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Kummer congruences for values of Bernoulli and Euler polynomials

by

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Dedicated to the memory of Bernard Dwork

1. Introduction. The Bernoulli polynomials $B_n(x)$ may be defined by

$$\left(\frac{t}{e^t-1}\right)e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1.1}$$

and their values at $x=0$ are called the Bernoulli numbers and denoted B_n. The strong form of the Kummer congruences (cf. [1]) states that if p is an odd prime, $c\equiv 0 \pmod{(p-1)p^a}$, and $p-1$ does not divide m, then

$$\Delta_c^k\left\{(1-p^{m-1})\frac{B_m}{m}\right\}\equiv 0 \ (\operatorname{mod} p^{k(a+1)}\mathbb{Z}_p), \tag{1.2}$$

where Δ_c is the forward difference operator with increment c and Δ_c^k denotes the kth compositional iterate of this operator.

In this paper we give a generalization to Bernoulli polynomials $B_n(x)$ where the argument x may be any p-adic integer. Specifically, we show in Theorem 3.2 below that

$$\Delta_c^k\left\{\frac{B_m(x)-p^{m-1}B_m(x')}{m}\right\}\equiv 0 \ (\operatorname{mod} p^{k(a+1)}\mathbb{Z}_p) \tag{1.3}$$

under the above hypotheses, as well as an extension to the case $p=2$. The map $x\mapsto x'$ appearing in (1.3) is Dwork's shift map, defined for $x\in\mathbb{Z}_p$ by the relation $px'-x=\mu_x\in\{0,1,\ldots,p-1\}$ (cf. [3], Ch. 8). A version of this result was first given in [4] in the case where $k=1$, $p\geq 5$, and $x\in\mathbb{Z}_p\cap\mathbb{Q}$. Our method is based on general properties of the p-adic Γ-transform recorded in [11] and yields a nontrivial analogous result in the case $p=2$. We also treat the Euler polynomials $H_n(u,x)$ in the same manner.

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Additionally, in each case we give an analogous congruence with Δ_c^k replaced by a binomial coefficient operator, as in [7].

In §4 we prove Kummer congruences for generalized Bernoulli polynomials $B_{n,\chi}(x)$ associated to a Dirichlet character χ. We demonstrate in Theorem 4.2 an analogue of (1.3) for $B_{m,\chi\omega^{-m}}(x)$ which holds for $x \in p\mathbb{Z}_p$ as long as χ is not a character of the second kind. The members of these congruences coincide with values of the two-variable p-adic L function defined by G. Fox in [6]. We also show how this congruence can be extended in certain cases where χ is a character of the first kind and x lies in $\mathbb{Z}_p^\times$.

2. Preliminaries. Throughout this paper p will denote a prime number, $\mathbb{Z}_p$ the ring of p-adic integers, and $\mathbb{Q}_p$ the field of p-adic numbers. If K is a finite extension of $\mathbb{Q}_p$ then $\mathfrak{O}_K$ will denote its ring of integers and $\mathfrak{O}_K^\times$ will denote the multiplicative group of units in $\mathfrak{O}_K$. Define the quantity q by setting $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. The Teichmüller character ω on $\mathbb{Z}_p^\times$ is defined by setting $\omega(x)$ to be the unique $\phi(q)$th root of unity congruent to x modulo $q\mathbb{Z}_p$. We use $\mathfrak{O}_K[T-1]$ and $\mathfrak{O}_K[[T-1]]$ to denote respectively the ring of polynomials and of formal power series in the indeterminate $T-1$ over $\mathfrak{O}_K$. We use "ord_p" to denote the additive valuation on K normalized by $\mathrm{ord}_p\, p = 1$. Finally, e^t denotes the exponential function defined by the power series $\sum_{n=0}^\infty t^n/n!$ for $\mathrm{ord}_p\, t > 1/(p-1)$.

If c is a nonnegative integer, the difference operator Δ_c operates on the sequence $\{a_m\}$ by

$$\Delta_c\, a_m = a_{m+c} - a_m. \tag{2.1}$$

The powers Δ_c^k of Δ_c are defined by $\Delta_c^0 =$ identity and $\Delta_c^k = \Delta_c \circ \Delta_c^{k-1}$ for positive integers k, so that

$$\Delta_c^k\, a_m = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} a_{m+jc} \tag{2.2}$$

for all nonnegative integers k. To define binomial coefficient operators $\binom{D}{k}$ associated to an operator D (cf. [7]), we write the binomial coefficient

$$\binom{X}{k} = \frac{X(X-1)\dots(X-k+1)}{k!} \tag{2.3}$$

for $k \ge 0$ as a polynomial in X, and replace X by D. Since the particular sequences considered in this paper have multiple indices, we shall always use the index m to denote the index on which an operator operates.

Define the linear operator φ by

$$\varphi h(T) = h(T) - \frac{1}{p} \sum_{\zeta^p=1} h(\zeta T). \tag{2.4}$$

This operator is well defined and stable on rational functions, and also on $\mathfrak{O}_K[[T-1]]$ (cf. [11], (2.14)). If $h(e^t) = \sum a_n t^n/n!$, write $(\varphi h)(e^t) = \sum \widehat{a}_n t^n/n!$. The following congruences for the numbers $\widehat{a}_n$ were proved in [11]:

THEOREM 2.1. *Let $h \in \mathfrak{O}_K[[T-1]]$ and write $\varphi h(e^t) = \sum_{n=0}^{\infty} \widehat{a}_n t^n/n!$. Then $\widehat{a}_n \in \mathfrak{O}_K$ for all n. Furthermore, if $c \equiv 0 \pmod{\phi(q)p^a}$ with $a \geq 0$ then*

$$\Delta_c^k \widehat{a}_m \equiv 0 \pmod{p^{ka^+}\mathfrak{O}_K}$$

for all $m, k \geq 0$, where $a^+ = a+1$ if $p > 2$ and $a^+ = a+3$ if $p = 2$, and also

$$\binom{p^{-r}\Delta_c}{k} \widehat{a}_m \in \mathfrak{O}_K$$

for $0 \leq r \leq a^+$ and all $m, k \geq 0$.

It will be observed from this theorem that the operator $(p^{-a^+}\Delta_c)^k$ is a polynomial of order k in Δ_c with leading coefficient p^{-ka^+} which sends $\widehat{a}_m$ into $\mathfrak{O}_K$, whereas the binomial coefficient operator $\binom{p^{-a^+}\Delta_c}{k}$ is a polynomial of order k in Δ_c with leading coefficient $p^{-ka^+}/k!$ which sends $\widehat{a}_m$ into $\mathfrak{O}_K$.

The proof of this theorem made use of the correspondence

$$\Lambda \leftrightarrow \mathfrak{O}_K[[T-1]] \tag{2.5}$$

where Λ denotes the set of all $\mathfrak{O}_K$-valued measures on $\mathbb{Z}_p$, under which each measure $\alpha \in \Lambda$ corresponds to the formal power series $h \in \mathfrak{O}_K[[T-1]]$ defined by

$$h(T) = \int_{\mathbb{Z}_p} T^x \, d\alpha(x). \tag{2.6}$$

From this it follows that

$$a_n = \int_{\mathbb{Z}_p} x^n \, d\alpha(x). \tag{2.7}$$

We also observed ([11], (2.14)) that

$$\varphi h(T) = \int_{\mathbb{Z}_p^\times} T^x \, d\alpha(x), \tag{2.8}$$

which implies

$$\widehat{a}_n = \int_{\mathbb{Z}_p^\times} x^n \, d\alpha(x). \tag{2.9}$$

Since

$$(1-\varphi)h(T) = \int_{p\mathbb{Z}_p} T^x \, d\alpha(x) = \int_{\mathbb{Z}_p} T^{px} \, d\alpha(px), \tag{2.10}$$

we see that $(1-\varphi)h(T) \in \mathfrak{O}_K[[T^p - 1]]$ according to the correspondence

(2.5), (2.6). Therefore there is a linear operator ψ on $\mathfrak{O}_K[[T-1]]$ such that $(\psi h)(T^p) = (1-\varphi)h(T)$. From (2.4) we see that the operator ψ may be defined by

$$\psi h(T) = \frac{1}{p}\sum_{Z^p=T} h(Z), \tag{2.11}$$

and therefore coincides on $\mathfrak{O}_K[[T-1]]$ with Dwork's ψ operator (cf. [3], Ch. 5). So if we write $(\psi h)(e^t) = \sum_{n=0}^{\infty} a_n^* t^n/n!$, then $a_n^* \in \mathfrak{O}_K$ for all n and

$$\widehat{a}_n = a_n - p^n a_n^*. \tag{2.12}$$

Now if $g \in \mathfrak{O}_K[[T-1]]$ and $g(e^t) = \sum_{n=0}^{\infty} u_n t^n/n!$, let us define polynomials $u_n(x) \in \mathfrak{O}_K[x]$ by

$$e^{xt}g(e^t) = \sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!}. \tag{2.13}$$

It is easily seen that with this definition we have $u_n(0) = u_n$, and in general

$$u_n(x) = \sum_{k=0}^{n} \binom{n}{k} u_{n-k}x^k. \tag{2.14}$$

The sequence $u_n(x)$ is therefore an example of an Appell family of polynomials, since the degree of u_n is n and $\frac{d}{dx}u_n(x) = nu_{n-1}(x)$ for all n. Furthermore, since

$$\psi(T^{p\tau}h(T)) = T^{\tau}(\psi h)(T) \tag{2.15}$$

for $\tau \in \mathbb{Z}_p$, we have

$$(a_n(p\tau))^* = a_n^*(\tau), \tag{2.16}$$

or equivalently,

$$\widehat{a_n(p\tau)} = a_n(p\tau) - p^n a_n^*(\tau), \tag{2.17}$$

for $\tau \in \mathbb{Z}_p$. These considerations demonstrate that congruences associated to a sequence $\{a_n\}$ produced by Theorem 2.1 extend immediately to $\{a_n(x)\}$ for $x \in p\mathbb{Z}_p$, as we record below:

THEOREM 2.2. *Let $h \in \mathfrak{O}_K[[T-1]]$ and write $\varphi h(e^t) = \sum_{n=0}^{\infty} \widehat{a}_n t^n/n!$. Then $\widehat{a}_n(x) \in \mathfrak{O}_K[x]$ for all n. Furthermore, if $c \equiv 0 \pmod{\phi(q)p^a}$ with $a \geq 0$ then for all $\tau \in \mathbb{Z}_p$,*

$$\Delta_c^k\{\widehat{a_m(p\tau)}\} \equiv 0 \pmod{p^{ka^+}\mathfrak{O}_K}$$

for all $m, k \geq 0$, and

$$\binom{p^{-r}\Delta_c}{k}\{\widehat{a_m(p\tau)}\} \in \mathfrak{O}_K$$

for $0 \leq r \leq a^+$ and all $m, k \geq 0$, where $\widehat{a_m(p\tau)}$ is as in (2.17).

3. Congruences for Bernoulli and Euler polynomials. Recall that Dwork's shift map $x \mapsto x'$ is defined for $x \in \mathbb{Z}_p$ by the relation $px' - x = \mu_x \in \{0, 1, \ldots, p-1\}$, so that μ_x is the representative of $-x \bmod p\mathbb{Z}_p$ which lies in $\{0, 1, \ldots, p-1\}$. The following lemma describes the action of Dwork's ψ operator on certain functions in terms of the shift map.

LEMMA 3.1. *For $x \in \mathbb{Z}_p$ and $(b, p) = 1$, we have formally*

$$\psi\left(\frac{T^{bx}}{T^b - c}\right) = c^{p-1-\mu_x} \frac{T^{bx'}}{T^b - c^p},$$

or equivalently,

$$\varphi\left(\frac{T^{bx}}{T^b - c}\right) = \frac{T^{bx}}{T^b - c} - c^{p-1-\mu_x} \frac{T^{bpx'}}{T^{bp} - c^p}.$$

Proof. Using (2.4) we compute

$$(3.1) \qquad (1-\varphi)\left(\frac{T^x}{T-c}\right) = \frac{1}{p} \sum_{\zeta^p=1} \frac{\zeta^x T^x}{\zeta T - c} = T^x \left(\frac{1}{p} \sum_{\zeta^p=1} \frac{\zeta^{-\mu_x}}{\zeta T - c}\right) = T^x \left(c^{p-1-\mu_x} \frac{T^{\mu_x}}{T^p - c^p}\right)$$

by considering the partial fraction decomposition of the latter rational function. The result in the case $b = 1$ follows immediately by noting that $x + \mu_x = px'$. To get the result for general b, note that $\varphi(h(T^b)) = (\varphi h)(T^b)$ for $(b, p) = 1$, since in this case the map $\zeta \mapsto \zeta^b$ permutes the solutions of $\zeta^p = 1$.

THEOREM 3.2. *Suppose $\phi(q)$ does not divide m. If $c \equiv 0 \pmod{\phi(q)p^a}$ with $a \geq 0$, then for all $x \in \mathbb{Z}_p$,*

$$\Delta_c^k \left\{ \frac{B_m(x) - p^{m-1} B_m(x')}{m} \right\} \equiv 0 \left(\operatorname{mod} \tfrac{1}{2} p^{ka^+} \mathbb{Z}_p\right)$$

and

$$\binom{p^{-r}\Delta_c}{k} \left\{ \frac{B_m(x) - p^{m-1} B_m(x')}{m} \right\} \in \frac{1}{2}\mathbb{Z}_p$$

for $0 \leq r \leq a^+$ and all $k > 0$.

Proof. Let b be any positive integer with $(b, p) = 1$, and define

$$(3.2) \qquad h(T) = \frac{bT^{bx}}{T^b - 1} - \frac{T^x}{T - 1}.$$

By writing

$$(3.3) \qquad h(T) = \frac{1}{T-1}\left(\frac{bT^{bx}}{\Phi_b(T)} - T^x\right)$$

with $\Phi_b(T) = T^{b-1} + T^{b-2} + \ldots + T + 1$ and observing that the latter factor in (3.3) lies in $\mathbb{Z}_p[[T - 1]]$ and has constant term zero, we see that

$h \in \mathbb{Z}_p[[T-1]]$. Applying Lemma 3.1 gives

$$(1-\varphi)h(T) = \frac{bT^{bpx'}}{T^{bp}-1} - \frac{T^{px'}}{T^p-1}, \tag{3.4}$$

and substitution of $T = e^t$ then yields

$$\varphi h(e^t) = \sum_{n=1}^{\infty}(b^n-1)\left(\frac{B_n(x)-p^{n-1}B_n(x')}{n}\right)\frac{t^{n-1}}{(n-1)!}. \tag{3.5}$$

So corresponding to the function $h(T)$ in (3.2), the congruences of Theorem 2.1 hold for the numbers

$$\widehat{a}_n = (b^{n+1}-1)\frac{B_{n+1}(x)-p^nB_{n+1}(x')}{n+1}. \tag{3.6}$$

Now supposing that $n+1$ is not divisible by $\phi(q)$, choose b so that $b^{n+1} \not\equiv 1 \pmod q$. Then since the congruences of Theorem 2.1 hold for the numbers $\widehat{a}_n$ in (3.6) associated to $(-1)^{p-1}b^{p^i}$ for $i = 1, 2, \ldots$, they hold for the numbers

$$(\omega(b)^{n+1}-1)\frac{B_{n+1}(x)-p^nB_{n+1}(x')}{n+1} \tag{3.7}$$

obtained upon passing to the p-adic limit (because $(-1)^{p-1}b^{p^i} \to \omega(b)$ in $\mathbb{Z}_p$). Put $m = n+1$. Since $\omega(b)^{m+jc} = \omega(b)^m$ for all integers j, we have

$$\Delta_c^k\left\{(\omega(b)^m-1)\frac{B_m(x)-p^{m-1}B_m(x')}{m}\right\} = (\omega(b)^m-1)\Delta_c^k\left\{\frac{B_m(x)-p^{m-1}B_m(x')}{m}\right\} \tag{3.8}$$

for all k, that is, the constant factor $\omega(b)^m - 1$ may be removed from each term in the congruences. Finally, noting that $\operatorname{ord}_p(\omega(b)^m-1) = \operatorname{ord}_p 2$ for all primes p gives the result of the theorem.

The generalized Euler polynomials $H_n(u,x)$ attached to an algebraic number $u \neq 1$ have been defined by

$$\left(\frac{1-u}{e^t-u}\right)e^{xt} = \sum_{n=0}^{\infty} H_n(u,x)\,\frac{t^n}{n!} \tag{3.9}$$

in [9]. For our purposes u will be an algebraic integer for which $1-u$ is a p-adic unit. When $p > 2$ and $u = -1$ one obtains the usual Euler polynomials $E_n(x) = H_n(-1,x)$. Theorem 2.1 may be applied to give congruences for $H_n(u,x)$ for $x \in \mathbb{Z}_p$.

THEOREM 3.3. *Let u be algebraic over $\mathbb{Q}_p$, and suppose that $1-u \in \mathfrak{O}_K^{\times}$, where $K = \mathbb{Q}_p(u)$. If $c \equiv 0 \pmod{\phi(q)p^a}$ then for all $x \in \mathbb{Z}_p$ we have*

$$\Delta_c^k\left\{H_m(u,x)-u^{p-1-\mu_x}\frac{1-u}{1-u^p}p^m H_m(u^p,x')\right\}\equiv 0 \pmod{p^{ka^+}\mathfrak{O}_K}$$

and

$$\binom{p^{-r}\Delta_c}{k}\left\{H_m(u,x)-u^{p-1-\mu_x}\frac{1-u}{1-u^p}p^m H_m(u^p,x')\right\}\in \mathfrak{O}_K$$

for $0\le r\le a^+$ *and all* $k>0$.

Proof. If $1-u\in\mathfrak{O}_K^\times$ then $h(T)=((1-u)/(T-u))\cdot T^x\in\mathfrak{O}_K[[T-1]]$ for all $x\in\mathbb{Z}_p$. From Lemma 3.1 we have

$$(1-\varphi)h(T)=u^{p-1-\mu_x}(1-u)\frac{T^{px'}}{T^p-u^p}. \tag{3.10}$$

Setting $T=e^t$ and expanding $\varphi h(e^t)=\sum_n \widehat{a}_n t^n/n!$ as formal power series gives

$$\widehat{a}_m=H_m(u,x)-u^{p-1-\mu_x}\frac{1-u}{1-u^p}p^m H_m(u^p,x'). \tag{3.11}$$

The theorem then follows from Theorem 2.1.

In many applications of Euler polynomials the parameter u is taken to be a nontrivial $(p-1)$st root of unity. We observe that in this case the numbers $\widehat{a}_m$ in the congruences simplify to

$$\widehat{a}_m=H_m(u,x)-u^{-\mu_x}p^m H_m(u,x'). \tag{3.12}$$

If in addition x is a rational number in $[0,1]$ with denominator dividing $p-1$ then $x'=x$ and we have

$$\widehat{a}_m=(1-u^{-\mu_x}p^m)H_m(u,x). \tag{3.13}$$

4. Congruences for generalized Bernoulli polynomials. For a primitive Dirichlet character χ of conductor $f=f_\chi$ the generalized Bernoulli polynomials $B_{n,\chi}(x)$ are defined by

$$\left(\sum_{a=1}^{f}\frac{\chi(a)te^{at}}{e^{ft}-1}\right)e^{xt}=\sum_{n=0}^{\infty}B_{n,\chi}(x)\frac{t^n}{n!}. \tag{4.1}$$

We begin this section by using Theorem 2.2 to produce congruences for $B_{n,\chi}(x)$ for $x\in p\mathbb{Z}_p$ and characters χ whose conductor is not a power of p. All the congruences of this section are extensions of congruences of the type given for $x=0$ which may be found in [2], [5], and [8].

THEOREM 4.1. *Suppose that* χ *is a primitive Dirichlet character whose conductor* f *is not a power of* p, *and put* $K=\mathbb{Q}_p(\chi)$. *If* $c\equiv 0 \pmod{\phi(q)p^a}$ *with* $a\ge 0$, *then for all* $\tau\in\mathbb{Z}_p$,

$$\Delta_c^k\left\{\frac{B_{m,\chi}(p\tau)-\chi(p)p^{m-1}B_{m,\chi}(\tau)}{m}\right\}\equiv 0 \pmod{p^{ka^+}\mathfrak{O}_K}$$

and

$$\binom{p^{-r}\Delta_c}{k}\left\{\frac{B_{m,\chi}(p\tau)-\chi(p)p^{m-1}B_{m,\chi}(\tau)}{m}\right\}\in\mathfrak{O}_K$$

for $0\le r\le a^+$ *and all* $m,k>0$.

Proof. Since f is not a power of p, we may write f in the form $f=dp^e$ with $e\ge 0$, $(d,p)=1$, and $d\ne 1$. Define the rational function

$$h_\chi(T)=\sum_{a=1}^{f}\frac{\chi(a)T^a}{T^f-1}. \tag{4.2}$$

We claim that $h_\chi\in\mathfrak{O}_K[[T-1]]$. To show this, observe that if $\zeta^{p^e}=1$, then $\sum_{a=1}^{f}\chi(a)\zeta^a=0$ by ([10], Lemma 4.7). Therefore $T^{p^e}-1$ divides $\sum_{a=1}^{f}\chi(a)T^a$ in $\mathfrak{O}_K[T-1]$. We also have

$$T^f-1=(T^{p^e}-1)(T^{(d-1)p^e}+T^{(d-2)p^e}+\ldots+T^{p^e}+1) \tag{4.3}$$

in $\mathfrak{O}_K[T-1]$, and the latter factor is a unit in $\mathfrak{O}_K[[T-1]]$ since its constant term is $d\in\mathfrak{O}_K^\times$. By dividing both numerator and denominator of (4.2) by $T^{p^e}-1$ we may then write $h_\chi(T)=g(T)/k(T)$ with g,k elements of $\mathfrak{O}_K[T-1]$ and k invertible in $\mathfrak{O}_K[[T-1]]$. This proves that $h_\chi\in\mathfrak{O}_K[[T-1]]$.

We now verify that $(1-\varphi)h_\chi(T)=\chi(p)h_\chi(T^p)$ as rational functions, by computing

$$\begin{aligned}(1-\varphi)h_\chi(T)&=\frac{1}{p}\sum_{\zeta^p=1}\left(\sum_{a=1}^{f}\frac{\chi(a)(\zeta T)^a}{(\zeta T)^f-1}\right)\\&=\frac{1}{p}\sum_{\zeta^p=1}\left(\sum_{a=1}^{fp}\frac{\chi(a)(\zeta T)^a}{(\zeta T)^{fp}-1}\right)\\&=\sum_{\substack{a=1\\p\mid a}}^{fp}\frac{\chi(a)T^a}{T^{fp}-1}=\sum_{b=1}^{f}\frac{\chi(pb)T^{pb}}{T^{fp}-1}=\chi(p)h_\chi(T^p),\end{aligned} \tag{4.4}$$

the second equality being obtained by multiplying each numerator and denominator by $1+(\zeta T)^f+(\zeta T)^{2f}+\ldots+(\zeta T)^{(p-1)f}$. Therefore if $h_\chi(e^t)=\sum a_nt^n/n!$ we have $a_n=B_{n+1,\chi}/(n+1)$ and $a_n^*=\chi(p)B_{n+1,\chi}/(n+1)$. The theorem then follows by taking $m=n+1$ and applying Theorem 2.2.

We say that χ is a character *of the first kind* if either $f=d$ or $f=dq$ with $(d,p)=1$; we say that χ is a character *of the second kind* if either $f=1$ or $f=qp^e$ with $e\ge 1$. If χ is a primitive Dirichlet character and n is an integer, the symbol χ_n will denote the character $\chi\omega^{-n}$, where ω is the Teichmüller character. Here we use Theorem 2.1 to produce congruences for $B_{n,\chi_n}(x)$ for $x\in p\mathbb{Z}_p$ and characters χ which are not of the second kind.

THEOREM 4.2. *Suppose that χ is a primitive Dirichlet character which is not of the second kind, and put $K = \mathbb{Q}_p(\chi)$. If $c \equiv 0 \pmod{\phi(q)p^a}$ with $a \geq 0$, then for all $\tau \in \mathbb{Z}_p$,*

$$\Delta_c^k\left\{\frac{B_{m,\chi_m}(p\tau) - \chi_m(p)p^{m-1}B_{m,\chi_m}(\tau)}{m}\right\} \equiv 0 \ \left(\operatorname{mod} \tfrac{1}{2}p^{ka^+}\mathfrak{O}_K\right)$$

and

$$\binom{p^{-r}\Delta_c}{k}\left\{\frac{B_{m,\chi_m}(p\tau) - \chi_m(p)p^{m-1}B_{m,\chi_m}(\tau)}{m}\right\} \in \frac{1}{2}\mathfrak{O}_K$$

for $0 \leq r \leq a^+$ and all $m, k > 0$.

Proof. First assume that the conductor $f = f_\chi$ of χ is not a power of p. Then the conductor $f = f_{\chi_m}$ of χ_m is also not a power of p for any m. Furthermore, $\omega^c = 1$, so $\chi_{m+jc} = \chi_m$ for all j. The theorem in this case follows immediately from Theorem 4.1.

We have now reduced to the case $f_\chi = q$, so that χ is a power of the Teichmüller character ω. In this case f_{χ_m} is either q or 1, and if $f_{\chi_m} = 1$, then $\phi(q)$ does not divide m, since χ is nontrivial. So in the case where $f_\chi = q$, $f_{\chi_m} = 1$, the theorem follows from Theorem 3.2 by observing that $\chi_{m+jc} = \chi_m = 1$ for all j and using the identity $B_{n,1}(x) = (-1)^n B_n(-x)$.

The remaining case is $f = f_\chi = f_{\chi_m} = q$. Let n be any positive integer with $f_{\chi_n} = q$, let b be a positive integer with $(b,p) = 1$, and define

$$h_{\chi_n}(T) = b\chi_n(b)T^{bx}\left(\sum_{a=1}^{f}\frac{\chi_n(a)T^{ab}}{T^{bf}-1}\right) - T^x\sum_{a=1}^{f}\frac{\chi_n(a)T^a}{T^f-1}, \tag{4.5}$$

where $x \in p\mathbb{Z}_p$. Equivalently we may write

$$\begin{aligned} h_{\chi_n}(T) &= \frac{b\chi_n(b)T^{bx}(\sum_{a=1}^{f}\chi_n(a)T^{ab}) - T^x\sum_{a=1}^{fb}\chi_n(a)T^a}{T^{bf}-1} \\ &= \frac{g(T)}{T^{bf}-1} \end{aligned} \tag{4.6}$$

with $g(T) \in \mathfrak{O}_K[[T-1]]$. If $\zeta^q = 1$, then

$$\begin{aligned} g(\zeta) &= b\chi_n(b)\zeta^{bx}\Big(\sum_{a=1}^{f}\chi_n(a)\zeta^{ab}\Big) - \zeta^x\sum_{a=1}^{fb}\chi_n(a)\zeta^a \\ &= b\Big(\zeta^{bx}\sum_{a=1}^{f}\chi_n(ab)\zeta^{ab} - \zeta^x\sum_{a=1}^{f}\chi_n(a)\zeta^a\Big) \\ &= b(\zeta^{bx} - \zeta^x)\Big(\sum_{a=1}^{f}\chi_n(a)\zeta^a\Big) = 0, \end{aligned} \tag{4.7}$$

since $(b-1)x \equiv 0 \pmod{q\mathbb{Z}_p}$ for $x \in p\mathbb{Z}_p$ and $(b,p)=1$. This shows that $T^f - 1$ divides $g(T)$ in $\mathfrak{O}_K[[T-1]]$. Since $T^{bf}-1$ is also divisible by T^f-1 in $\mathfrak{O}_K[T-1]$ and the quotient is a unit in $\mathfrak{O}_K[[T-1]]^\times$, we see that $h_{\chi_n} \in \mathfrak{O}_K[[T-1]]$.

We now obtain $a_{m-1} = (b^m\chi_n(b)-1)B_{m,\chi_n}(x)/m$ by using equation (4.1) to expand $h_{\chi_n}(e^t) = \sum a_m t^m/m!$. Furthermore, as in (4.4) we can compute

$$(4.8)\quad (1-\varphi)h_{\chi_n}(T) = b\chi_n(b)\Bigg(\sum_{\substack{a=1\\ p|a}}^{f} \frac{\chi_n(a)T^{b(a+x)}}{T^{bf}-1}\Bigg) - \sum_{\substack{a=1\\ p|a}}^{f} \frac{\chi_n(a)T^{a+x}}{T^f-1} = 0,$$

since $\chi_n(a)=0$ if p divides a; therefore $\widehat{a}_m = a_m$ for all m.

Since χ is nontrivial, we may choose a value of b for which $(b,p)=1$ and $\chi(b)\neq 1$. The congruences of Theorem 2.1 hold for the numbers $\widehat{a}_{m-1}$ associated to b and to $(-1)^{p-1}b^{p^i}$ for $i=1,2,\ldots$, so they also hold for the numbers

$$(4.9)\qquad (\omega(b)^m\chi_n(b)-1)B_{m,\chi_n}(x)/m$$

obtained upon passing to the p-adic limit. Since $\omega^m = \omega^{m+jc}$ for all j, the constant factor $\omega(b)^m\chi_n(b)-1$ may be factored out of each term in the congruences. Now set $n=m$. Since $\chi(b)$ and $\omega(b)$ are both nonzero, $\omega(b)^m\chi_m(b)-1$ equals $\chi(b)-1$. Observing that $\mathrm{ord}_p(\chi(b)-1) = \mathrm{ord}_p 2$ for all primes p, putting $x=p\tau$, and observing that $\chi_m(p)=0$ completes the proof.

In [6] G. Fox studied a two-variable p-adic L function $L_p(s,\tau,\chi)$ which is defined for s, τ lying in the p-adic completion $\mathbb{C}_p$ of an algebraic closure of $\mathbb{Q}_p$ and satisfying $\mathrm{ord}_p\,\tau \geq 0$ and $\mathrm{ord}_p\, s \geq 1/(p-1) - \mathrm{ord}_p\, q$. The members of the congruences in the above theorem are values of this L-function at negative integer values of s. Specifically, Fox showed that if n is a positive integer then

$$(4.10)\qquad L_p(1-n,\tau,\chi) = \frac{1}{n}(B_{n,\chi_n}(p\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}(\tau)).$$

In both (4.10) and Theorem 4.2 the main term Bernoulli polynomial has an argument whose p-adic ordinal must be at least 1. We conclude with an analogue of Theorem 3.2 for values of generalized Bernoulli polynomials at an argument which is not restricted to lie in $p\mathbb{Z}_p$, extending Theorem 4.1. Unfortunately this requires certain impositions upon the conductor of χ. For this result we make use of the decomposition of χ as $\chi = \chi_{(0)}\chi_{(p)}$, where the conductor f_0 of $\chi_{(0)}$ is relatively prime to p and the conductor f_p of $\chi_{(p)}$ is a power of p (cf. [10], p. 23).

THEOREM 4.3. *Suppose that χ is a primitive Dirichlet character such that $f_0 \neq 1$ and $f_p = 1$ or p, and put $K = \mathbb{Q}_p(\chi)$. Let $x \in \mathbb{Z}_p$ be such that f_0 divides μ_x in $\mathbb{Z}$. If $c \equiv 0 \pmod{\phi(q)p^a}$ with $a \geq 0$, then*

$$\Delta_c^k \left\{ \frac{B_{m,\chi}(x) - \chi_{(p)}(\mu_x)\chi_{(0)}(p)p^{m-1}B_{m,\chi_{(0)}}(x')}{m} \right\} \equiv 0 \pmod{p^{ka^+}\mathfrak{O}_K}$$

and

$$\binom{p^{-r}\Delta_c}{k} \left\{ \frac{B_{m,\chi}(x) - \chi_{(p)}(\mu_x)\chi_{(0)}(p)p^{m-1}B_{m,\chi_{(0)}}(x')}{m} \right\} \in \mathfrak{O}_K$$

for $0 \leq r \leq a^+$ and all $m, k > 0$.

Proof. Suppose that $x \in \mathbb{Z}_p$ is such that f_0 divides μ_x in $\mathbb{Z}$. Define the function

$$h_\chi(T, x) = \sum_{a=1}^{f} \frac{\chi(a)T^{a+x}}{T^f - 1}. \tag{4.11}$$

As the product of T^x and the function $h_\chi(T)$ in (4.2), this function lies in $\mathfrak{O}_K[[T-1]]$. We compute

$$\begin{aligned} (1-\varphi)h_\chi(T,x) &= \frac{1}{p}\sum_{\zeta^p=1}\left(\sum_{a=1}^{f} \frac{\chi(a)(\zeta T)^{a+x}}{(\zeta T)^f - 1}\right) \\ &= \frac{1}{p}\sum_{\zeta^p=1}\left(\sum_{a=1}^{fp} \frac{\chi(a)(\zeta T)^{a+x}}{(\zeta T)^{fp} - 1}\right) = \sum_{\substack{a=1 \\ p \mid a+x}}^{fp} \frac{\chi(a)T^{a+x}}{T^{fp}-1}. \end{aligned} \tag{4.12}$$

Since $(f_0, f_p) = 1$, we have $\chi(a) = \chi_{(0)}(a)\chi_{(p)}(a)$ for all a. If p divides $a + x$ in $\mathbb{Z}_p$ then $a \equiv \mu_x \pmod{p}$ and therefore $\chi_{(p)}(a) = \chi_{(p)}(\mu_x)$. Then by writing $pb = a + x$ as b runs from x' to $x' + f - 1$, equation (4.12) becomes

$$\begin{aligned} (1-\varphi)h_\chi(T,x) &= \chi_{(p)}(\mu_x) \sum_{\substack{a=1 \\ p \mid a+x}}^{fp} \frac{\chi_{(0)}(a)T^{a+x}}{T^{fp}-1} \\ &= \chi_{(p)}(\mu_x) \sum_{c=0}^{f-1} \frac{\chi_{(0)}(\mu_x + pc)T^{p(c+x')}}{T^{fp}-1} \\ &= \chi_{(p)}(\mu_x) \sum_{c=0}^{f-1} \frac{\chi_{(0)}(pc)T^{p(c+x')}}{T^{fp}-1} \qquad \text{(since } f_0 \mid \mu_x) \\ &= \chi_{(p)}(\mu_x)\chi_{(0)}(p)h_{\chi_{(0)}}(T^p, x'). \end{aligned} \tag{4.13}$$

Therefore if $h_\chi(e^t, x) = \sum a_n t^n/n!$ we have $a_n = B_{n+1,\chi}(x)/(n+1)$ for all n and $a_n^* = \chi_{(p)}(\mu_x)\chi_{(0)}(p)B_{n+1,\chi_{(0)}}(x')/(n+1)$. The theorem then follows by taking $m = n + 1$ and applying Theorem 2.1.

By using the identity $B_{n,\chi}(-x) = (-1)^n \chi(-1) B_{n,\chi}(x)$ and the fact that $(-x)' = 1 - x'$ and $\mu_{-x} = p - \mu_x$ when $\mu_x \neq 0$ we obtain the following variation of Theorem 4.3.

Corollary 4.4. *Suppose that χ is a primitive Dirichlet character such that $f_0 \neq 1$ and $f_p = 1$ or p, and put $K = \mathbb{Q}_p(\chi)$. Let $x \in \mathbb{Z}_p$ be such that f_0 divides $p - \mu_x$ in $\mathbb{Z}$. If $c \equiv 0 \pmod{\phi(q)p^a}$ with $a \geq 0$, then*

$$\Delta_c^k \left\{ \frac{B_{m,\chi}(x) - \chi_{(p)}(\mu_x)\chi_{(0)}(p)p^{m-1}B_{m,\chi_{(0)}}(x'-1)}{m} \right\} \equiv 0 \pmod{p^{ka^+}\mathfrak{O}_K}$$

and

$$\binom{p^{-r}\Delta_c}{k} \left\{ \frac{B_{m,\chi}(x) - \chi_{(p)}(\mu_x)\chi_{(0)}(p)p^{m-1}B_{m,\chi_{(0)}}(x'-1)}{m} \right\} \in \mathfrak{O}_K$$

for $0 \leq r \leq a^+$ and all $m, k > 0$.

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