

Products of an arbitrary number of Hecke eigenforms

by

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1. Introduction. Let

$$E_k(z) = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

be the normalized Eisenstein series of weight k for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, where $q = e^{2\pi iz}$. The identities

$$(1) \quad E_4 \cdot E_4 = E_8, \quad E_4 \cdot E_6 = E_{10}, \quad E_4 \cdot E_{10} = E_{14}, \quad E_6 \cdot E_8 = E_{14}$$

are well-known and follow from the fact that the vector space of modular forms of level 1 and weight k is 1-dimensional for $k \in \{4, 6, 8, 10, 14\}$. Elementary proofs of these identities can be found in [7].

Comparing Fourier coefficients of the power series in the products, we get interesting identities involving the divisor functions $\sigma_k(n) = \sum_{d|n} d^k$. For example, $E_4 \cdot E_4 = E_8$ gives

$$(2) \quad \sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m).$$

Similar identities can be found by comparing coefficients in the other products.

Let Δ_l denote the unique cuspidal eigenform of weight l in $\mathcal{S}_l(\Gamma)$ for $l \in \{12, 16, 18, 20, 22, 26\}$. Along with the identities in (1) we have

$$(3) \quad \begin{aligned} E_4 \cdot \Delta_{12} &= \Delta_{16}, & E_4 \cdot \Delta_{16} &= \Delta_{20}, & E_4 \cdot \Delta_{18} &= \Delta_{22}, \\ E_4 \cdot \Delta_{22} &= \Delta_{26}, & E_6 \cdot \Delta_{12} &= \Delta_{18}, & E_6 \cdot \Delta_{16} &= \Delta_{22}, \\ E_6 \cdot \Delta_{20} &= \Delta_{26}, & E_8 \cdot \Delta_{12} &= \Delta_{20}, & E_8 \cdot \Delta_{18} &= \Delta_{26}, \\ E_{10} \cdot \Delta_{12} &= \Delta_{22}, & E_{10} \cdot \Delta_{16} &= \Delta_{26}, & E_{14} \cdot \Delta_{12} &= \Delta_{26}. \end{aligned}$$

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These give all cases where the product of two Hecke eigenforms is an eigenform. This was proven both by Duke in [2] and by Ghatge in [4] using the Rankin–Selberg method. That is, the product of two Hecke eigenforms is a Hecke eigenform if and only if it is forced to be by dimension considerations. In [3] the first author found all cases where the product of two Hecke eigenforms is another Hecke eigenform for the congruence subgroup $\Gamma_0(p)$, for $p \geq 5$ prime.

It is still possible that the product of more than two eigenforms is an eigenform. In particular, if the product of two or more Eisenstein series is another Eisenstein series, then we may obtain identities similar to (2). The goal of this paper is to show that this does not happen. That is, the product of an arbitrary number of Hecke eigenforms is never a Hecke eigenform except in the cases obtained from (1) and (3). Note that $E_4^2 \cdot E_6 = E_{14}$, but this is trivially obtained from the identities $E_4 \cdot E_4 = E_8$ and $E_6 \cdot E_8 = E_{14}$. Other such products are obtained similarly.

In the first cases we prove that the product of an arbitrary number of Eisenstein series is never an Eisenstein series except when trivially obtained from (1). We consider products of the form

$$E_{k_1} \cdots E_{k_n} = E_l$$

where $l = k_1 + \cdots + k_n$ and $n > 1$. Assume that $k_i \leq k_j$ when $i < j$.

THEOREM 1. *Let E_k be the normalized Eisenstein series of weight k for the full modular group Γ . If*

$$E_{k_1} \cdots E_{k_n} = E_l$$

then $l \in \{8, 10, 14\}$.

The proof uses the nature of the zeros of the Eisenstein series and the growth of the coefficients

$$C_k = \frac{(2\pi i)^k}{\zeta(k)(k-1)!}.$$

The second case is to prove that the product of a cuspidal eigenform $f \in \mathcal{S}_l(\Gamma)$ and an arbitrary number of Eisenstein series is not a cuspidal eigenform.

THEOREM 2. *Let E_k be the normalized Eisenstein series of weight k for the full modular group Γ and let f be a Hecke eigenform in $\mathcal{S}_l(\Gamma)$. If*

$$f \cdot E_{k_1} \cdots E_{k_n}$$

is a Hecke eigenform then $l + \sum_{i=1}^n k_i \in \{16, 18, 20, 22, 26\}$.

Note that this is sufficient to prove our claim since if more than one of the factors is a cusppform, then the product cannot be an eigenform.

2. Proof of Theorem 1. Note that repeat factors can occur as in $E_4^2 = E_8$ and $E_4^2 \cdot E_6 = E_{14}$. The first lemma states that only E_4 can occur as a repeat factor and it can only occur with multiplicity two.

LEMMA 1. *If $E_{k_1} \cdots E_{k_n} = E_l$ where $k_1 \leq \cdots \leq k_n$, then*

$$k_1 < \cdots < k_n \quad \text{or} \quad 4 = k_1 = k_2 < k_3 < \cdots < k_n.$$

Proof. Let $v_z(f)$ be the order of vanishing of the modular form f at the point z . We have

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\varrho(f) + \sum'_z v_z(f) = \frac{k}{12},$$

where the sum is over the points in the fundamental domain other than i and $\varrho = e^{2\pi i/3}$ and k is the weight of the modular form [5]. If we write $k = 12r + s$, where r is an integer and $s \in \{0, 4, 6, 8, 10, 14\}$, then $E_k(z)$ has exactly r zeros on the unit circle between i and ϱ , and each of these zeros is a simple zero [6].

Let a_r be the number of k_i 's congruent to r modulo 12. Noting that $E_k(i) = 0$ if and only if $k \equiv 2 \pmod{4}$ and that i is a simple zero, we have

$$(4) \quad a_2 + a_6 + a_{10} = \begin{cases} 0 & \text{if } l \equiv 0 \pmod{4}, \\ 1 & \text{if } l \equiv 2 \pmod{4}. \end{cases}$$

In particular, we cannot have repeated factors of E_k for $k \equiv 2, 10 \pmod{12}$.

Similarly, if we note that $E_k(\varrho) = 0$ if and only if $k \equiv 2, 4 \pmod{6}$, and the zero is a simple zero when $k \equiv 4 \pmod{6}$ and a double zero when $k \equiv 2 \pmod{6}$, then we have

$$(5) \quad 2a_2 + a_4 + 2a_8 + a_{10} = \begin{cases} 0 & \text{if } l \equiv 0 \pmod{6}, \\ 1 & \text{if } l \equiv 4 \pmod{6}, \\ 2 & \text{if } l \equiv 2 \pmod{6}. \end{cases}$$

In particular, we cannot have repeated factors of E_k for $k \equiv 2, 8 \pmod{12}$, and we cannot have more than two factors of E_k for $k \equiv 4 \pmod{12}$.

It remains to show that there cannot be any repeat factors of E_k for $k \equiv 0, 4 \pmod{12}$ and $k \geq 12$. This follows from the fact that $E_k(e^{i\theta})$ has the required number of zeros on the interval $\theta \in (\pi/2, 2\pi/3)$, and each of these zeros is a simple zero. ■

Recall that the Fourier expansion of the Eisenstein series is

$$E_k(z) = 1 + C_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $C_k = (2\pi i)^k / \zeta(k)(k-1)!$. If we have

$$E_{k_1} \cdots E_{k_n} = E_l,$$

then by comparing the q -coefficient in the product we have

$$(6) \quad C_{k_1} + \cdots + C_{k_n} = C_l.$$

The following two lemmas concern the growth of the coefficients C_k .

LEMMA 2. *If $10 \leq l < k$ or $l = 8 \& k \geq 12$ or $l = 6 \& k \geq 14$ or $l = 4 \& k \geq 20$ then*

$$|C_k| \leq 2^{(l-k)/2} |C_l|.$$

Proof. As

$$1 \leq \zeta(k) \leq 1 + \int_1^\infty x^{-k} dx = \frac{k}{k-1}$$

for k even, we have

$$\frac{|C_{k+2}|}{|C_k|} \leq \frac{(2\pi)^2 \zeta(k)}{k(k+1)\zeta(k+2)} \leq \frac{(2\pi)^2}{(k+1)(k-1)}.$$

Therefore for $k \geq 10$, $|C_{k+2}| < \frac{1}{2}|C_k|$. By induction, the result holds for all k .

We obtain the other cases by examining the list

$$C_4 = 240, \quad C_6 = -504, \quad C_8 = 480, \quad C_{10} = -264, \quad C_{12} = \frac{65520}{691},$$

$$C_{14} = -24, \quad C_{16} = \frac{16320}{3617}, \quad C_{18} = -\frac{28728}{43867}, \quad C_{20} = \frac{13200}{174611}. \quad \blacksquare$$

LEMMA 3. *The series $\sum_{j=2}^\infty C_{2j}$ is absolutely convergent and for $m \geq 5$ we have*

$$\sum_{j=m}^\infty |C_{2j}| < 2|C_{2m}|.$$

Proof. Absolute convergence follows immediately from Lemma 2 and the ratio test. For the upper bound,

$$\begin{aligned} \sum_{j=m}^\infty |C_{2j}| &= |C_{2m}| + |C_{2m+2}| + |C_{2m+4}| + \cdots \\ &< |C_{2m}| + \frac{1}{2}|C_{2m}| + \frac{1}{4}|C_{2m}| + \cdots \\ &= |C_{2m}| \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) = 2|C_{2m}|. \quad \blacksquare \end{aligned}$$

The following lemma establishes that in all but a finite number of cases the product of an arbitrary number of Eisenstein series is not an Eisenstein series.

LEMMA 4. *If $k_1 \geq 10$ or $k_1 \in \{4, 6, 8\}$ and $k_2 \geq 12$ then*

$$E_{k_1} \cdots E_{k_n} \neq E_l.$$

Proof. Suppose that

$$E_{k_1} \cdots E_{k_n} = E_l.$$

Equating the q -coefficients we have

$$C_{k_1} + \cdots + C_{k_n} = C_l.$$

For $k_1 \geq 10$, we find from Lemma 1 that the k_i are distinct and therefore

$$\begin{aligned} |C_{k_1}| &\leq |C_{k_2}| + |C_{k_3}| + \cdots + |C_l| \\ &< |C_{k_1}|(2^{(k_1-k_2)/2} + 2^{(k_1-k_3)/2} + \cdots + 2^{(k_1-l)/2}) \\ &< |C_{k_1}| \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) = |C_{k_1}|, \end{aligned}$$

which is a contradiction.

For $k_1 \in \{4, 6, 8\}$ and $k_2 \geq 12$ we have

$$|C_{k_1}| \leq |C_{k_2}| + |C_{k_3}| + \cdots + |C_l| < \sum_{m=k_2/2}^{\infty} |C_{2m}| < 2|C_{k_2}| < 200.$$

However, for $k_1 \in \{4, 6, 8\}$ we have $|C_{k_1}| \geq 240$. ■

The remaining cases are

$$(k_1, k_2) \in \{(4, 4), (4, 6), (4, 8), (4, 10), (6, 8), (6, 10), (8, 10)\}.$$

The cases $(k_1, k_2) = (4, 8)$ or $(8, 10)$ cannot happen, since this would contradict (5). That is, the order of vanishing at $\varrho = e^{2\pi i/3}$ would be too great to be an Eisenstein series. Similarly, the case $(k_1, k_2) = (6, 10)$ cannot happen as this would contradict (4). That is, the order of vanishing at i would be too great.

Note that $(k_1, k_2) \in \{(4, 4), (4, 6), (4, 10), (6, 8)\}$ occur in (1). It remains to show that there are no other such identities with these as factors.

LEMMA 5. *Let*

$$E_{k_1} \cdots E_{k_n} = E_l$$

where $(k_1, k_2) \in \{(4, 4), (4, 6), (4, 10), (6, 8)\}$. If $(k_1, k_2) \in \{(4, 6), (4, 10), (6, 8)\}$ then $n = 2$ and we have $E_4 \cdot E_6 = E_{10}$, $E_4 \cdot E_{10} = E_{14}$, or $E_6 \cdot E_8 = E_{14}$. If $k_1 = k_2 = 4$ then either $n = 2$ and we have $E_4 \cdot E_4 = E_8$, or $n = 3$ and we have $E_4 \cdot E_4 \cdot E_6 = E_{14}$.

Proof. First consider $(k_1, k_2) = (6, 8)$. Suppose that

$$E_6 \cdot E_8 \cdot E_{k_3} \cdots E_{k_n} = E_l.$$

Rewrite the above equation by substituting E_{14} for $E_6 \cdot E_8$. Note that none of the other factors can be E_{14} by Lemma 1. So this equation can be rewritten with the smallest factor greater than or equal to E_{10} , but we have shown this cannot happen. Therefore, there can be no other factors in the product,

and $(k_1, k_2) = (6, 8)$ implies that the only identity is $E_6 \cdot E_8 = E_{14}$. The other cases are handled similarly. ■

This finishes the proof of Theorem 1. ■

3. Proof of Theorem 2. We begin with a preliminary lemma.

LEMMA 6. *Let f be a cusp form and a Hecke eigenform of weight l for Γ . If $f \cdot E_{k_1} \cdots E_{k_n}$ is an eigenform then $k_j \in \{4, 6, 8, 10, 14\}$ for $1 \leq j \leq n$.*

Proof. Let $f \cdot g = h$, where $f = \sum_{m=1}^{\infty} a_m q^m$ with $a_1 = 1$, $g = 1 + \sum_{m=1}^{\infty} \gamma_m q^m$ and $h = \sum_{m=1}^{\infty} b_m q^m$ with a_j and b_j algebraic integers for all j . Since $\gamma_1 = b_2 - a_2$ and $\gamma_m = b_{m+1} - a_{m+1} - a_m \gamma_1 - a_{m-1} \gamma_2 - \cdots - a_2 \gamma_{m-1}$ for $m \geq 2$, by induction it follows that γ_j is also an algebraic integer for every j .

As f and $f \cdot E_{k_1} \cdots E_{k_n}$ are eigenforms, they have algebraic integer coefficients. From the previous paragraph, we deduce that $E_{k_1} \cdots E_{k_n}$ also has algebraic integer coefficients. Since each E_{k_j} has rational Fourier coefficients, it follows that $E_{k_1} \cdots E_{k_n}$ has Fourier coefficients in \mathbb{Z} .

Let p be the largest prime that occurs in the denominators of the reduced forms of all of the C_{k_j} 's and suppose exactly m of these C_{k_j} 's have a factor of p in the denominator. Without loss of generality we can suppose that these are C_{k_1}, \dots, C_{k_m} . The q^m th Fourier coefficient of $E_{k_1} \cdots E_{k_n}$ is

$$(6) \quad \sum_j C_{k_j} \sigma_{k_j-1}(m) + \cdots + \sum_{\substack{j_1, \dots, j_r \\ m_1 + \dots + m_r = m}} C_{k_{j_1}} \cdots C_{k_{j_r}} \sigma_{k_{j_1}-1}(m_1) \cdots \sigma_{k_{j_r}-1}(m_r) \\ + \cdots + \sum_{j_1, \dots, j_m} C_{k_{j_1}} \cdots C_{k_{j_m}}.$$

If p^r is the highest power of p that occurs in the denominators of all of the above terms then only $C_{k_1} \cdots C_{k_m}$ has a factor of p^r in the denominator. It follows that the reduced form of the rational number (6) must have a factor of p^r in its denominator, and so cannot be an integer. Therefore we must have $C_{k_j} \in \mathbb{Z}$ for every j and the lemma follows from Lemma 2. ■

Hence we may consider products of the form $h = f \cdot E_4^a \cdot E_6^b \cdot E_8^c \cdot E_{10}^d \cdot E_{14}^e$. Since we have the identities in (1), we only need to consider products of the form $h = f \cdot E_4^b \cdot E_6^c$, where f is a cusp form in $\mathcal{S}_l(\Gamma)$.

Note that $E_4^b = 1 + \sum_{n=1}^{\infty} b_n q^n$, where

$$\begin{aligned} b_1 &= 240b, \\ b_2 &= 240(120b^2 - 111b), \\ b_3 &= 240(9600b^3 - 26640b^2 + 17068b), \end{aligned}$$

and $E_6^c = 1 + \sum_{n=1}^{\infty} c_n q^n$, where

$$\begin{aligned} c_1 &= -504c, \\ c_2 &= 504(252c^2 - 285c), \\ c_3 &= -504(42336c^3 - 143640c^2 + 101548c). \end{aligned}$$

We can expand $h = f \cdot E_4^b \cdot E_6^c = (\sum_{n=1}^{\infty} a_n q^n)(1 + \sum_{n=1}^{\infty} b_n q^n)(1 + \sum_{n=1}^{\infty} c_n q^n)$ as

$$\begin{aligned} h &= q + (a_2 + (b_1 + c_1))q^2 + (a_3 + a_2(b_1 + c_1) + (b_2 + b_1c_1 + c_2))q^3 \\ &\quad + (a_4 + a_3(b_1 + c_1) + a_2(b_2 + b_1c_1 + c_2) + (b_3 + b_2c_1 + b_1c_2 + c_3))q^4 \\ &\quad + \dots \end{aligned}$$

Since h is an eigenform for the weight $l + 4b + 6c$ operator $T(2)$ and f is an eigenform for the weight l operator $T(2)$, we have

$$\begin{aligned} (a_2 + (b_1 + c_1))^2 - 2^{l+4b+6c-1} \\ = a_2^2 - 2^{l-1} + a_3(b_1 + c_1) + a_2(b_2 + b_1c_1 + c_2) + (b_3 + b_2c_1 + b_1c_2 + c_3). \end{aligned}$$

Solving for 2^l we obtain

$$\begin{aligned} 2^l &= \frac{2}{2^{4b+6c} - 1} (a_3(-b_1 - c_1) + a_2(2b_1 + 2c_1 - b_2 - b_1c_1 - c_2) \\ &\quad + (b_1 + c_1)^2 - (b_3 + b_2c_1 + b_1c_2 + c_3)). \end{aligned}$$

From the Ramanujan–Pettersson estimate $|a_p| \leq 2p^{(l-1)/2}$ (see [1]) we have

$$\begin{aligned} 2^l &\leq \frac{2}{2^{4b+6c} - 1} (2 \cdot 3^{(l-1)/2} |b_1 + c_1| \\ &\quad + 2 \cdot 2^{(l-1)/2} |2b_1 + 2c_1 - b_2 - b_1c_1 - c_2| \\ &\quad + |(b_1 + c_1)^2 - (b_3 + b_2c_1 + b_1c_2 + c_3)|). \end{aligned}$$

Therefore l must satisfy at least one of the following inequalities:

$$(7) \quad \left(\frac{2}{\sqrt{3}}\right)^l \leq \frac{12}{\sqrt{3}(2^{4b+6c} - 1)} |b_1 + c_1| = \frac{12}{\sqrt{3}(2^{4b+6c} - 1)} |240b - 504c|,$$

$$\begin{aligned} (8) \quad (\sqrt{2})^l &\leq \frac{2^{3/2} \cdot 3}{2^{4b+6c} - 1} |2b_1 + 2c_1 - b_2 - b_1c_1 - c_2| \\ &= \frac{2^{3/2} \cdot 3}{2^{4b+6c} - 1} |27120b + 142632c - 28800b^2 \\ &\quad + 120960bc - 127008c^2|, \end{aligned}$$

$$\begin{aligned}
 (9) \quad 2^l &\leq \frac{2 \cdot 3}{2^{4b+6c} - 1} |(b_1 + c_1)^2 - (b_3 + b_2c_1 + b_1c_2 + c_3)| \\
 &= \frac{2 \cdot 3}{2^{4b+6c} - 1} |6451200b^2 + 20805120bc - 72140544c^2 \\
 &\quad - 4096320b - 2304000b^3 + 14515200cb^2 - 30481920bc^2 \\
 &\quad + 51180192c + 21337344c^3|.
 \end{aligned}$$

If we let $x = \max\{b, c\}$, then we have

$$l \leq \sup \left\{ \frac{\log\left(\frac{6048x}{\sqrt{3}(16^x-1)}\right)}{\log(2/\sqrt{3})}, \frac{\log\left(\frac{2^{3/2} \cdot 1085598x^2}{16^x-1}\right)}{\log(\sqrt{2})}, \frac{\log\left(\frac{792836144x^3}{16^x-1}\right)}{\log(2)} \right\}.$$

If $x \geq 7$ then $l < 12$, and since there are no cusp forms of weight l for $l < 12$, there are no solutions. So we only have the cases where $E_4^b \cdot E_6^c$ with $b, c \leq 6$. By substituting directly into inequalities (7), (8), and (9) we see that $l < 12$ in all cases except

$$(b, c) \in \{(0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (3, 0), (4, 0)\}.$$

The cases $(b, c) \in \{(0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}$ were handled in [2]. And so we are left with the cases $(b, c) = (0, 2)$ where $l < 18$, $(b, c) = (1, 2)$ where $l < 13$, $(b, c) = (0, 3)$ where $l < 18$, and $(b, c) = (0, 4)$ where $l < 13$. These cases give the possible eigenforms

$$\Delta_{12} \cdot E_6^2, \quad \Delta_{16} \cdot E_6^2, \quad \Delta_{12} \cdot E_6^2 \cdot E_4, \quad \Delta_{12} \cdot E_4^3, \quad \Delta_{16} \cdot E_4^3, \quad \Delta_{12} \cdot E_4^4.$$

The products $\Delta_{16} \cdot E_6^2$ and $\Delta_{12} \cdot E_6^2 \cdot E_4$ are the same, as are the products $\Delta_{16} \cdot E_4^3$ and $\Delta_{12} \cdot E_4^4$. So we are left with checking if the products

$$\begin{aligned}
 \Delta_{12} \cdot E_6^2 &= q - 1032q^2 + 245196q^3 + 10965568q^4 + \dots, \\
 \Delta_{12} \cdot E_6^2 \cdot E_4 &= q - 792q^2 - 324q^4 + 67590208q^4 + \dots, \\
 \Delta_{12} \cdot E_4^3 &= q + 696q^2 + 162252q^3 + 12831808q^4 + \dots, \\
 \Delta_{12} \cdot E_4^4 &= q + 936q^2 + 331452q^3 + 53282368q^4 + \dots
 \end{aligned}$$

are eigenforms. Since these products are not eigenvectors of the Hecke operator $T(2)$, they are not Hecke eigenforms. ■

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