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Minimal polynomials of some beta-numbers and Chebyshev polynomials

by

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1. Introduction. For $\beta > 1$, the *β-transformation* $T_\beta : x \mapsto \beta x \pmod 1$ is multiplication by β modulo 1. Then the *β-expansion* $d_\beta(x) = (x_i)_{i\geq 1}$ of $x \in [0,1]$ is defined by $x_i = \lfloor \beta T_\beta^{i-1}(x)\rfloor$. While Rényi [15] introduced the β-transformation and proved that it is ergodic, Parry [13] found its invariant measure and characterized possible sequences that can be a β-expansion. It was shown that the β-expansion of 1 bounds, in a sense, the β-expansion of any other $x \in [0,1)$ in terms of lexicographic order. To be more precise, let us fix $\beta > 1$ and assume that $s = x_1x_2\cdots$ is an infinite word over the alphabet $\{0,1,\ldots,\lfloor\beta\rfloor\}$. In addition, put

$$d_\beta^*(1) := \lim_{\varepsilon\searrow 0} d_\beta(1-\varepsilon);$$

we have $d_\beta^*(1) = (e_1\cdots e_{m-1}(e_m-1))^\omega$ if $d_\beta(1) = e_1\cdots e_m0^\omega$ and $d_\beta^*(1) = d_\beta(1)$ otherwise. Here $a^\omega := aa\cdots$. Then Parry showed that $s = d_\beta(x)$ for some $x\in[0,1)$ if and only if $\sigma^n(s) < d_\beta^*(1)$ for all $n \geq 0$, where σ is the shift.

If $d_\beta(1)$ is eventually periodic, say $d_\beta(1) = e_1\cdots e_n(e_{n+1}\cdots e_{n+p})^\omega$, then we call β a *beta-number*. In this case, β is the dominant root of

$$\Big(x^{n+p} - \sum_{i=1}^{n+p} e_i x^{n+p-i}\Big) - \Big(x^n - \sum_{i=1}^{n} e_i x^{n-i}\Big) = 0.$$

In particular, if $d_\beta(1)$ is finite, i.e., $d_\beta(1) = e_1\cdots e_n0^\omega$, then we call β a *simple beta-number* and β is the dominant root of

$$x^n - \sum_{i=1}^{n} e_i x^{n-i} = 0.$$

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These two polynomials above are called the *beta-polynomial* of β, or simply the *β-polynomial* if β is clear from the context. The term *characteristic polynomial* is also used in the literature.

We focus on a special class of β-expansions. We study the real $\beta > 1$ for which $d_\beta^*(1)$ encodes a rational rotation on $\mathbb{R}/\mathbb{Z}$. In this case, β is an algebraic integer. For the case of irrational rotations, see [4, 6]. An arithmetic study was also pursued there and the current work is in fact motivated by those papers.

Let $\alpha > 0$ and $\varrho \in [0, 1]$. We consider the following two infinite sequences called *lower* and *upper mechanical words* with *slope* α and *intercept* ϱ: for $n \geq 0$,

$$\begin{aligned} s_{\alpha,\varrho}(n) &= \lfloor \alpha(n+1) + \varrho \rfloor - \lfloor \alpha n + \varrho \rfloor, \\ s'_{\alpha,\varrho}(n) &= \lceil \alpha(n+1) + \varrho \rceil - \lceil \alpha n + \varrho \rceil. \end{aligned}$$

Note that these are infinite words over the alphabet $A = \{\lceil \alpha \rceil - 1, \lceil \alpha \rceil\}$. For an irrational α, the words are aperiodic and called *Sturmian words*. On the other hand, a rational α produces purely periodic words, whose shortest period words are called *Christoffel words*. Our main concern is the rational case. We now describe it in more detail.

Let $\alpha = p/q > 0$ with $\gcd(p, q) = 1$ and $b = \lceil \alpha \rceil$. Then $A = \{b - 1, b\}$ and the lengths of the Christoffel words, say $t_{p,q}$, $t'_{p,q}$, are q. So $s_{\alpha,0} = t_{p,q}^\omega$ and $s'_{\alpha,0} = (t'_{p,q})^\omega$. We also have

$$t_{p,q} = (b-1)z_{p,q}b, \qquad t'_{p,q} = bz_{p,q}(b-1),$$

for some word $z_{p,q}$, called a *central word*. We see that $z_{p,q}$ is a *palindrome*, i.e., $z_{p,q}$ is equal to its reversal (see [12]). The motivation of this paper comes from the next proposition. For any word $w = a_0a_1 \cdots a_{n-1}$ with $a_i \in \mathbb{Z}$, we denote by $\overrightarrow{w}$ the vector $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n$.

Proposition 1.1 ([4, 6]). *For $\alpha > 0$, there exists a unique $\beta > 1$ for which $d_\beta^*(1) = s'_{\alpha,0}$. Define $\Delta : (0, \infty) \to (1, \infty)$ by $\Delta(\alpha) = \beta$. Then*

(a) *At an irrational $\alpha > 0$, Δ is continuous, and $\Delta(\alpha)$ is a transcendental number.*
(b) *Δ is left-continuous but not right-continuous at every rational.*
(c) *If $\alpha = p/q$ with $\gcd(p, q) = 1$ and $b = \lceil \alpha \rceil$, then the $\Delta(\alpha)$-polynomial is*

$$x^q - \overrightarrow{bz_{p,q}b} \cdot (x^{q-1}, x^{q-2}, \ldots, 1).$$

(d) *For the same rational α as above, let $\Delta(\alpha+) := \lim_{x\to\alpha+} \Delta(x)$. Then the $\Delta(\alpha+)$-polynomial is*

$$x^{q+1} - \overrightarrow{bz_{p,q}b} \cdot (x^q, x^{q-1}, \ldots, x) - x + 1.$$

For a rational $\alpha > 0$, $\Delta(\alpha)$ is called a *lower self-Christoffel number*, and the right limit $\Delta(\alpha+)$ an *upper self-Christoffel number*.

We are now in a position to state our main theorem.

THEOREM. *Let p_0/q_0 be a fixed rational with $0 < p_0 \le q_0$ and $\gcd(p_0, q_0) = 1$. For a positive integer b, suppose $p/q = b - 1 + p_0/q_0$. Then there exists an effectively computable B such that for all $b \ge B$, we have*:

(a) $x^q - \overrightarrow{bz_{p,q}b} \cdot (x^{q-1}, x^{q-2}, \dots, 1)$ *is irreducible over* $\mathbb{Q}$,

(b) $x^{q+1} - \overrightarrow{bz_{p,q}b} \cdot (x^q, x^{q-1}, \dots, x) - x + 1$ *is irreducible over* $\mathbb{Q}$.

Thus the polynomials mentioned are eventually the minimal polynomials of $\Delta(p/q)$ and $\Delta(p/q+)$. The above theorem was in fact proved in [7] but we did not know from which value b on the polynomials were irreducible. Here we give another proof involving Chebyshev polynomials. The new proof enables us to effectively find B for which $b \ge B$ implies that the corresponding polynomials are irreducible over $\mathbb{Q}$. The effective procedure for finding B is given in Section 4. These quantitative results have some connections with other fields of number theory, which will be discussed in Section 4.

2. Preliminaries. In this section we briefly review some concepts and known results to be used in the main proof. First we recall some definitions in number theory.

Among algebraic integers, a *Pisot number* is $\alpha > 1$ all of whose conjugates lie inside the unit circle. Suppose $g(x) = a_n x^n + \dots + a_1 x + a_0 = a_n \prod_{i=1}^n (x - \alpha_i) \in \mathbb{Z}[x]$ with $a_n \neq 0$. Then the *Mahler measure* of g is the positive number defined by

$$M(g) = |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}.$$

In particular, cyclotomic polynomials have Mahler measure 1. The most famous problem on Mahler measures, posed by Lehmer [10], is whether or not 1 is an accumulation point of the set $\{M(g) \mid g \in \mathbb{Z}[x]\}$. Though it is still open, an answer can be given for a special class of polynomials. Let $f \in \mathbb{R}[x]$ and $\deg(f) = n$. If f satisfies $f(x) = x^n f(x^{-1})$ then we say that f is *reciprocal*. Now the Mahler measures of nonreciprocal polynomials cannot be less than the smallest Pisot number:

THEOREM 2.1 ([17]). *Let $p(x) \in \mathbb{Z}[x]$ and let $\theta_0 = 1.32472\ldots$ be the smallest Pisot number, i.e., the real zero of $x^3 - x - 1$. If $M(p) < \theta_0$, then $p(x)$ is a reciprocal polynomial.*

Let T_n and U_n be the nth *Chebyshev polynomials of the first* and the *second kind* respectively. Among many equivalent definitions we adopt the following simple equations, which are suitable for our proof:

$$T_n(\cos\theta) := \cos n\theta, \qquad U_n(\cos\theta) := \frac{\sin(n+1)\theta}{\sin\theta}.$$

For instance, $T_2(x) = 2x^2 - 1$ and $U_3(x) = 8x^3 - 4x$.

In the analysis of our polynomials, we shall encounter some reciprocal polynomials, and they will be converted into more convenient ones via a transformation introduced in [3].

Suppose that $p(z) = \sum_{i=0}^{2n} a_i z^i \in \mathbb{R}[z]$ is a nonzero reciprocal polynomial with $a_i = a_{2n-i}$, $i = 0, \ldots, n$. If $a_{2n} = a_{2n-1} = \cdots = a_{n+k+1} = 0$ but $a_{n+k} \neq 0$, then

$$\begin{aligned} p(z) = \sum_{i=0}^{2n} a_i z^i &= z^n \left[a_{n+k}\left(z^k + \frac{1}{z^k}\right) + \cdots + a_{n+1}\left(z + \frac{1}{z}\right) + a_n \right] \\ &= a_{n+k} z^n \prod_{i=1}^{k} \left(z + \frac{1}{z} - \alpha_i\right) = a_{n+k} z^{n-k} \prod_{i=1}^{k} (z^2 - \alpha_i z + 1) \end{aligned}$$

for some $\alpha_i \in \mathbb{C}$, $i = 1, \ldots, k$. Given a reciprocal polynomial p as above, the *Chebyshev transform* $\mathcal{T}$ of p is defined by

$$\mathcal{T}p(x) := a_{n+k} \prod_{i=1}^{k} (x - \alpha_i).$$

THEOREM 2.2 ([3]). *The Chebyshev transform $\mathcal{T}$ is a linear isomorphism of the space of real reciprocal polynomials*, $\{p(z) = \sum_{i=0}^{2n} a_i z^i \in \mathbb{R}[z] : a_i = a_{2n-i},\ i = 0, \ldots, n\}$, *into the space of real polynomials of degree at most n.*

Note that

$$|p(e^{i\theta})| = |\mathcal{T}p(2\cos\theta)|.$$

Chebyshev polynomials now naturally appear via Chebyshev transforms.

LEMMA 2.3 ([9]). *If $p(z) = z^{2n} + 1$ and $q(z) = z^{2n} + z^{2n-2} + \cdots + z^2 + 1$, then $\mathcal{T}p(x) = 2T_n(x/2)$ and $\mathcal{T}q(x) = U_n(x/2)$.*

The following geometric study on beta-polynomials will be useful below. The reader should note that all the bounds given in the proposition tend to 1 as β increases.

PROPOSITION 2.4 ([7]).

(a) *If β is an upper self-Christoffel number and $\gamma \neq \beta$ is a zero of the β-polynomial, then*

$$|\gamma| \leq \frac{\beta + \sqrt{\beta^2 + 4\beta}}{2\beta}.$$

(b) *If β is a lower self-Christoffel number and $\gamma \neq \beta$ is a zero of the β-polynomial, then*
$$\frac{2\beta+1-\sqrt{8\beta+1}}{2\beta} \le |\gamma| \le \frac{2\beta+1+\sqrt{8\beta+1}}{2\beta}.$$

3. Proof. Throughout this section, the notations appearing in the Theorem will be used without explicit mention. First we consider lower self-Christoffel numbers.

Put $f(x) = x^q - \overrightarrow{bz_{p,q}b} \cdot (x^{q-1}, x^{q-2}, \dots, 1)$ and suppose that $f(x) = g(x)h(x)$ over $\mathbb{Q}$ and $g(\beta) \neq 0 = h(\beta)$. Then Theorem 2.1 together with Proposition 2.4 implies that g is eventually reciprocal as b increases. If γ is a zero of g, then so is γ^{-1}. Thus
$$\begin{aligned}\gamma^q &= \overrightarrow{bz_{p,q}b} \cdot (\gamma^{q-1}, \gamma^{q-2}, \dots, 1) = \gamma^{q-1}\overrightarrow{bz_{p,q}b} \cdot (\gamma^{1-q}, \gamma^{2-q}, \dots, 1)\\ &= \gamma^{q-1}\gamma^{-q} = \gamma^{-1},\end{aligned}$$
where we use the fact that $bz_{p,q}b$ is a palindrome. So $\gamma^{q+1} = 1$.

Assume that $q = 2n+1$ and $\gamma_j = e^{i\theta_j} = e^{2j\pi i/(q+1)}$ is a zero of g for some $j = 1, \dots, q$. We then consider the polynomial
$$\begin{aligned}r(z) &:= z^q - f(z) = \overrightarrow{bz_{p,q}b} \cdot (z^{2n}, z^{2n-1}, \dots, 1)\\ &= b(z^{2n} + z^{2n-1} + \dots + 1) - \sum_{k=1}^{n} a_k z^k (z^{2n-2k} + 1), \qquad a_k \in \{0, 1/2, 1\}.\end{aligned}$$
Together with the equation
$$z^{2n}+z^{2n-1}+\dots+1 = (z^{2n}+z^{2n-2}+\dots+z^2+1)+z(z^{2n-2}+z^{2n-4}+\dots+z^2+1),$$
Lemma 2.3 guarantees that the Chebyshev transform of $r(z)$ is
$$\mathcal{T}r(x) = b\left(U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right)\right) - \sum_{k=1}^{n} 2a_k T_{n-k}\left(\frac{x}{2}\right).$$
To compute $\mathcal{T}r(2\cos\theta_j)$, we use the fact that $T_{n-k}(\cos\theta_j) = \cos(n-k)\theta_j$ and that
$$\begin{aligned}(1)\qquad U_n(\cos\theta_j) + U_{n-1}(\cos\theta_j) &= \frac{\sin(n+1)\theta_j + \sin n\theta_j}{\sin\theta_j}\\ &= \frac{2\sin\frac{(2n+1)\theta_j}{2}\cos\frac{\theta_j}{2}}{\sin\theta_j} = \frac{\sin\frac{(2n+1)\theta_j}{2}}{\sin\frac{\theta_j}{2}}.\end{aligned}$$
Thus one finds that
$$1 = |\mathcal{T}r(2\cos\theta_j)| \ge b\,\frac{\left|\sin\frac{(2n+1)\theta_j}{2}\right|}{\left|\sin\frac{\theta_j}{2}\right|} - \sum_{k=1}^{n} 2a_k.$$

Since $0 < \theta_j/2 < \pi$ and $(2n+1)\theta_j/2 = (2n+1)j\pi/(2n+2)$ cannot be an integer multiple of π for any $j = 1, \dots, 2n+1$, this leads to a contradiction as b increases.

If $q = 2n$, then

$$\begin{aligned} r(z) = z^q - f(z) &= \overrightarrow{bz_{p,q}b} \cdot (z^{2n-1}, z^{2n-2}, \dots, 1) \\ &= b(z^{2n-1} + z^{2n-2} + \cdots + 1) - \sum_{k=1}^{n-1} a_k z^k (z^{2n-2k-1} + 1), \qquad a_k \in \{0,1\}. \end{aligned}$$

If $n = 1$, then the sum is understood to be zero. Define a polynomial $r_1(z)$ by

$$\begin{aligned} r_1(z) := \frac{r(z)}{z+1} &= b(z^{2n-2} + z^{2n-4} + \cdots + z^2 + 1) \\ &\quad - \sum_{k=1}^{n-1} a_k z^k (z^{2n-2k-2} - z^{2n-2k-3} + \cdots - z + 1). \end{aligned}$$

Its Chebyshev transform is

$$\mathcal{T} r_1(x) = bU_{n-1}\left(\frac{x}{2}\right) - \sum_{k=1}^{n-1} a_k \left(U_{n-k-1}\left(\frac{x}{2}\right) - U_{n-k-2}\left(\frac{x}{2}\right)\right).$$

If $\gamma_j = e^{i\theta_j} = e^{2j\pi i/(q+1)}$ is a zero of g for some $j = 1, \dots, q$, then

$$\begin{aligned} U_{n-k-1}(\cos\theta_j) - U_{n-k-2}(\cos\theta_j) &= \frac{\sin(n-k)\theta_j - \sin(n-k-1)\theta_j}{\sin\theta_j} \\ &= \frac{2\cos\frac{(2n-2k-1)\theta_j}{2}\sin\frac{\theta_j}{2}}{\sin\theta_j} = \frac{\cos\frac{(2n-2k-1)\theta_j}{2}}{\cos\frac{\theta_j}{2}}. \end{aligned} \tag{2}$$

Since $\gamma_j \neq -1$, there is a constant M so that

$$M \geq |\mathcal{T} r_1(2\cos\theta_j)| \geq b\,\frac{|\sin n\theta_j|}{|\sin\theta_j|} - \sum_{k=1}^{n} \frac{a_k}{|\cos\frac{\theta_j}{2}|}.$$

For any $j = 1, \dots, 2n$, the number $n\theta_j = 2nj\pi/(2n+1)$ is never an integer multiple of π and the number $\theta_j/2 = j\pi/(2n+1)$ cannot be $\pi/2$. Now we get a contradiction for every sufficiently large b. This proves part (a) of the Theorem.

The crucial ingredient in the previous proof is to split a given beta-polynomial into a reciprocal polynomial and a monomial. This technique also works for upper self-Christoffel numbers.

Put $f(x) = x^{q+1} - \overrightarrow{bz_{p,q}b} \cdot (x^q, x^{q-1}, \dots, x) - x + 1$ and suppose that $f(x) = g(x)h(x)$ over $\mathbb{Q}$ and $g(\beta) \neq 0 = h(\beta)$. Then the same reasoning as before shows that g is eventually reciprocal as b increases. If γ is a zero of g, then so is γ^{-1}. Hence

$$\begin{aligned}\gamma &= \gamma^{q+1} - \overrightarrow{bz_{p,q}b} \cdot (\gamma^q, \gamma^{q-1}, \dots, \gamma) + 1 \\ &= \gamma^{q+1}[\gamma^{-q-1} - \overrightarrow{bz_{p,q}b} \cdot (\gamma^{-q}, \gamma^{-q+1}, \dots, \gamma^{-1}) + 1] = \gamma^{q+1}\gamma^{-1} = \gamma^q,\end{aligned}$$

which gives $\gamma^{q-1} = 1$.

First, we suppose that $q = 2n-1$ and $\gamma_j = e^{i\theta_j} = e^{2j\pi i/(q-1)}$ is a zero of g for some $j = 1, \dots, q-2$. We then put

$$\begin{aligned} r(z) := f(z) + z &= z^{2n} - \overrightarrow{bz_{p,q}b} \cdot (z^{2n-1}, z^{2n-2}, \dots, z) + 1 \\ &= z^{2n} + z^{2n-1} + \dots + 1 - (b+1)z(z^{2n-2} + \dots + 1) \\ &\quad + \sum_{k=1}^{n} a_k z^k (z^{2n-2k} + 1), \qquad a_k \in \{0, 1/2, 1\}. \end{aligned}$$

Similar arguments to those above yield the Chebyshev transform

$$\begin{aligned} \mathcal{T}r(x) = U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right) - (b+1)\left(U_{n-1}\left(\frac{x}{2}\right) + U_{n-2}\left(\frac{x}{2}\right)\right) \\ + \sum_{k=1}^{n} 2a_k T_{n-k}\left(\frac{x}{2}\right). \end{aligned}$$

Using $T_{n-k}(\cos\theta_j) = \cos(n-k)\theta_j$ and (1) one thus finds that

$$1 = |\mathcal{T}r(2\cos\theta_j)| \geq (b+1)\,\frac{\left|\sin\frac{(2n-1)\theta_j}{2}\right|}{\left|\sin\frac{\theta_j}{2}\right|} - \frac{\left|\sin\frac{(2n+1)\theta_j}{2}\right|}{\left|\sin\frac{\theta_j}{2}\right|} - \sum_{k=1}^{n} 2a_k. \tag{3}$$

Since $0 < \theta_j/2 < \pi$ and $(2n-1)\theta_j/2 = (2n-1)j\pi/(2n-2)$ cannot be an integer multiple of π for any $j = 1, \dots, 2n-3$, we eventually get a contradiction as b increases.

If $q = 2n$, then

$$\begin{aligned} r(z) = f(z) + z &= z^{2n+1} - \overrightarrow{bz_{p,q}b} \cdot (z^{2n}, z^{2n-1}, \dots, z) + 1 \\ &= z^{2n+1} + z^{2n} + \dots + 1 - (b+1)z(z^{2n-1} + \dots + 1) \\ &\quad + \sum_{k=1}^{n} a_k z^k (z^{2n+1-2k} + 1), \qquad a_k \in \{0, 1\}. \end{aligned}$$

This can be rewritten in the form

$$\begin{aligned} r_1(z) &:= \frac{r(z)}{z+1} \\ &= z^{2n} + z^{2n-2} + \dots + z^2 + 1 - (b+1)z(z^{2n-2} + z^{2n-4} + \dots + z^2 + 1) \\ &\quad + \sum_{k=1}^{n} a_k z^k (z^{2n-2k} - z^{2n-2k-1} + \dots - z + 1). \end{aligned}$$

Hence we find

$$\mathcal{T}r_1(x) = U_n\left(\frac{x}{2}\right) - (b+1)U_{n-1}\left(\frac{x}{2}\right) + \sum_{k=1}^{n} a_k\left(U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right)\right).$$

If $\gamma_j = e^{i\theta_j} = e^{2j\pi i/(q-1)}$ is a zero of g for some $j = 1, \ldots, q-2$, then one can verify as in (2) that

$$U_{n-k}(\cos\theta_j) - U_{n-k-1}(\cos\theta_j) = \frac{\cos\frac{(2n-2k+1)\theta_j}{2}}{\cos\frac{\theta_j}{2}}.$$

Since $\gamma_j \neq -1$, there is a constant M so that

$$M \geq |\mathcal{T}r_1(2\cos\theta_j)| \geq (b+1)\frac{|\sin n\theta_j|}{|\sin\theta_j|} - \frac{|\sin(n+1)\theta_j|}{|\sin\theta_j|} - \sum_{k=1}^{n}\frac{a_k}{|\cos\frac{\theta_j}{2}|}.$$

For any $j = 1, \ldots, 2n-2$, the number $n\theta_j = 2nj\pi/(2n-1)$ is never an integer multiple of π and the number $\theta_j/2 = j\pi/(2n-1)$ cannot be $\pi/2$. Now we get a contradiction for every sufficiently large b.

4. Discussion and further studies. Suppose that an integer $b \geq 1$ is divisible by a prime p but not by p^2. Then the Eisenstein criterion shows that the beta-polynomial of $\Delta((bq-1)/q)$ is irreducible for any $q \geq 1$. Now we introduce another interesting connection between prime numbers and irreducibility for self-Christoffel numbers. This is related to *Mersenne primes*, i.e., primes of the form $2^n - 1$. A famous open question in number theory is:

Are there infinitely many Mersenne primes?

We need a classical irreducibility criterion for polynomials in $\mathbb{Z}[x]$, which appears for example in [14].

THEOREM 4.1. *Let $\beta_1, \ldots, \beta_n$ be the zeros of some $f(x) \in \mathbb{Z}[x]$ of degree n. If there exists an integer b such that $f(b)$ is prime*, $f(b-1) \neq 0$ *and*

$$\Re(\beta_i) < b - 1/2 \quad \text{for } 1 \leq i \leq n, \tag{4}$$

then $f(x)$ is irreducible over $\mathbb{Q}$.

We consider the beta-polynomials of $\Delta(1/q)$ and $\Delta(1/q+)$. Denote them by $f_q^l(x)$ and $f_q^u(x)$ respectively, and take $b = 2$. Then $f_q^l(1) = f_q^u(1) = -1$. If $q \geq 3$ then $\Delta(1/q) < 3/2$, and if $q \geq 4$ then $\Delta(1/q+) < 3/2$. Since beta-numbers are *Perron numbers* [11], condition (4) holds for lower (resp. upper) self-Christoffel numbers whenever $q \geq 3$ (resp. $q \geq 4$). We also note that

$$f_q^l(2) = 2^{q-1} - 1, \qquad f_q^u(2) = 2^q - 3.$$

Now one can say the following:

1. *If there exist infinitely many Mersenne primes, then $f^l_q(x)$ is irreducible for infinitely many $q \geq 3$.*
2. *If there exist infinitely many primes of the form $2^n - 3$, then $f^u_q(x)$ is irreducible for infinitely many $q \geq 4$.*

Using a computer the author checked the irreducibility of beta-polynomials of self-Christoffel numbers. To be more precise, suppose $\alpha = p/q+b-1$, where $1 \leq p < q \leq 200$, $\gcd(p,q) = 1$ and $1 \leq b \leq 150$. For given $1/2 \leq p/q \leq 199/200$, the irreducibility of beta-polynomials of $\Delta(\alpha)$ and $\Delta(\alpha+)$ was checked for $1 \leq b \leq 150$. When some beta-polynomial is reducible, the case was recorded as "(p,q,b)" in a row. As a result, a table was obtained, which comprises 5642 (resp. 3422) reducible cases for lower (resp. upper) self-Christoffel numbers. This is too huge to be included here. We refer to [8] instead. For example, if $\alpha = 6/11 + b - 1$ then the beta-polynomial of $\Delta(\alpha)$ is irreducible for all $b = 1, 2$, $4 \leq b \leq 150$, and the beta-polynomial of $\Delta(\alpha+)$ is irreducible for all $2 \leq b \leq 150$. On the other hand, if $\alpha = 4/9+b-1$ then the beta-polynomials of both $\Delta(\alpha)$ and $\Delta(\alpha+)$ are irreducible for all $1 \leq b \leq 150$. We will use this table below implicitly.

Suppose $\alpha = p/q+b-1$. If we follow the proof of the Theorem described in Section 3, then we can find effectively the smallest possible constant B for which $b \geq B$ implies that the beta-polynomial of $\Delta(\alpha)$ or $\Delta(\alpha+)$ is irreducible. This procedure is demonstrated in the next example.

Example 4.2. Let $\alpha = 4/7 + b - 1$ and let us find B for the upper self-Christoffel number $\Delta(\alpha+)$. First we must determine B_1 so that the Mahler measure of g is less than θ_0 given in Theorem 2.1. One can see that it is enough that

$$\left(\frac{B_1 + \sqrt{B_1^2 + 4B_1}}{2B_1}\right)^5 < \theta_0.$$

So $B_1 \geq 17$. Second we also find B_2 which contradicts inequality (3). Computation shows that $B_2 \geq 11$. Let $B_3 = \max\{B_1, B_2\} = 17$. Then for $b \geq B_3$ the beta-polynomial of $\Delta((4/7 + b - 1)+)$ is irreducible. One readily notes that the constant B_3 works not only for $4/7$ but for all $1/7, 2/7, \ldots, 6/7$. To find the smallest B instead of B_3, we consult the table of [8]. It suffices to check whether or not $(1,7,b), (2,7,b), \ldots, (6,7,b)$ for $1 \leq b \leq 16$ are on the list. But none of them is, even for $1 \leq b \leq 150$. Now we can state the following (i.e., in this case the beta-polynomial of $\Delta(\alpha+)$ is irreducible for every $b \geq B = 1$):

For any integer $p \geq 1$ which is not a multiple of 7, the beta-polynomial of $\Delta(p/7+)$ is irreducible.

Some calculations tempt us to conjecture that all lower self-Christoffel numbers are Pisot numbers. But this is false as explained below.

Flatto *et al.* [5] considered the dynamical zeta-function

$$\zeta_\beta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{P_n}{n}\, z^n\right)$$

for the β-transformation T_β, where P_n is the number of fixed points of T_β^n. They wrote it in terms of the β-expansions of 1:

1. If β is not a simple beta-number with $d_\beta(1) = e_1 e_2 \cdots$, then

$$\zeta_\beta(z) = \frac{1}{1 - \sum_{i=1}^{\infty} e_i z^i}.$$

2. If β is a simple beta-number with $d_\beta(1) = e_1 \cdots e_n$, then

$$\zeta_\beta(z) = \frac{1 - z^n}{1 - \sum_{i=1}^{n} e_i z^i}.$$

In both cases, $\zeta_\beta(z)$ is meromorphic in the open unit disk. It has a simple pole at $z = 1/\beta$ and no other pole in $\{z : |z| \le 1/\beta\}$. Then $M(\beta)$ was defined by the second smallest modulus of the poles of $\zeta_\beta(z)$. If no pole other than $1/\beta$ exists in the open unit disk, then $M(\beta) = 1$. The main interest was the behavior of the function $M(\beta)$. The next theorem hints at the possible abundance of non-Pisot lower self-Christoffel numbers.

Theorem 4.3 ([5]). *There exists $\varepsilon > 0$ such that $M(\beta) < 1$ for every $\beta \in (1, 1+\varepsilon)$.*

In particular, if the Mersenne prime conjecture is true then there are infinitely many non-Pisot lower self-Christoffel numbers. There still remains a question in this direction.

Question 1. *Let p/q be a fixed rational with $0 < p \le q$ and $\gcd(p,q) = 1$. If $\alpha = b - 1 + p/q$ for $b \in \mathbb{N}$, is $\Delta(\alpha)$ eventually a Pisot number as b increases?*

Together with the Theorem, an affirmative answer to Question 1 would produce many concrete examples of *irreducible Pisot type beta-substitutions.* See [1] for details.

While our main interest in this paper is irreducible beta-polynomials, Boyd [2] focused on reducible beta-polynomials. Specifically, he dealt with the situation where β is a Pisot number. It is well known that every Pisot number is a beta-number [16]. Let β be a Pisot number and $f(x)$ be the β-polynomial. Now we suppose that $f(x) = g(x)h(x)$ is a nontrivial factorization over $\mathbb{Q}$, where $h(x)$ is the minimal polynomial of β. Boyd called $g(x)$ the *complementary factor.* Although extensive calculations made some mathematicians suspect that $g(x)$ should be a product of cyclotomic polynomials, Boyd showed by systematic computation that there exist many Pisot numbers β for which their complementary factors are noncyclotomic or even nonreciprocal.

Motivated by Boyd's work, the author checked the complementary factors of all reducible cases listed in [8] for both lower (5642) and upper (3422) self-Christoffel numbers. Surprisingly enough, all the complementary factors are (either cyclotomic polynomials or) products of cyclotomic polynomials.

For lower self-Christoffel numbers in [8], all irreducibles of the complementary factors are among the Φ_n with $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26, 28$, or 30, where Φ_n is the nth cyclotomic polynomial. For upper self-Christoffel numbers in [8], all irreducibles of the complementary factors are among the Φ_n with $n = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12$, or 14.

A natural question arises.

QUESTION 2. *For β a lower or an upper self-Christoffel number, if the β-polynomial is reducible, is its complementary factor a product of cyclotomic polynomials?*

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