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On an asymptotic formula of Ramanujan for a certain theta-type series

by

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To the memory of Professor Takayoshi Mitsui

1. Introduction. Let τ be a positive real parameter, $s = \sigma + it$ a complex variable, and $\Gamma(s)$ and $\zeta(s)$ denote the gamma function and the Riemann zeta-function respectively. In Chapter 15 of his celebrated notebook [Ra] (see also Berndt [Be]), Ramanujan suggested a way of computing in exact form the error term of the asymptotic formula

$$\sum_{n=1}^{\infty} \frac{1}{e^{n^2\tau} - 1} = R(\tau) + o(1)$$

as $\tau \to +0$, where

$$R(\tau) = \frac{\pi^2}{6\tau} + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\zeta\left(\frac{1}{2}\right) + \frac{1}{4}.$$

The computation was carried out by Berndt and Evans [BeEv] in terms of Poisson's summation device. They showed

$$\sum_{n=1}^{\infty} \frac{1}{e^{n^2\tau} - 1} = R(\tau) + S(\tau), \tag{1.1}$$

where

$$S(\tau) = \sqrt{\frac{\pi}{2\tau}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cdot \frac{\cos(2\pi\sqrt{\pi n/\tau} + \pi/4) - e^{-2\pi\sqrt{\pi n/\tau}}\cos(\pi/4)}{\cosh(2\pi\sqrt{\pi n/\tau}) - \cos(2\pi\sqrt{\pi n/\tau})}.$$

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In view of $e^{-x} = \cosh x - \sinh x$, the last sum is rewritten as

$$S(\tau) = \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left\{\frac{\sinh(2\pi\sqrt{\pi n/\tau}) - \sin(2\pi\sqrt{\pi n/\tau})}{\cosh(2\pi\sqrt{\pi n/\tau}) - \cos(2\pi\sqrt{\pi n/\tau})} - 1\right\}.$$

Still another alternative expression of $S(\tau)$ was found by Klusch [Kl, p. 60, 2(B)], who also gave its various interesting applications involving arithmetical functions. A χ-analogue formula of (1.1) was derived by Egami [Ega]. The methods applied by Klusch and Egami are slightly different from each other, but both based on the Mellin transform technique. It is worth while noting that this technique has the advantage over heuristic treatments in studying certain mean values of zeta-functions (see [Ka3], [Ka4]), and power series and asymptotic series associated with zeta-functions (see [Ka1], [Ka2], [Ka5], [Ka6]).

It is the purpose of the present paper to generalize the formula (1.1) in two directions (Theorems 1 and 2 below) by continuing the previous work by Berndt and Evans [BeEv], Klusch [Kl] and Egami [Ega]. The key method for our treatment of theta-type series is a Mellin transform technique (see (3.2) and (4.1)), similar to that of [Ega]. It should be noticed that the functional properties of the generalized Hurwitz zeta-function (see Theorems 3 and 4 in Section 2) and the Lerch zeta-function (see (2.8)) play important parts in the proofs.

In order to describe our results we prepare several notations and terminology. Let r be a positive integer, α a positive real parameter, and let $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n be Pochhammer's symbol. We first introduce the generalized Hurwitz zeta-function $\zeta_r(s,\alpha)$ defined by

$$\zeta_r(s,\alpha) = \sum_{n=0}^{\infty}\frac{(r)_n}{n!}(n+\alpha)^{-s}, \tag{1.2}$$

which converges absolutely for $\operatorname{Re} s > r$, since

$$\frac{(r)_n}{n!} = O(n^{r-1}) \qquad (n\to\infty), \tag{1.3}$$

while continues to a meromorphic function over the whole s-plane (see Theorem 3). Note that (1.2) is a particular case $\underline{\omega} = (1,\dots,1)$ of Barnes' multiple zeta-function $\zeta_r(s,\alpha\,|\,\underline{\omega})$ (cf. Barnes [Ba]) and so $\zeta_1(s,\alpha) = \zeta(s,\alpha)$ is the usual Hurwitz zeta-function, because the number of the r-tuple $(n_1,\dots,n_r)$ of non-negative integers satisfying $n_1+\dots+n_r = n$ is equal to $(r)_n/n!$. We next associate with $\zeta_r(s,\alpha)$ Nörlund's generalized Bernoulli polynomial $B_k^{(r)}(\alpha)$ $(k=0,1,\dots)$ defined by the Taylor series expansion

$$\left(\frac{z}{e^z-1}\right)^r e^{\alpha z} = \sum_{k=0}^{\infty}\frac{B_k^{(r)}(\alpha)}{k!}z^k \qquad (|z|<2\pi) \tag{1.4}$$

(cf. Nörlund [Nö, pp. 145, 77]). Note that $B_k^{(1)}(\alpha) = B_k(\alpha)$ is the usual Bernoulli polynomial, and so $B_k^{(1)}(0) = B_k$ is the usual Bernoulli number (cf. Erdélyi [Er1, p. 36, 1.13, (2) and (4)]). The properties of $\zeta_r(s,\alpha)$ and its connection with $B_k^{(r)}(\alpha)$ were first studied in a more general case of $\zeta_r(s,\alpha\,|\,\underline{\omega})$ by Barnes [Ba] (see also Theorem 3). We set

$$a(x) = 2\pi\sqrt{\pi x}, \qquad b_k(y) = \frac{1}{2}\pi\left(k+\frac{1}{2}\right) - 2\pi y \tag{1.5}$$

for $x, y > 0$ and $k = 0, 1, \ldots,$ and

$$\begin{aligned} f_k(x,y) &= \frac{\cos(a(x)+b_k(y)) - e^{-a(x)}\cos b_k(y)}{\cosh a(x) - \cos a(x)} \\ &= \frac{\sinh a(x)\cos b_k(y) - \sin a(x)\sin b_k(y)}{\cosh a(x) - \cos a(x)} - \cos b_k(y), \end{aligned} \tag{1.6}$$

where the second equality follows from $e^{-a(x)} = \cosh a(x) - \sinh a(x)$. Then our first main result can be stated as

THEOREM 1. *Let $\zeta_r(s,\alpha)$ and $B_k^{(r)}(\alpha)$ be defined as in* (1.2) *and* (1.4) *respectively. Then for any $\tau > 0$, any integer $r \ge 1$ and any real α with $0 < \alpha \le r$, we have the formula*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{e^{-\alpha n^2\tau}}{(1-e^{-n^2\tau})^r} &= \frac{1}{2}(-1)^{r+1}\sum_{h=0}^{r} \frac{B_{2h}B_{r-h}^{(r)}(\alpha)}{(2h)!(r-h)!}\left(\frac{4\pi^2}{\tau}\right)^h \\ &\quad + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\zeta_r\left(\frac{1}{2},\alpha\right) + S_r(\tau;\alpha), \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} S_r(\tau;\alpha) &= \frac{1}{\tau}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{r-1} \frac{(-1)^{r-k-1}B_{r-k-1}^{(r)}(\alpha)}{k!(r-k-1)!(2\pi\tau)^k} \\ &\quad \times \left(\tau^2\frac{\partial}{\partial\tau}\right)^k \left\{\sqrt{\tau}\sum_{n=1}^{\infty}\frac{1}{n^{k+1/2}}f_k\left(\frac{n}{\tau},\alpha n\right)\right\} \end{aligned} \tag{1.8}$$

with the notations in (1.5) *and* (1.6).

REMARK 1. By introducing the new variable $t = 1/\tau$, the iterated differentiation on the right-hand side of (1.8) is written in a more convenient form $(\tau^2\partial/\partial\tau)^k = (-\partial/\partial t)^k$.

REMARK 2. The inner infinite sum on the right-hand side of (1.8) is regarded as a convergent asymptotic series in the ascending order of τ when τ is small, since the exact order of each term is

$$f_k(n/\tau,\alpha n) = 2e^{-a(n/\tau)}\cos(a(n/\tau)+b_k(\alpha n)) + O(e^{-2a(n/\tau)})$$

as $\tau \to +0$, with the implied O-constant being absolute.

The case $r = 1$ of Theorem 1 reduces to

Corollary 1.1. *For any $\tau > 0$ and $0 < \alpha \le 1$ we have*

$$\sum_{n=1}^{\infty} \frac{e^{-\alpha n^2\tau}}{1-e^{-n^2\tau}} = \frac{\pi^2}{6\tau} + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\zeta\left(\frac{1}{2},\alpha\right) + \frac{1}{2}\left(\alpha - \frac{1}{2}\right) + \sqrt{\frac{\pi}{2\tau}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} f_0\left(\frac{n}{\tau}, \alpha n\right). \tag{1.9}$$

Remark. The case $\alpha = 1$ of this corollary implies (1.1).

Let us denote by $\zeta_r(s) = \zeta_r(s, r)$ and $B_k^{(r)} = B_k^{(r)}(0)$ $(k \ge 0)$ the generalized Riemann zeta-function and the generalized Bernoulli number respectively. It is readily seen from (1.4) that $B_k^{(r)}(r) = (-1)^k B_k^{(r)}$ for $k = 0, 1, 2, \dots$ Then the case $\alpha = r$ of Theorem 1 reduces to

Corollary 1.2. *For any $\tau > 0$ we have*

$$\sum_{n=1}^{\infty} \frac{1}{(e^{n^2\tau}-1)^r} = \frac{1}{2}\sum_{h=0}^{r} \frac{(-1)^{h+1}B_{2h}B_{r-h}^{(r)}}{(2h)!(r-h)!}\left(\frac{4\pi^2}{\tau}\right)^h + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\zeta_r\left(\frac{1}{2}\right) + S_r(\tau),$$

where

$$S_r(\tau) = \frac{1}{\tau}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{r-1}\frac{B_{r-k-1}^{(r)}}{k!(r-k-1)!(2\pi\tau)^k} \times \left(\tau^2\frac{d}{d\tau}\right)^k\left\{\sqrt{\tau}\sum_{n=1}^{\infty}\frac{1}{n^{k+1/2}} f_k\left(\frac{n}{\tau}\right)\right\}$$

with the notation $f_k(x) = f_k(x, 0)$.

We next proceed to state our second main result. Let λ be a real parameter. The Lerch zeta-function $\phi(\lambda, \alpha, s)$ is defined by

$$\phi(\lambda, \alpha, s) = \sum_{n=0}^{\infty} e^{2\pi i\lambda n}(n+\alpha)^{-s} \tag{1.10}$$

for $\operatorname{Re} s > 1$, and its meromorphic continuation over the whole s-plane (cf. Lerch [Le]). Note that this reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$ if λ is an integer. We set

$$g_0(x,y) = \frac{\sin(a(x)+b_0(y)) - e^{-a(x)}\sin b_0(y)}{\cosh a(x) - \cos a(x)} = \frac{\sinh a(x)\sin b_0(y) + \sin a(x)\cos b_0(y)}{\cosh a(x) - \cos a(x)} - \sin b_0(y) \tag{1.11}$$

with the notations in (1.5). Then our second main result can be stated as

THEOREM 2. *For any $\tau > 0$ and any real α and λ with $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$, we have the formula*

$$\sum_{n=1}^{\infty} \frac{e^{-\alpha n^2 \tau}}{1 - e^{2\pi i \lambda - n^2\tau}} = \varepsilon(\lambda)\frac{\pi^2}{6\tau} + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\phi\left(\lambda, \alpha, \frac{1}{2}\right) - \frac{1}{2}\phi(\lambda, \alpha, 0) + S(\tau; \lambda, \alpha) + iT(\tau; \lambda, \alpha), \tag{1.12}$$

where $\varepsilon(\lambda)$ is 0 or 1 according as $0 < \lambda < 1$ or $\lambda = 1$, and

$$S(\tau; \lambda, \alpha) = \frac{1}{2}\sqrt{\frac{\pi}{2\tau}}\left\{\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+\lambda}} f_0\left(\frac{n+\lambda}{\tau}, \alpha(n+\lambda)\right) + {\sum_{n=0}^{\infty}}' \frac{1}{\sqrt{n+1-\lambda}} f_0\left(\frac{n+1-\lambda}{\tau}, \alpha(n+1-\lambda)\right)\right\} \tag{1.13}$$

and

$$T(\tau; \lambda, \alpha) = \frac{1}{2}\sqrt{\frac{\pi}{2\tau}}\left\{\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+\lambda}} g_0\left(\frac{n+\lambda}{\tau}, \alpha(n+\lambda)\right) - {\sum_{n=0}^{\infty}}' \frac{1}{\sqrt{n+1-\lambda}} g_0\left(\frac{n+1-\lambda}{\tau}, \alpha(n+1-\lambda)\right)\right\} \tag{1.14}$$

with the notations in (1.6) *and* (1.11). *Here the primed summation symbols indicate that the terms corresponding to $n = 0$ are to be omitted if $\lambda = 1$.*

REMARK 1. It is known that $\phi(\lambda, \alpha, 0) = 1/(1 - e^{2\pi i \lambda})$ if λ is not an integer (cf. Apostol [Ap1, p. 164]), while $\phi(\lambda, \alpha, 0) = \zeta(0, \alpha) = 1/2 - \alpha$ if λ is an integer (see (2.2) below).

REMARK 2. The case $\lambda = 1$ of Theorem 2 implies again Corollary 1.1.

Taking the real and imaginary parts of both sides of (1.12) we immediately obtain

COROLLARY 2.1. *Under the same assumptions as in Theorem 2 we have*

$$\sum_{n=1}^{\infty} \frac{e^{-\alpha n^2 \tau}(1 - e^{-n^2\tau}\cos(2\pi\lambda))}{1 - e^{-n^2\tau}\cos(2\pi\lambda) + e^{-2n^2\tau}} = \varepsilon(\lambda)\frac{\pi^2}{6\tau} + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\operatorname{Re}\phi\left(\lambda, \alpha, \frac{1}{2}\right) - \frac{1}{2}\operatorname{Re}\phi(\lambda, \alpha, 0) + S(\tau; \lambda, \alpha)$$

and

$$\sum_{n=1}^{\infty} \frac{e^{-(\alpha+1) n^2 \tau}\sin(2\pi\lambda)}{1 - e^{-n^2\tau}\cos(2\pi\lambda) + e^{-2n^2\tau}} = \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\operatorname{Im}\phi\left(\lambda, \alpha, \frac{1}{2}\right) - \frac{1}{2}\operatorname{Im}\phi(\lambda, \alpha, 0) + T(\tau; \lambda, \alpha).$$

We lastly mention that Corollary 2.1 implies Egami's χ-analogue formula of (1.1). Let q be a positive integer, χ a primitive Dirichlet character modulo q, and $L(s,\chi)$ denote the Dirichlet L-function attached to χ. We use the notations

$$E(\chi)=\begin{cases}1 & \text{if } \chi \text{ is principal,}\\ 0 & \text{otherwise,}\end{cases} \qquad W(\chi)=i^{\delta(\chi)}\sqrt{q}\,g(\chi)^{-1}$$

with $\delta(\chi)=(1-\chi(-1))/2$, where $g(\chi)=\sum_{a=1}^{q}\chi(a)e^{2\pi ia/q}$ denotes Gauss's sum. Then we can show

Corollary 2.2 ([Ega, Theorem]). *For any $\tau>0$ and any primitive Dirichlet character χ modulo q (≥ 1) we have the formula*

$$\begin{aligned}\sum_{r=1}^{q}\chi(r)\sum_{n=1}^{\infty}\frac{e^{-(r/q)n^2\tau}}{1-e^{-n^2\tau}} &= E(\chi)\frac{q\pi^2}{6\tau}-\frac{1}{2}L(0,\chi)+\frac{1}{2}\sqrt{\frac{q\pi}{\tau}}L\Big(\frac{1}{2},\chi\Big)+\frac{1}{2}\sqrt{\frac{q\pi}{\tau}}W(\overline{\chi})\\ &\quad\times\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{\sqrt{n}}\left\{\frac{\sinh(2\pi\sqrt{\pi n/\tau})-\chi(-1)\sin(2\pi\sqrt{\pi n/\tau})}{\cosh(2\pi\sqrt{\pi n/\tau})-\cos(2\pi\sqrt{\pi n/\tau})}-1\right\}.\end{aligned}$$

Remark. Egami stated this formula with the variable $q\tau$ instead of τ.

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We prepare several necessary properties of $\zeta_r(s,\alpha)$ and $\phi(\lambda,\alpha,s)$ in the next section. Theorems 1 and 2 are proved in Sections 3 and 4 respectively. The final section is devoted to showing Corollary 2.2.

2. Properties of $\zeta_r(s,\alpha)$ and $\phi(\lambda,\alpha,s)$. Several basic properties of $\zeta_r(s,\alpha)$ and $\phi(\lambda,\alpha,s)$ are summarized in this section. We first give some functional properties of $\zeta_r(s,\alpha)$ in the following Theorems 3 and 4.

Theorem 3. *For any $\alpha>0$ the zeta-function $\zeta_r(s,\alpha)$, defined by* (1.2), *continues to a meromorphic function over the whole s-plane, having the only singularities at $s=h$ $(h=1,\dots,r)$, which are all simple poles with the residues*

$$\operatorname{Res}_{s=h}\zeta_r(s,\alpha)=\frac{(-1)^{r-h}B_{r-h}^{(r)}(\alpha)}{(h-1)!(r-h)!}\qquad (h=1,\dots,r), \tag{2.1}$$

where $B_k^{(r)}(\alpha)$ is defined by (1.4). *Moreover the particular values of $\zeta_r(s,\alpha)$*

at non-positive integers are given by

$$\zeta_r(-j,\alpha)=\frac{(-1)^r j! B^{(r)}_{r+j}(\alpha)}{(r+j)!}\qquad (j=0,1,2,\ldots). \tag{2.2}$$

Let λ be a real parameter, and $\psi(\lambda,s)$ denote the exponential zeta-function defined by

$$\psi(\lambda,s)=\sum_{n=1}^{\infty}e^{2\pi i\lambda n}n^{-s}\quad (=e^{2\pi i\lambda}\phi(\lambda,1,s))$$

for $\operatorname{Re}s>1$, and its meromorphic continuation over the whole s-plane.

THEOREM 4. *For any real α with $0<\alpha\le r$ the zeta-function $\zeta_r(s,\alpha)$ satisfies the functional equation*

$$\begin{aligned}\zeta_r(1-s,\alpha)=&\sum_{k=0}^{r-1}\frac{(-1)^{r-k-1}B^{(r)}_{r-k-1}(\alpha)}{k!(r-k-1)!}\cdot\frac{\Gamma(s+k)}{(2\pi)^{s+k}}\\ &\times\{e^{-\pi i(s+k)/2}\psi(\alpha,s+k)+e^{\pi i(s+k)/2}\psi(-\alpha,s+k)\}.\end{aligned}\tag{2.3}$$

REMARK 1. Theorem 3 was originally proved in a more general setting by Barnes [Ba]; however, it seems that the functional equation (2.3) has not appeared in the literature. For convenience of the reader we give complete proofs of Theorems 3 and 4.

REMARK 2. The existence of the functional equation for $\zeta_r(s,\alpha)$ was kindly suggested by the late Professor Takayoshi Mitsui as his comment to the author's talk, which was given at the annual meeting of the Mathematical Society of Japan held in the autumn of 1990.

Proof of Theorems 3 and 4. Suppose first that $\operatorname{Re}s>r$. Then multiplying both sides of

$$(n+\alpha)^{-s}=\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-(n+\alpha)x}x^{s-1}\,dx\qquad (n\ge 0)$$

by $(r)_n/n!$ and summing over $n=0,1,2,\ldots$, we obtain

$$\zeta_r(s,\alpha)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{e^{-\alpha x}x^{s-1}}{(1-e^{-x})^r}\,dx. \tag{2.4}$$

Here the inversion of the order of summation and integration is justified by Lebesgue's Lemma together with (1.3) and $(1-e^{-x})^{-r}\sim x^{-r}$ $(x\to+0)$. Next let $\mathcal{C}$ be the contour which starts from infinity, proceeds along the positive real axis to a small δ $(0<\delta<2\pi)$, rounds the origin counterclockwise, and returns to infinity along the positive real axis. Then by a standard argument, (2.4) can be further transformed as

$$\zeta_r(s,\alpha)=\frac{1}{\Gamma(s)(e^{2\pi is}-1)}\int_{\mathcal{C}}\frac{e^{-\alpha z}z^{s-1}}{(1-e^{-z})^r}\,dz, \tag{2.5}$$

where $\arg z$ varies from 0 to 2π round $\mathcal{C}$. Here we used the fact that the integrand in (2.5) is of order $O(|z|^{\sigma-r-1})$ as $z\to 0$ and $O(e^{-\alpha\operatorname{Re}z}|z|^{\sigma-1})$ as $\operatorname{Re}z\to\infty$ respectively. Since the integral in (2.5) converges absolutely for all complex s, it defines an entire function of the variable s. The formula (2.5) therefore provides the meromorphic continuation of $\zeta_r(s,\alpha)$ over the whole s-plane.

We now compute the residues and particular values of $\zeta_r(s,\alpha)$. Let $I(s)$ denote the integral on the right-hand side of (2.5) (without the factor on the left of the integral). Then for any integer h it follows from (1.4) that

$$I(h)=\begin{cases}2\pi i\dfrac{(-1)^{r-h}B^{(r)}_{r-h}(\alpha)}{(r-h)!} & \text{if } h\le r,\\ 0 & \text{if } h>r.\end{cases}$$

The only singularities of $\zeta_r(s,\alpha)$ are thus at $s=h$ $(h=1,\dots,r)$, which are all simple poles with residues (2.1). It is also seen that the particular values of $\zeta_r(s,\alpha)$ at non-positive integers are given by (2.2).

We next proceed to prove Theorem 4. Let N be a positive integer, and $\mathcal{C}_N$ the contour which starts from infinity, proceeds along the positive real axis to $(2N+1)\pi$, encircles the origin counter-clockwise, and returns to infinity along the positive real axis. Let $I_N(s)$ be the integral obtained by replacing the contour $\mathcal{C}$ of $I(s)$ by $\mathcal{C}_N$. Then the difference $I_N(s)-I(s)$ is equal to $2\pi i$ times the sum of the residues of the poles at $z=\pm 2\pi in$ $(n=1,\dots,N)$ of the integrand. It follows from (1.4) that

$$\begin{aligned}\operatorname{Res}_{z=\pm 2\pi in}&\frac{e^{-\alpha z}z^{s-1}}{(1-e^{-z})^r}\\ &=e^{\mp 2\pi i\alpha n}(-1)^{r-1}\sum_{k=0}^{r-1}\frac{(1-s)_k B^{(r)}_{r-k-1}(\alpha)}{k!(r-k-1)!}\,(2\pi e^{(1\mp\frac12)\pi i}n)^{s-k-1}\end{aligned} \tag{2.6}$$

for $n=1,2,\dots$ The order of the integrand in $I_N(s)$ is $O(e^{-\alpha\operatorname{Re}z}|z|^{\sigma-1})$ if $\operatorname{Re}z\ge 0$ and $O(e^{(r-\alpha)\operatorname{Re}z}|z|^{\sigma-1})$ if $\operatorname{Re}z\le 0$ respectively, and hence $\lim_{N\to\infty}I_N(s)=0$ for $0<\alpha\le r$, provided $\operatorname{Re}s<0$. This with (2.6) shows

$$\begin{aligned}-I(s)=2\pi i(-1)^{r-1}&\sum_{k=0}^{r-1}\frac{(1-s)_k B^{(r)}_{r-k-1}(\alpha)}{k!(r-k-1)!}\,(2\pi e^{\pi i})^{s-k-1}\\ &\times\{e^{\pi i(s-k-1)/2}\psi(\alpha,1-s+k)+e^{-\pi i(s-k-1)/2}\psi(-\alpha,1-s+k)\}\end{aligned}$$

for $\operatorname{Re} s < 0$ and $0 < \alpha \le r$. Hence from (2.5), noting

$$(z)_k \Gamma(z) = \Gamma(z+k) \quad (k = 0, 1, 2, \ldots), \tag{2.7}$$

we arrive at the functional equation (2.3) with the variable s instead of $1-s$. The temporary assumption $\operatorname{Re} s < 0$ is removed by the analytic continuations of $\psi(\pm\alpha, s)$. The proof of Theorems 3 and 4 is now complete. ■

An argument similar to the preceding proof yields the functional equation of the Lerch zeta-function $\phi(\lambda, \alpha, s)$, which asserts that, for $0 < \lambda \le 1$ and $0 < \alpha \le 1$,

$$\begin{aligned} \phi(\lambda, \alpha, 1-s) = \frac{\Gamma(s)}{(2\pi)^s} \{ & e^{\pi i s/2 - 2\pi i \alpha\lambda} \phi(-\alpha, \lambda, s) \\ & + e^{-\pi i s/2 + 2\pi i \alpha(1-\lambda)} \phi(\alpha, 1-\lambda, s)\} \end{aligned} \tag{2.8}$$

(cf. Lerch [Le]). Here the latter Lerch zeta-function $\phi(\alpha, 1-\lambda, s)$ on the right-hand side is to be regarded as $\psi(\alpha, s)$ if $\lambda = 1$.

In the remainder of this section the vertical behaviours of $\phi(\lambda, \alpha, s)$ and $\zeta_r(s, \alpha)$ are considered. We first show

LEMMA 1. *Define*

$$\mu(\sigma; \lambda, \alpha) = \limsup_{t \to \pm\infty} \frac{\log |\phi(\lambda, \alpha, \sigma + it)|}{\log |t|}.$$

Then for any α *and* λ *with* $0 < \alpha \le 1$ *and* $0 < \lambda \le 1$ *we have the bounds*

$$\mu(\sigma; \lambda, \alpha) \le \begin{cases} 1/2 - \sigma & \text{if } \sigma \le 0, \\ (1-\sigma)/2 & \text{if } 0 \le \sigma \le 1, \\ 0 & \text{if } \sigma \ge 1. \end{cases} \tag{2.9}$$

Proof. The upper bounds for the first and third cases except the points $\sigma = 0$ and 1 follow respectively from the functional equation (2.8) and the series representation (1.10). Since $\mu(\sigma; \lambda, \alpha)$ is continuous with respect to σ (cf. Titchmarsh [Ti1, p. 299, 9.41 (1) below]), it is seen that the bounds in (2.9) are also valid for $\sigma = 0$ and 1. To treat the remaining range, we use the formula

$$\phi(\lambda, \alpha+1, s) - \phi(\lambda, \alpha, s) = -s \int_0^1 \phi(\lambda, \alpha + x, s+1)\, dx, \tag{2.10}$$

which is obtained for $\operatorname{Re} s > 1$ by integrating both sides of (1.10), and then for all s by analytic continuation. This and the relation

$$e^{2\pi i\lambda} \phi(\lambda, \alpha+1, s) = \phi(\lambda, \alpha, s) - \alpha^{-s}$$

show that $\phi(\lambda, \alpha, s)$ is of finite order (cf. [Ti1, p. 298, 9.4]) in the strip $0 \le \sigma \le 1$ if $0 < \lambda < 1$. When $\lambda = 1$ we integrate by parts the right-hand side of (2.10) to obtain

$$-\alpha^{-s} = -s\zeta(s+1,\alpha) + s(s+1)\int_0^1 (1-x)\zeta(s+2,\alpha+x)\,dx,$$

which shows that $\zeta(s,\alpha)$ is of finite order for $0 \le \sigma \le 1$. The convexity principle (cf. [Ti1, p. 299, 9.41 (1)]) therefore yields the bound for the second case of Lemma 1. ∎

We next show

LEMMA 2. *Define*

$$\nu_r(\sigma;\alpha) = \limsup_{t\to\pm\infty} \frac{\log|\zeta_r(\sigma+it,\alpha)|}{\log|t|}. \tag{2.11}$$

Then for any positive integer r and any $\alpha > 0$ we have the bounds

$$\nu_r(\sigma;\alpha) \le \begin{cases} r-1/2-\sigma & \text{if } \sigma \le 0,\\ (1-1/(2r))(r-\sigma) & \text{if } 0 \le \sigma \le r,\\ 0 & \text{if } \sigma \ge r.\end{cases}$$

Proof. The upper bounds for the first and third cases follow respectively from (2.3) and (1.2), together with the continuity of $\nu_r(\sigma;\alpha)$. It is seen from (2.3) and Lemma 1 that $\zeta_r(s,\alpha)$ is of finite order for $0 \le \sigma \le r$ (see the definition of $\psi(\lambda,s)$). The convexity principle therefore yields the bound for the second case of Lemma 2. ∎

3. Proof of Theorem 1. We set

$$\Theta_r(\tau;\alpha) = \sum_{n=1}^{\infty} \frac{e^{-\alpha n^2\tau}}{(1-e^{-n^2\tau})^r}$$

for $\tau > 0$, a positive integer r and $0 < \alpha \le r$. Then this is transformed by applying the Mellin inversion formula

$$e^{-(\alpha+m)n^2\tau} = \frac{1}{2\pi i}\int_{(\sigma_0)} \Gamma(s)((\alpha+m)n^2\tau)^{-s}\,ds \tag{3.1}$$

for $m \ge 0$ and $n > 0$, where σ_0 is a constant satisfying $\sigma_0 > r$ and (σ_0) denotes the vertical straight line from $\sigma_0 - i\infty$ to $\sigma_0 + i\infty$. Multiplying both sides of (3.1) by $(r)_m/m!$ and summing up with $m = 0, 1, 2, \ldots$, we get

$$\frac{e^{-\alpha n^2\tau}}{(1-e^{-n^2\tau})^r} = \frac{1}{2\pi i}\int_{(\sigma_0)} \Gamma(s)\zeta_r(s,\alpha)(n^2\tau)^{-s}\,ds,$$

where the interchange of the order of summation and integration is justified by absolute convergence (see (1.3)). We further sum up both sides of this equality for $n = 1, 2, \ldots$ to obtain the formula

$$\Theta_r(\tau;\alpha) = \frac{1}{2\pi i} \int_{(\sigma_0)} \Gamma(s)\zeta_r(s,\alpha)\zeta(2s)\tau^{-s}\, ds, \tag{3.2}$$

which is the key to the following derivation.

Let σ_1 be a constant satisfying $\sigma_1 < 0$ and ε a small positive number. Then we can move the path of integration in (3.2) from (σ_0) to (σ_1), since the integrand is of order $O(e^{-\pi|t|/2}|t|^{\sigma-1/2+\nu_r(\sigma;\alpha)+\nu_1(2\sigma;1)+\varepsilon})$ as $t \to \pm\infty$ (see Lemma 2). Collecting the residues of the poles at $s = h$ $(h = 0, 1, \ldots, r)$ and $1/2$ of the integrand, noting (2.1) and

$$\zeta(2h) = \frac{(-1)^{h+1}(2\pi)^{2h}}{2(2h)!} B_{2h} \qquad (h = 0, 1, 2, \ldots) \tag{3.3}$$

(cf. [Er1, pp. 34–35, 1.12 (18) and (21)]), we obtain the first assertion (1.7) of Theorem 1 with

$$S_r(\tau;\alpha) = \frac{1}{2\pi i} \int_{(\sigma_1)} \Gamma(s)\zeta_r(s,\alpha)\zeta(2s)\tau^{-s}\, ds. \tag{3.4}$$

The second assertion (1.8) can be derived as follows. We change the variable s into $1/2 - s/2$ in the integral on the right-hand side of (3.4), and then substitute the functional equation for $\zeta_r(1/2 - s/2, \alpha)$ (see (2.3)) and

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s) \tag{3.5}$$

(cf. [Ti2, p. 16, Chapter II, (2.1.8)]). Noting the facts (2.7) and

$$\Gamma(1/2 - s/2)\Gamma(1/2 + s/2) = \frac{\pi}{\cos(\pi s/2)}, \tag{3.6}$$

we obtain

$$S_r(\tau;\alpha) = \sum_{k=0}^{r-1} \frac{(-1)^{r-k-1} B_{r-k-1}^{(r)}(\alpha)}{k!(r-k-1)!} X_k, \tag{3.7}$$

where

$$\begin{aligned} X_k = {} & \frac{(2\pi)^{1/2-k}}{4\pi i\sqrt{\tau}} \\ & \times \int_{(\sigma_2)} (s/2 + 1/2)_k \Gamma(s)\zeta(s)\{e^{-\pi i(s/2+1/2+k)/2}\psi(\alpha, s/2 + 1/2 + k) \\ & + e^{\pi i(s/2+1/2+k)/2}\psi(-\alpha, s/2 + 1/2 + k)\}((2\pi)^{3/2}\tau^{-1/2})^{-s}\, ds \end{aligned}$$

with the constant $\sigma_2 = 1 - 2\sigma_1$ (> 1). The integral on the right-hand side is further modified by substituting the series representations $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$ and $\psi(\pm\alpha, s/2 + 1/2 + k) = \sum_{n=1}^{\infty} e^{\pm 2\pi i n\alpha} n^{-s/2-1/2-k}$ (both of which converge absolutely on the line $\operatorname{Re} s = \sigma_2$), and by changing the order of summation and integration. Hence

$$(3.8)\quad X_k = \frac{(2\pi)^{1/2-k}}{2\sqrt{\tau}} \times \Big\{ e^{-\pi i(k+1/2)/2} \sum_{m,n=1}^{\infty} \frac{e^{2\pi i n\alpha}}{n^{k+1/2}} F_k((2\pi)^{3/2} e^{\pi i/4} m(n/\tau)^{1/2}) + e^{\pi i(k+1/2)/2} \sum_{m,n=1}^{\infty} \frac{e^{-2\pi i n\alpha}}{n^{k+1/2}} F_k((2\pi)^{3/2} e^{-\pi i/4} m(n/\tau)^{1/2}) \Big\}$$

for $k = 0, 1, \dots, r-1$, where

$$(3.9)\qquad F_k(z) = \frac{1}{2\pi i} \int_{(\sigma_2)} (s/2+1/2)_k \Gamma(s) z^{-s}\, ds \qquad (|\arg z| < \pi/2).$$

A simple expression for $F_k(z)$ is given by

LEMMA 3. *For any complex z, we have*

$$F_k(z) = \left(-\frac{1}{2}\right)^k z^{2k+1} \left(\frac{d}{zdz}\right)^k \frac{e^{-z}}{z} \qquad (k = 0, 1, 2, \dots).$$

Proof. By substituting the duplication formula

$$\Gamma(s) = 2^{s-1}\pi^{-1/2}\Gamma(s/2)\Gamma(s/2+1/2)$$

(cf. [Er1, p. 5, 1.2 (15)]) into the integral in (3.9) while noting (2.7), it is seen that

$$F_k(z) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{2\pi i} \int_{(\sigma_2)} \Gamma(s/2)\Gamma(s/2+1/2+k)(z/2)^{-s}\, ds = \frac{2}{\sqrt{\pi}} \left(\frac{1}{2}z\right)^{k+1/2} K_{k+1/2}(z)$$

for $|\arg z| < \pi/2$, where $K_\nu(z)$ denotes the modified Bessel function of the third kind (cf. [Er2, p. 5, 7.2.2 (13)]). Here the second equality follows from the fact that the pair

$$x^\nu K_\nu(x), \qquad 2^{s+\nu-2}\Gamma(s/2)\Gamma(s/2+\nu) \qquad (\operatorname{Re} s > \max(0, 2\nu))$$

is a pair of Mellin transforms (see [Ti3, p. 197, Chapter VII, (7.9.12)]). The assertion is then a consequence of the formula

$$K_{k+1/2}(z) = (-1)^k \sqrt{\frac{\pi}{2z}} z^{k+1} \left(\frac{d}{zdz}\right)^k \frac{e^{-z}}{z} \qquad (k = 0, 1, 2, \dots)$$

for any complex z (cf. [Er2, p. 10, 7.2.6 (43)]). The proof of Lemma 3 is complete. ∎

We write $c^{\pm} = (2\pi)^{3/2}e^{\pm\pi i/4}$ for simplicity. Then for the variable $z = c^{\pm}mn^{1/2}\tau^{-1/2}$ it is seen that

$$\frac{d}{zdz} = -2(c^{\pm}mn^{1/2})^{-2}\tau^2\frac{d}{d\tau},$$

and hence from Lemma 3,

$$F_k(c^{\pm}mn^{1/2}\tau^{-1/2}) = \tau^{-k-1/2}\left(\tau^2\frac{d}{d\tau}\right)^k\sqrt{\tau}\exp(-c^{\pm}mn^{1/2}\tau^{-1/2})$$

for $k = 0, 1, 2, \ldots$ and $m, n = 1, 2, \ldots$ This equality allows us to sum up the infinite sums on the right-hand side of (3.8) for $m = 1, 2, \ldots$, and therefore

$$X_k = \frac{1}{2}(2\pi)^{1/2-k}\tau^{-k-1}\left(\tau^2\frac{d}{d\tau}\right)^k\left\{\sqrt{\tau}\sum_{n=1}^{\infty}\frac{1}{n^{k+1/2}}(X_{k,n}^{-} + X_{k,n}^{+})\right\},$$

where

$$\begin{aligned} X_{k,n}^{\pm} &= \frac{\exp\bigl(-c^{\pm}n^{1/2}\tau^{-1/2} \mp \frac{1}{2}\pi i\bigl(k+\frac{1}{2}\bigr) \pm 2\pi i\alpha n\bigr)}{1-\exp(-c^{\pm}n^{1/2}\tau^{-1/2})} \\ &= \frac{\exp(-2a(n/\tau)e^{\pm\pi i/4} \mp b_k(\alpha n))}{1-\exp(-2a(n/\tau)e^{\pm\pi i/4})} \end{aligned}$$

with the notations in (1.5). Here the inversion of the order of summation and differentiation is ensured by absolute convergence. Noting the fact

$$X_{k,n}^{\pm} = \frac{\exp(-a(n/\tau)e^{\pm\pi i/4} \mp b_k(\alpha n))}{2\sinh(a(n/\tau)e^{\pm\pi i/4})}$$

and using the identity

$$2\sinh z\sinh w = \cosh(z+w) - \cosh(z-w),$$

we find that $X_{k,n}^{-} + X_{k,n}^{+}$ is equal to the first expression for $f_k(n/\tau, \alpha n)$ in (1.6). The assertion (1.8) of Theorem 1 is thus deduced from (3.7). The proof of Theorem 1 is complete. ∎

4. Proof of Theorem 2. The skeleton of the proof of Theorem 2 is the same as that of the preceding proof, so the details are omitted in what follows.

We set

$$\Theta(\tau;\lambda,\alpha) = \sum_{n=1}^{\infty}\frac{e^{-\alpha n^2\tau}}{1-e^{2\pi i\lambda - n^2\tau}}$$

for $\tau > 0$, $0 < \alpha \le 1$ and $0 < \lambda \le 1$. The Mellin transform formula for $\Theta(\tau;\lambda,\alpha)$ is obtained by multiplying both sides of (3.1) by $e^{2\pi i\lambda m}$, summing up over $m = 0, 1, 2, \ldots$, and then summing up the resulting expression for

$n = 1, 2, \ldots$ This yields

$$\Theta(\tau;\lambda,\alpha) = \frac{1}{2\pi i}\int_{(\sigma_0)} \Gamma(s)\phi(\lambda,\alpha,s)\zeta(2s)\tau^{-s}\,ds, \tag{4.1}$$

where σ_0 is a constant satisfying $\sigma_0 > 1$. Let σ_1 be a constant satisfying $\sigma_1 < 0$ and ε a small positive number. Then we can move the path of integration in (4.1) from (σ_0) to (σ_1), since the integrand is of order $O(e^{-\pi|t|/2}|t|^{\sigma-1/2+\mu(\sigma;\lambda,\alpha)+\mu(2\sigma;1,1)+\varepsilon})$ as $t \to \pm\infty$ (see Lemma 1). Collecting the residues of the poles at $s = 1, 1/2$ and 0 of the integrand, noting (3.3) for $h = 0, 1$ and $\operatorname{Res}_{s=1} \phi(\lambda,\alpha,s) = \varepsilon(\lambda)$, we obtain

$$\Theta(\tau;\lambda,\alpha) = \varepsilon(\lambda)\frac{\pi^2}{6\tau} + \frac{1}{2}\sqrt{\frac{\pi}{\tau}}\phi\Big(\lambda,\alpha,\frac{1}{2}\Big) - \frac{1}{2}\phi(\lambda,\alpha,0) + U(\tau;\lambda,\alpha),$$

where

$$U(\tau;\lambda,\alpha) = \frac{1}{2\pi i}\int_{(\sigma_1)} \Gamma(s)\phi(\lambda,\alpha,s)\zeta(2s)\tau^{-s}\,ds. \tag{4.2}$$

This establishes the first assertion (1.12) of Theorem 2.

We next proceed to prove the second assertion (1.13) and (1.14). The integral in (4.2) is transformed by changing the variable s into $1/2 - s/2$, applying the functional equations (2.8) and (3.5) while noting (3.6), and hence

$$\begin{aligned} &U(\tau;\lambda,\alpha) \\ &\quad = \frac{1}{2\pi i}\sqrt{\frac{\pi}{2\tau}}\int_{(\sigma_2)} \Gamma(s)\zeta(s)\{e^{\pi i(s+1)/4-2\pi i\alpha\lambda}\phi(-\alpha,\lambda,s/2+1/2) \\ &\qquad + e^{-\pi i(s+1)/4+2\pi i\alpha(1-\lambda)}\phi(\alpha,1-\lambda,s/2+1/2)\}((2\pi)^{3/2}\tau^{-1/2})^{-s}\,ds, \end{aligned}$$

where $\sigma_2 = 1 - 2\sigma_1$ (> 1). We substitute the series representations for $\zeta(s)$, $\phi(-\alpha,\lambda,s/2+1/2)$ and $\phi(\alpha,1-\lambda,s/2+1/2)$ (all of which converge absolutely on the line $\operatorname{Re} s = \sigma_2$) into the integrand above, change the order of summation and integration, and then evaluate the resulting expression for each term by applying Lemma 3 with $k = 0$. This yields

$$\begin{aligned} &U(\tau;\lambda,\alpha) \\ &\quad = \frac{1}{2}\sqrt{\frac{\pi}{2\tau}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+\lambda}}\cdot\frac{\exp\{-2a\big(\frac{n+\lambda}{\tau}\big)e^{-\pi i/4} + b_0(\alpha(n+\lambda))\}}{1-\exp\{-2a\big(\frac{n+\lambda}{\tau}\big)e^{-\pi i/4}\}} \\ &\qquad + \frac{1}{2}\sqrt{\frac{\pi}{2\tau}}{\sum_{n=0}^{\infty}}'\frac{1}{\sqrt{n+1-\lambda}}\cdot\frac{\exp\{-2a\big(\frac{n+1-\lambda}{\tau}\big)e^{\pi i/4} - b_0(\alpha(n+1-\lambda))\}}{1-\exp\{-2a\big(\frac{n+1-\lambda}{\tau}\big)e^{\pi i/4}\}} \end{aligned}$$

with the notations of (1.5). The second assertion (1.13) and (1.14) of Theorem 2 is thus deduced from this formula and the relations

$$\frac{\exp(-2a(x)e^{\mp\pi i/4}+b_0(y))}{1-\exp(-2a(x)e^{\mp\pi i/4})}=f_0(x,y)\pm ig_0(x,y).$$

The proof of Theorem 2 is complete. ∎

5. Proof of Corollary 2.2. To prove Corollary 2.2 we first note the equality

$$\sum_{r=1}^{q}\chi(r)e^{2\pi inr/q}=\overline{\chi}(n)g(\chi)$$

for any integer n and any primitive character χ modulo q (cf. [Ap2, p. 171, 8.10, Theorem 8.19]). This immediately implies

$$\sum_{r=1}^{q}\chi(r)\cos b_0\left(\frac{nr}{q}\right)=\frac{i^{\delta(\chi)}}{\sqrt{2}}\overline{\chi}(-n)g(\chi),$$

$$\sum_{r=1}^{q}\chi(r)\sin b_0\left(\frac{nr}{q}\right)=\frac{i^{\delta(\chi)}}{\sqrt{2}}\overline{\chi}(n)g(\chi),$$

which gives

$$\sum_{r=1}^{q}\chi(r)f\left(\frac{n}{\tau},\frac{nr}{q}\right)=\frac{i^{\delta(\chi)}}{\sqrt{2}}g(\chi)\overline{\chi}(-n)\left\{\frac{\sinh a(n/\tau)-\chi(-1)\sin a(n/\tau)}{\cosh a(n/\tau)-\cos a(n/\tau)}-1\right\}.$$

We use this equality and the well-known relation

$$L(s,\chi)=q^{-s}\sum_{r=1}^{q}\chi(r)\zeta(s,r/q)$$

(cf. [Ap2, p. 249, 12.1]) to sum up both sides of (1.9) with $\alpha=r/q$ ($r=1,\ldots,q$) multiplied by $\chi(r)$. This, together with the facts $\zeta(0,\alpha)=1/2-\alpha$ and $\sqrt{q}\,W(\overline{\chi})=i^{\delta(\chi)}\chi(-1)g(\chi)$, establishes Corollary 2.2. ∎

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