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On the Siegel–Tatuzawa–Hoffstein theorem

by

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1. Introduction. Let χ be a real primitive Dirichlet character of conductor k (> 1). The well-known Siegel theorem (Siegel [8]) says that for any $\varepsilon > 0$ there exists a positive number $c(\varepsilon)$ such that

$$L(1,\chi) > \frac{c(\varepsilon)}{k^{\varepsilon}},$$

where $c(\varepsilon)$ is an ineffective constant depending upon ε. Estermann [2], Chowla [1], Goldfeld [3] and Goldfeld and Schinzel [4] have given several proofs of Siegel's theorem.

Tatuzawa [9] proved that if $0 < \varepsilon < 1/11.2$ and $k > e^{1/\varepsilon}$, then, with at most one exception,

$$L(1,\chi) > \frac{0.655\varepsilon}{k^{\varepsilon}}.$$

Hoffstein [5] proved that if $0 < \varepsilon < 1/(6\log 10)$ and $k > e^{1/\varepsilon}$, then, with at most one exception,

$$L(1,\chi) > \min\left\{\frac{1}{7.735\log k}, \frac{\varepsilon}{0.349k^{\varepsilon}}\right\}.$$

Ji and Lu [6] proved that if $0 < \varepsilon < 1/(6\log 10)$ and $k > e^{1/\varepsilon}$, then, with at most one exception,

$$L(1,\chi) > \min\left\{\frac{1}{7.7388\log k}, \frac{32.260\varepsilon}{k^{\varepsilon}}\right\}.$$

In this paper the following theorem is proved.

THEOREM 1. *Let $0 < \varepsilon < 1/(6\log 10)$ and χ be a real primitive Dirichlet character modulo k with $k > e^{1/\varepsilon}$. Then, with at most one exception,*

$$L(1,\chi) > \min\left\{\frac{1}{7.732\log k}, \frac{1.5\cdot 10^{6}\varepsilon}{k^{\varepsilon}}\right\}.$$

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In this paper we follow the proof of Ji and Lu [6], with the following improvement: First, β and β_1 do not play the same roles in Ji and Lu [6], but they do in this paper (see (5) in the next section). Secondly, we use the following fact: the function $xd_F^{-A_2x}$ increases for $0 < x \le 1/(A_2 \log d_F)$ and decreases for $x \ge 1/(A_2 \log d_F)$. Thus for $a \le x \le b$ we have $xd_F^{-A_2x} \ge \min\{ad_F^{-A_2a}, bd_F^{-A_2b}\}$. This improves $xd_F^{-A_2x} \ge ad_F^{-A_2b}$ for $a \le x \le b$. Thirdly, we give a result with five parameters. Then by carefully choosing these parameters we find a numerical result.

In fact we prove a general Siegel–Tatuzawa–Hoffstein type theorem in the next section (Theorem 2). By taking explicit values of the five parameters we can obtain many results similar to Theorem 1.

2. Proofs

LEMMA 1 (Hoffstein [5, Lemma 2]). *Let $c = 6 - 4\sqrt{2}$. Let K $(\ne \mathbb{Q})$ be an algebraic number field and let d_K denote the absolute value of the discriminant of K. Then $\zeta_K(s)$ has at most one real simple zero β with*

$$1 - \beta < \frac{c}{\log d_K}.$$

LEMMA 2 (Ji and Lu [6, Lemma 2]). *Let K be an algebraic number field of degree $n > 1$ and assume that for each $m \ge 1$ there exists at least one integral ideal of K of norm m^2 (e.g. K is a quadratic or a biquadratic bicyclic number field). Assume also that $1/2 < \beta < 1$ and $\zeta_K(\beta) \le 0$. Then the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ of K satisfies*

$$\kappa_K \ge (1-\beta)\left(x^{\beta-1}\left(\frac{\pi^2}{6} - \frac{n+2}{[\sqrt{x}\,]}\right) - 2\,\frac{d_K}{x^{3/2}}\,\frac{\zeta^n(3/2)(n+1)!}{(4n-3)\pi^n}\right) \qquad (x \ge 1).$$

LEMMA 3 (Ramaré [7]). *If χ is a primitive Dirichlet character modulo k, then*

$$|L(1,\chi)| \le (\log k + 1.4165)/2.$$

Lemma 3 follows from [7, Corollary 1] and $5 - 2\log 6 < 1.4165$.

Let $c = 6 - 4\sqrt{2}$. Let A_1, A_2, r, u, v be parameters with

$$A_1 > \frac{2}{3}, \qquad 9A_2 \log 10 - 6\log 10 - rA_2 > 0,$$
$$0 < r < \frac{1}{2A_2}, \qquad u \ge \frac{2.206rA_2 - 0.103}{1 - 2rA_2}, \qquad u \ge 1, \qquad v \ge 1.$$

Let

$$D_1 = \frac{\pi^2}{6} - \frac{4}{[10^{3A_1}]} - \frac{12\zeta^2(3/2)}{5\pi^2}\,\frac{e^{rA_1}}{10^{9A_1-6}},$$

$$D_2 = \frac{\pi^2}{6} - \frac{6}{[10^{6A_2}]} - \frac{240\zeta^4(3/2)}{13\pi^4}\,\frac{e^{2rA_2}}{10^{18A_2-12}},$$

$$B_1 = \frac{2D_2}{D_1}\min\biggl\{\frac{D_1^2}{2D_2}\, re^{-rA_1}, 0.475e^{rA_1-4rA_2}, 0.1188\,\frac{c}{r}\, e^{rA_1-cA_2},$$
$$\frac{u}{u+1.103}\, e^{r(A_1-2A_2-2A_2u)}, \frac{cv}{2r(v+1)(v+1.103)}\, e^{rA_1-cA_2}\biggr\},$$
$$B_2 = \frac{2D_2}{D_1}\min\biggl\{\frac{1}{u+1.103}\, e^{r(A_1-2A_2)+(1-2rA_2)u},$$
$$\frac{c}{2r(v+1)(v+1.103)}\, e^{rA_1-cA_2+v}\biggr\}.$$

To prove Theorem 1, we first prove the following general Siegel–Tatuzawa–Hoffstein type theorem.

THEOREM 2. *Let $0 < \varepsilon < 1/(6\log 10)$ and B_1, B_2 be two constants depending only on the parameters A_1, A_2, r, u, v. Let χ be a real primitive Dirichlet character modulo k with $k > e^{1/\varepsilon}$. Then, with at most one exception*,

$$L(1,\chi) > \min\biggl\{\frac{B_1}{\log k}, \frac{\varepsilon B_2}{k^\varepsilon}\biggr\}.$$

Theorem 1 follows from Theorem 2 by taking $A_1 = 1$, $A_2 = 0.78$, $r = 0.086$, $u = 19.23$ and $v = 22$.

Proof of Theorem 2. Let χ_1 be a real primitive Dirichlet character of least conductor $k_1 > e^{1/\varepsilon}$ such that

$$L(1,\chi_1) \le \frac{B_1}{\log k_1} \le D_1 re^{-rA_1}\, \frac{1}{\log k_1} \tag{1}$$

if it exists. Let χ_1 be the Kronecker symbol of the quadratic number field $K_1 = \mathbb{Q}(\sqrt{d_1})$, where d_1 is the fundamental discriminant with $|d_1| = k_1$. If $L(s,\chi_1) \ne 0$ for $1 - r/\log k_1 < s < 1$, let $s_1 = 1 - r/\log k_1$; then $\zeta_{K_1}(s_1) \le 0$. Take $x = k_1^{A_1}$ and $n = 2$. By Lemma 2 and the assumptions $A_1 > 2/3$ and $k_1 > e^{1/\varepsilon} > 10^6$ we have

$$\begin{aligned} L(1,\chi_1) &\ge (1-s_1)x^{s_1-1}\biggl(\frac{\pi^2}{6} - \frac{4}{[\sqrt{x}\,]} - \frac{12\zeta^2(3/2)}{5\pi^2}\,\frac{k_1}{x^{3/2}}\, x^{1-s_1}\biggr) \\ &= \frac{r}{\log k_1}\, e^{-rA_1}\biggl(\frac{\pi^2}{6} - \frac{4}{[k_1^{A_1/2}]} - \frac{12\zeta^2(3/2)}{5\pi^2}\, e^{rA_1}k_1^{1-3A_1/2}\biggr) \\ &> D_1 re^{-rA_1}\,\frac{1}{\log k_1}, \end{aligned}$$

contradicting (1). So $L(s,\chi_1)$ has a real zero β_1 with

$$1 - \beta_1 < \frac{r}{\log k_1}. \tag{2}$$

Let χ be another real primitive Dirichlet character modulo $k > e^{1/\varepsilon}$ and $\chi \ne \chi_1$. By the choice of k_1 we have $k \ge k_1$. Let χ be the Kronecker symbol of the quadratic number field $K = \mathbb{Q}(\sqrt{d})$, where d is the fundamental discriminant with $|d| = k$. Let $F = \mathbb{Q}(\sqrt{d}, \sqrt{d_1})$. Then, by the theory of biquadratic number fields, there exists another quadratic number field $K_2 = \mathbb{Q}(\sqrt{d_2})$ such that

$$\zeta_F(s) = \zeta(s)L(s,\chi)L(s,\chi_1)L(s,\chi_2),$$

where d_2 is the fundamental discriminant with $d_2 \mid dd_1$ and χ_2 is the Kronecker symbol of the quadratic number field K_2.

Similarly to the above arguments, if $L(s,\chi) \neq 0$ for $1 - r/\log k < s < 1$, then

$$L(1,\chi) > D_1 r e^{-rA_1} \frac{1}{\log k} \geq \frac{B_1}{\log k}.$$

Now we assume that $L(s,\chi)$ has a real zero β with

$$1 - \beta < \frac{r}{\log k} \leq \frac{r}{\log k_1}. \tag{3}$$

By Lemma 1 and the equalities $\zeta_F(\beta_1) = \zeta_F(\beta) = 0$ we have

$$\max\{1-\beta_1, 1-\beta\} \geq \frac{c}{\log d_F}. \tag{4}$$

Let $x = d_F^{A_2}$ and $n = 4$. Noting that $9A_2 \log 10 - 6\log 10 - rA_2 > 0$, $d_F \geq kk_1 \geq k_1^2$ and $k_1 > 10^6$, by Lemma 2, (2) and the equality $\zeta_F(\beta_1) = 0$ we have

$$\begin{aligned}\kappa_F &\geq (1-\beta_1) d_F^{(\beta_1-1)A_2} \left(\frac{\pi^2}{6} - \frac{6}{[d_F^{A_2/2}]} - \frac{240\zeta^4(3/2)}{13\pi^4}\, d_F^{1-3A_2/2+(1-\beta_1)A_2} \right) \\ &\geq (1-\beta_1) d_F^{(\beta_1-1)A_2} \left(\frac{\pi^2}{6} - \frac{6}{[d_F^{A_2/2}]} - \frac{240\zeta^4(3/2)}{13\pi^4}\, d_F^{1-3A_2/2+rA_2/\log k_1} \right) \\ &\geq (1-\beta_1) d_F^{(\beta_1-1)A_2} D_2.\end{aligned}$$

Similarly, by Lemma 2, (3) and the equality $\zeta_F(\beta) = 0$ we have

$$\kappa_F \geq (1-\beta) d_F^{(\beta-1)A_2} D_2.$$

Since $\kappa_F = L(1,\chi)L(1,\chi_1)L(1,\chi_2)$, we obtain

$$L(1,\chi)L(1,\chi_1)L(1,\chi_2) \geq D_2 \max\{(1-\beta) d_F^{(\beta-1)A_2}, (1-\beta_1) d_F^{(\beta_1-1)A_2}\}. \tag{5}$$

Since $d_F \leq (kk_1)^2$, and the function $x d_F^{-A_2 x}$ increases for $x \leq 1/(A_2 \log d_F)$ and decreases for $x \geq 1/(A_2 \log d_F)$, by (2)–(5) we get

$$\begin{aligned}L(1,\chi)L(1,\chi_1)L(1,\chi_2) &\geq D_2 \min\left\{ \frac{r}{\log k_1}\, d_F^{-rA_2/\log k_1}, \frac{c}{\log d_F}\, d_F^{-cA_2/\log d_F} \right\} \\ &\geq D_2 \min\left\{ \frac{r}{\log k_1}\, (kk_1)^{-2rA_2/\log k_1}, \frac{c}{2\log(kk_1)}\, e^{-cA_2} \right\}.\end{aligned}$$

Lemma 3 yields

$$(6)\qquad L(1,\chi_2) \le \frac{1}{2}(\log|d_2| + 1.4165) \le \frac{1}{2}(\log(kk_1) + 1.4165).$$

By (1) and (6) we have

$$\begin{aligned}
&L(1,\chi)\\
&\ge D_2 L(1,\chi_1)^{-1} L(1,\chi_2)^{-1} \min\left\{\frac{r}{\log k_1}(kk_1)^{-2rA_2/\log k_1}, \frac{c}{2\log(kk_1)}e^{-cA_2}\right\}\\
&\ge \frac{2D_2}{rD_1}e^{rA_1}\frac{\log k_1}{\log(kk_1) + 1.4165}\min\left\{\frac{r(kk_1)^{-2rA_2/\log k_1}}{\log k_1}, \frac{ce^{-cA_2}}{2\log(kk_1)}\right\}\\
&= \frac{2D_2}{D_1}\min\left\{\frac{e^{rA_1 - 2rA_2 - 2rA_2\log k/\log k_1}}{\log(kk_1) + 1.4165}, \frac{ce^{rA_1 - cA_2}\log k_1}{2r(\log(kk_1) + 1.4165)\log(kk_1)}\right\}.
\end{aligned}$$

Let

$$(7)\qquad x = \frac{\log k}{\log k_1}.$$

If $1 \le x \le u$, then

$$\begin{aligned}
&(\log k + \log k_1 + 1.4165)^{-1} e^{-2rA_2\log k/\log k_1}\\
&\qquad = (\log k)^{-1} x(x + 1 + 1.4165/\log k_1)^{-1} e^{-2rA_2 x}\\
&\qquad \ge (\log k)^{-1} x(x + 1 + 1.4165/(6\log 10))^{-1} e^{-2rA_2 x}\\
&\qquad \ge (\log k)^{-1} x(x + 1.103)^{-1} e^{-2rA_2 x}\\
&\qquad \ge (\log k)^{-1} \min\{2.103^{-1} e^{-2rA_2}, u(u + 1.103)^{-1} e^{-2rA_2 u}\}\\
&\qquad \ge (\log k)^{-1} \min\{0.475 e^{-2rA_2}, u(u + 1.103)^{-1} e^{-2rA_2 u}\}.
\end{aligned}$$

If $x > u$, then the inequalities $0 < r < (2A_2)^{-1}$, $\log k \ge \log k_1 > 1/\varepsilon$ and

$$u \ge \frac{2.206rA_2 - 0.103}{1 - 2rA_2}$$

imply

$$\begin{aligned}
&(\log k + \log k_1 + 1.4165)^{-1} e^{-2rA_2\log k/\log k_1}\\
&\qquad = (\log k_1)^{-1} e^{-\log k/\log k_1}(x + 1 + 1.4165/\log k_1)^{-1} e^{(1-2rA_2)x}\\
&\qquad \ge \frac{\varepsilon}{k^\varepsilon}(x + 1 + 1.4165/(6\log 10))^{-1} e^{(1-2rA_2)x}\\
&\qquad \ge \frac{\varepsilon}{k^\varepsilon}(x + 1.103)^{-1} e^{(1-2rA_2)x}\\
&\qquad > \frac{\varepsilon}{k^\varepsilon}(u + 1.103)^{-1} e^{(1-2rA_2)u}.
\end{aligned}$$

In the first inequality, we use the fact that $\log k \ge \log k_1 > 1/\varepsilon$ and that the function $te^{(\log k)/t}$ decreases for $0 < t \le \log k$. Hence

$$\frac{2D_2}{D_1}\,\frac{e^{rA_1-2rA_2-2rA_2\log k/\log k_1}}{\log(kk_1)+1.4165} \ge \min\left\{\frac{B_1}{\log k}, \frac{\varepsilon B_2}{k^{\varepsilon}}\right\}.$$

If $1 \le x \le v$, then

$$\begin{aligned}(\log k_1)&(\log k+\log k_1+1.4165)^{-1}(\log k+\log k_1)^{-1}\\ &= (\log k)^{-1}(x+1+1.4165/\log k_1)^{-1}(x+1)^{-1}x\\ &\ge (\log k)^{-1}(x+1+1.4165/(6\log 10))^{-1}(x+1)^{-1}x\\ &\ge (\log k)^{-1}\min\{0.2377, (v+1.103)^{-1}(v+1)^{-1}v\}.\end{aligned}$$

If $x > v$, then

$$\begin{aligned}(\log k_1)&(\log k+\log k_1+1.4165)^{-1}(\log k+\log k_1)^{-1}\\ &= (\log k_1)^{-1}e^{-\log k/\log k_1}e^{x}(x+1+1.4165/\log k_1)^{-1}(x+1)^{-1}\\ &\ge \frac{\varepsilon}{k^{\varepsilon}}\,e^{x}(x+1.103)^{-1}(x+1)^{-1}\\ &\ge \frac{\varepsilon}{k^{\varepsilon}}\,e^{v}(v+1.103)^{-1}(v+1)^{-1}.\end{aligned}$$

Here we use the fact that $\log k \ge \log k_1 > 1/\varepsilon$ and that the function $te^{(\log k)/t}$ decreases for $0 < t \le \log k$. Hence

$$\frac{2D_2}{D_1}\,\frac{ce^{rA_1-cA_2}\log k_1}{2r(\log(kk_1)+1.4165)\log(kk_1)} \ge \min\left\{\frac{B_1}{\log k}, \frac{\varepsilon B_2}{k^{\varepsilon}}\right\}.$$

Therefore

$$L(1,\chi) > \min\left\{\frac{B_1}{\log k}, \frac{\varepsilon B_2}{k^{\varepsilon}}\right\}.$$

This completes the proof of Theorem 2.

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