0\le t\le 1/2$ only. It is clear that

$$S_x(t)=\frac{1}{x}\sum_{n\le x}\frac{\tau^2(n,x^t)}{\tau^2(n)}-\frac{1}{x}\sum_{n\le x}\frac{\tau^2(n,x^t)-\tau^2(n,n^t)}{\tau^2(n)}. \tag{4.2}$$

As in the proof of the DDT theorem, the last sum in (4.2) can be estimated by

$$\frac{2}{x}\sum_{n\le x}\frac{\tau(n,x^t)-\tau(n,n^t)}{\tau(n)}=\mathrm{O}\Big(\frac{1}{\sqrt{\ln x}}\Big).$$

Furthermore we have

$$\frac{1}{x}\sum_{n\le x}\frac{\tau^2(n,x^t)}{\tau^2(n)}=T(x,t)-\frac{1}{x}\sum_{k\le x^t}\sum_{n\le x/k}\frac{1}{\tau^2(nk)}, \tag{4.3}$$

where

$$T(x,t) := \frac{2}{x}\sum_{l\le x^t}\sum_{\substack{m\le l\\(m,l)=1}}\sum_{k\le x^t/l}\sum_{n\le x/klm} 1/(\tau^2(nklm)).$$

For $0\le t\le 1/2$, Corollary 3.4 with $i=0$ and Lemma 3.2 imply

$$\sum_{k\le x^t}\sum_{n\le x/k}\frac{1}{\tau^2(nk)} \ll \frac{x}{\ln^{3/4} x}\sum_{k\le x^t}\frac{g_1(k)+h_1(k)}{k}$$
$$\ll \frac{x}{\ln^{3/4} x}\biggl(\exp\biggl(\sum_{p\le x^t}\frac{g_1(p)}{p}\biggr)+\exp\biggl(\sum_{p\le x^t}\frac{h_1(p)}{p}\biggr)\biggr) \ll \frac{x}{\sqrt{\ln x}}.$$

Substituting the above estimate into (4.3), we conclude from (4.2) that

$$S_x(t) = T(x,t) + \mathrm{O}(1/\sqrt{\ln x}). \tag{4.4}$$

In order to evaluate the inner sum in $T(x,t)$ we apply Corollary 3.4. Then

$$T(x,t) = T_1(x,t) + \mathrm{O}(R_1(x,t)), \tag{4.5}$$

where

$$T_1(x,t) := \frac{2\,L_0}{\Gamma(1/4)}\sum_{l\le x^t}\frac{1}{l}\sum_{\substack{m\le l\\(m,l)=1}}\frac{1}{m}\sum_{k\le x^t/l}\frac{g_1(klm)}{k\ln^{3/4}\frac{xe}{klm}},$$

$$R_1(x,t) := \sum_{l\le x^t}\frac{1}{l}\sum_{\substack{m\le l\\(m,l)=1}}\frac{1}{m}\sum_{k\le x^t/l}\frac{h_1(klm)}{k\ln^{7/4}\frac{xe}{klm}}.$$

Let us estimate $R_1(x,t)$. By Corollary 3.4, the multiplicative function $h_1(n)$ meets the conditions of Lemma 3.3, which implies the inequality

$$\sum_{k\le u} h_1(klm) \ll \frac{u}{\ln^{3/4}(eu)}\hat{h}_1(lm), \qquad u\ge 1,$$

with a multiplicative function $\hat{h}_1(n)$ satisfying

$$\hat{h}_1(p^k) = \frac{1}{(k+1)^2} + \mathrm{O}\Bigl(\frac{1}{p^{\sigma_0}}\Bigr). \tag{4.6}$$

Hence, Corollary 3.8 and Lemma 3.6 yield

$$\sum_{k\le x^t/l}\frac{h_1(klm)}{k\ln^{7/4}\frac{xe}{klm}}$$
$$\ll \hat{h}_1(lm)\biggl(I\biggl(\ln\frac{x^t}{l},\ln\frac{ex}{lm},\frac{3}{4},\frac{7}{4}\biggr)+\ln^{-3/4}\frac{ex^t}{l}\cdot\ln^{-7/4}\frac{ex^{1-t}}{m}\biggr)$$
$$\ll \hat{h}_1(lm)\ln^{-3/2}\frac{ex^{1-t}}{m}.$$

Therefore

$$R_1(x,t) \ll \sum_{l\le x^t} \frac{\hat h_1(l)}{l} \sum_{m\le l} \frac{\hat h_1(m)}{m} \ln^{-3/2} \frac{ex^{1-t}}{m}.$$

The estimate (4.6) and Lemma 3.2 imply

$$\sum_{m\le u} \hat h_1(m) \ll \frac{u}{\ln u} \exp\biggl(\sum_{p\le u} \frac{\hat h_1(p)}{p}\biggr) \ll \frac{u}{\ln^{3/4}(eu)}.$$

Then by repeated application of Corollary 3.8 and Lemma 3.6 we deduce

$$R_1(x,t) \ll \frac{1}{\ln^{3/4} x} \sum_{l\le x^t} \frac{\hat h_1(l)}{l} \ln^{-1/2} \frac{ex^{1-t}}{l} \ll \frac{1}{\ln x}. \tag{4.7}$$

Next we have to evaluate $T_1(x,t)$. Corollary 3.4 together with Lemmas 3.7 and 3.6 yiels

$$T_1(x,t) = T_2(x,t) + \mathrm{O}(R_2(x,t)), \tag{4.8}$$

where

$$T_2(x,t) := \frac{2\,L_0L_1}{\Gamma^2(1/4)} \sum_{l\le x^t} \frac{g_2(l)}{l} \sum_{\substack{m\le l\\ (m,l)=1}} \frac{g_2(m)}{m}\, I\biggl(\ln\frac{x^t}{l}, \ln\frac{ex}{lm}, \frac{3}{4}, \frac{3}{4}\biggr),$$

$$R_2(x,t) := \sum_{l\le x^t} \frac{h_2(l)}{l} \sum_{m\le l} \frac{h_2(m)}{m} \ln^{-3/4} \frac{ex^{1-t}}{m}.$$

$R_2(x,t)$ may be handled in much the same way as $R_1(x,t)$. So we have

$$R_2(x,t) \ll \frac{1}{\ln^{1/2} x} \sum_{l\le x^t} \frac{h_2(l)}{l} \ll \frac{1}{\ln^{1/4} x}. \tag{4.9}$$

Now let us consider $T_2(x,t)$. Interchanging summation and integration in the inner sum, we have

$$T_2(x,t) = \frac{2\,L_0L_1}{\Gamma^2(1/4)} \sum_{l\le x^t} \frac{g_2(l)}{l} \int_0^{\ln(x^t/l)} \frac{V_l(v)}{(1+v)^{3/4}}\, dv,$$

where

$$V_l(v) := \sum_{m\le l} \frac{g_2(m)\chi_0(m,l)}{m \ln^{3/4}\frac{K}{m}}, \quad K := \frac{x}{l} e^{1-v}.$$

Lemmas 3.5–3.7 lead to the estimate

$$V_l(v) = \frac{g_3(l)L_2}{\Gamma(1/4)} \int_0^{\ln l} \frac{du}{(1+u)^{3/4}(\ln K - u)^{3/4}} + \mathrm{O}\biggl(\frac{h_3(l)}{\ln^{3/4} K}\biggr).$$

Thus, setting

$$T_3(x,t) := \sum_{l\le x^t} \frac{g_2(l)g_3(l)}{l} \int\limits_0^{\ln(x^t/l)} \frac{dv}{(1+v)^{3/4}} \int\limits_0^{\ln l} \frac{(1+u)^{-3/4}\,du}{(\ln(ex/l)-u-v)^{3/4}},$$

we have

$$T_2(x,t) = \frac{2\,L_0L_1L_2}{\Gamma^3(1/4)} T_3(x,t) + \mathrm{O}(R_3(x,t)),$$

where

$$R_3(x,t) \ll \sum_{l\le x^t} \frac{g_2(l)h_3(l)}{l} \int\limits_0^{\ln(x^t/l)} \frac{dv}{(1+v)^{3/4}\ln^{3/4} K} \ll \frac{1}{\ln^{1/4} x}.$$

The latter inequality follows from Lemma 3.2, since $g_2(p)h_3(p) = 1/4 + \mathrm{O}(p^{-\sigma_0})$. Combining these estimates with (4.4), (4.5), (4.7), (4.8) and (4.9) we get

$$S_x(t) = \frac{2\,L_0L_1L_2}{\Gamma^3(1/4)} T_3(x,t) + \mathrm{O}\left(\frac{1}{\ln^{1/4} x}\right). \tag{4.10}$$

Now we have to evaluate $T_3(x,t)$. It may be easily checked that the multiplicative function $g_4(m) := g_2(m)g_3(m)$ satisfies the conditions of Lemma 3.3 with $\varkappa = 1/4$. Hence

$$M(u) := \sum_{n\le u} g_4(n) = \frac{u}{\ln^{3/4}(eu)}\left(\frac{L_4}{\Gamma(1/4)} + \mathrm{O}\left(\frac{1}{\ln(eu)}\right)\right) \tag{4.11}$$

for any $u \ge 1$ and

$$L_4 := \prod_p \left(\sum_{m=0}^{\infty} \frac{g_4(p^m)}{p^m}\right)\left(1-\frac{1}{p}\right)^{1/4}.$$

Set

$$F(w) := \int\limits_0^{t\ln x - w} \frac{dv}{(1+v)^{3/4}} \int\limits_0^{w} \frac{du}{(1+u)^{3/4}(1+\ln x - u - v - w)^{3/4}}.$$

Then

$$T_3(x,t) = \sum_{l\le x^t} \frac{g_4(l)}{l} F(\ln l) = \int\limits_{0-}^{t\ln x} e^{-w} F(w)\, dM(e^w).$$

Integrating the integral by parts and applying the estimate (4.11) we get

$$T_3(x,t) = \frac{L_4}{\Gamma(1/4)} \int\limits_0^{t\ln x} \frac{F(w)\,dw}{(1+w)^{3/4}} + \mathrm{O}\left(\int\limits_0^{t\ln x} \frac{F(w)+|F'(w)|}{(1+w)^{7/4}}\,dw\right). \tag{4.12}$$

By (3.8) we have

$$F(w)=\int_0^{t\ln x-w}\frac{I(w,1+\ln x-v-w,3/4,3/4)}{(1+v)^{3/4}}\,dv$$

and

$$F'(w)=\frac{3}{4}\int_0^{t\ln x-w}\frac{I(w,1+\ln x-v-w,3/4,7/4)}{(1+v)^{3/4}}\,dv$$
$$+\frac{I(t\ln x-w,1+\ln x-2w,3/4,3/4)}{(1+w)^{3/4}}-\frac{I(w,1+(1-t)\ln x,3/4,3/4)}{(1+t\ln x-w)^{3/4}}.$$

Estimating these integrals by means of Lemma 3.6 we deduce that

$$F(w)+|F'(w)|\ll\frac{1}{(1+w)^{3/4}\sqrt{1+\ln x-2w}}+\frac{1}{\ln^{1/4}x}$$

and consequently the remainder term in (4.12) is $\mathrm{O}(\ln^{-1/4}x)$. Hence, after routine transformations of the integral in the main term, the relation (4.12) can be rewritten as

$$T_3(x,t)=\frac{L_4}{\Gamma(1/4)}\Psi(\alpha_x,t)+\mathrm{O}\biggl(\frac{1}{\ln^{1/4}x}\biggr),$$

where $\alpha_x:=\ln^{-1}x$ and

$$\Psi(\alpha,t):=\int_0^t\frac{dw}{(\alpha+w)^{3/4}}\int_0^{t-w}\frac{dv}{(\alpha+v)^{3/4}}\int_0^w\frac{du}{((\alpha+u)(\alpha+1-u-v-w))^{3/4}}.$$

From this and (4.10) we obtain

$$S_x(t)=\frac{2\,L_0L_1L_2L_4}{\Gamma^4(1/4)}\Psi(\alpha_x,t)+\mathrm{O}\biggl(\frac{1}{\ln^{1/4}x}\biggr). \tag{4.13}$$

In order to evaluate $\Psi(\alpha_x,t)$ we will use the estimates of its derivative contained in Lemma 3.9. For any $\alpha\geq 0$ we have $\Psi(\alpha,0)=0$ and

$$\frac{\partial\Psi(\alpha,t)}{\partial t}=\psi(\alpha,t),\quad t\in(0,\,1/2].$$

By Lemma 3.9 this yields

$$|\Psi(\alpha_x,t)-\Psi(0,t)|=\biggl|\int_0^t(\psi(\alpha_x,u)-\psi(0,u))\,du\biggr| \tag{4.14}$$

$$\ll\alpha_x^{1/4}\int_0^t\frac{du}{\sqrt{u}}\ll\frac{1}{\ln^{1/4}x}. \tag{4.15}$$

It remains to evaluate the coefficient in the main term of (4.13). For brevity let us denote

$$K_1(p) := \sum_{k=0}^{\infty} \Phi(1/p, 2, k+1)\, p^{-k},$$

$$K_2(p) := \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \Phi(1/p, 2, k+m+1)\, p^{-k-m}.$$

By the definitions of the multiplicative functions g_0, g_1, g_2, g_4 we have

$$L_0 L_1 L_2 L_4 = \prod_p \Phi(1/p, 2, 1) \cdot \frac{K_1(p)}{\Phi(1/p, 2, 1)} \cdot \frac{K_2(p)}{K_1(p)} \cdot \Big(2 - \frac{K_1(p)}{K_2(p)}\Big)\Big(1 - \frac{1}{p}\Big)$$

$$= \prod_p (2K_2(p) - K_1(p))(1 - 1/p).$$

Consider the sum $K_2(p)$. Changing summation indices yields

$$K_2(p) = \sum_{n=0}^{\infty} (n+1) \frac{\Phi(1/p, 2, n+1)}{p^n} = \frac{K_1(p)}{2} + \sum_{n=0}^{\infty} (n+1/2) \frac{\Phi(1/p, 2, n+1)}{p^n}.$$

Expanding the function Φ and collecting terms in the latter sum we obtain

$$K_2(p) = \frac{K_1(p)}{2} + \sum_{m=0}^{\infty} \frac{p^{-m}}{(m+1)^2} \sum_{n=0}^{m} (n+1/2) = \frac{1}{2}(K_1(p) + \frac{p}{p-1}).$$

This gives $L_0 L_1 L_2 L_4 = 1$ and together with (4.13)–(4.14) completes the proof of Theorem 2.1.

Finally, since

$$I'(t) = \frac{2}{\Gamma^4(1/4)} \frac{\partial \Psi(0, t)}{\partial t} = \frac{2}{\Gamma^4(1/4)} \psi(0, t),$$

the representation (2.1) follows from (3.12).

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Gintautas Bareikis, Algirdas Mačiulis (corresponding author)
Department of Mathematics and Informatics
Vilnius University
Naugarduko 24
LT-03225 Vilnius, Lithuania
E-mail: gintautas.bareikis@mif.vu.lt
algirdas.maciulis@mif.vu.lt

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