Heine's basic transform and
a permutation group for q-harmonic series

by

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1. Introduction. More than 150 years ago, E. Heine [He1], [He2] considered the series

$$
{}_2\phi_1\!\left(\begin{array}{c} q^a,\, q^b \\ q^c \end{array}\;\middle|\; q, z\right) = 1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}\, z
$$
$$
+ \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}\, z^2 + \dots,
$$
$$
|q| < 1, \quad |z| < 1, \quad c \neq 0, -1, -2, \dots,
$$

and proved several results for it. In particular, he obtained the transformation formula

$$
(1) \qquad {}_2\phi_1\!\left(\begin{array}{c} q^a,\, q^b \\ q^c \end{array}\;\middle|\; q, z\right) = \frac{(q^a;q)_\infty (q^b z;q)_\infty}{(q^c;q)_\infty (z;q)_\infty} \cdot {}_2\phi_1\!\left(\begin{array}{c} q^{c-a},\; z \\ q^b z \end{array}\;\middle|\; q, q^a\right),
$$

where $(x;q)_\infty := \prod_{n=1}^\infty (1 - xq^{n-1})$. Today generalized Heine series become an actively investigated part of modern mathematics and a lot of papers and monographs (see, e.g., [Ex], [Fi], [GR]) are devoted to their study. The aim of this note is to make use of Heine's transform (1) to deduce a sharp irrationality measure for the q-harmonic series

$$
(2) \qquad h_p(1) := \sum_{n=1}^\infty \frac{1}{p^n - 1} = \sum_{n=1}^\infty \frac{q^n}{1 - q^n}
$$
$$
= \frac{q}{1-q} \cdot {}_2\phi_1\!\left(\begin{array}{c} q,\, q \\ q^2 \end{array}\;\middle|\; q, q\right), \qquad p = 1/q \in \mathbb{Z} \setminus \{0, \pm 1\}.
$$

As is easily seen, the series $h_p(1)$ is irrational as a function of p. The irrationality of the number $h_2(1)$ (i.e., $q = 1/2$ in (2)) was first proved by P. Erdős [Er], who also posed the problem of extending his result to an arbitrary integer $p \in \mathbb{Z} \setminus \{0, \pm 1\}$. This problem (even in more general

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settings) was solved by J.-P. Bézivin [Bé] and in a quantitative form by
P. Borwein [Bo]. A sharp irrationality measure

$$(3) \qquad \mu(h_p(1)) \leq \frac{2\pi^2}{\pi^2 - 2} = 2.50828476\ldots$$

was obtained by P. Bundschuh and K. Väänänen [BV] with the help of
explicit Padé approximations to the q-logarithm function and a certain q-
arithmetic observation. (Here $\mu = \mu(\alpha)$ denotes the irrationality exponent
of an irrational number α that is the least possible exponent such that
for any $\varepsilon > 0$ the inequality $|\alpha - a/b| \leq b^{-(\mu+\varepsilon)}$ has only finitely many
solutions in integers a, b.) The works [BV], [MV], and [Ass] contain estimates
for irrationality exponents of the q-logarithm values. Finally, introducing
a q-arithmetic approach in [Zu1] resulted in a slight improvement of the
estimate (3) and of the estimate in [Ass] for a q-analogue of $\log 2$ (the result
for $\mu(h_p(1))$ in [Zu1] is wrong due to a computational error; see the remark
at the end of Section 3 below).

The "ordinary" arithmetic approach occurs as a part of the group-
structure approach proposed by G. Rhin and C. Viola in [RV1], [RV2] for
obtaining quantitative results for the values $\zeta(2)$ and $\zeta(3)$ of Riemann's
zeta function. Recently, the author [Zu2] extended the method of [RV1] to a
suitable q-analogue of $\zeta(2)$. The permutation group in [RV1], [Zu2] is rather
rich to get nice estimates for irrationality exponents in both ordinary and
q-(basic) cases. A simpler group (of order 12) for the q-harmonic series that
appears below also leads to a quantitative result.

THEOREM 1. *The irrationality exponent of $h_p(1)$ satisfies the estimate*

$$\mu(h_p(1)) \leq 2.46497868\ldots$$

However, the group has no ordinary analogue, hence it has not appeared
before in an arithmetic study. As pointed out by the referee, it is worth
remarking that the estimate in Theorem 1 is uniform in p.

2. q-Basis. Throughout the paper $p = 1/q \in \mathbb{Z} \setminus \{0, \pm 1\}$. As usual (see,
e.g., [GR]), define the *shifted q-factorial*

$$(a; q)_0 = 1, \qquad (a; q)_n := \prod_{\nu=1}^{n} (1 - aq^{\nu-1}) \quad \text{for } n = 1, 2, \ldots,$$

and *Jackson's q-gamma function*

$$\Gamma_q(t) := \frac{(q; q)_\infty}{(q^t; q)_\infty} (1 - q)^{1-t}.$$

Then q-extensions of factorial and binomial coefficients read as follows:

$$[n]_q! := \Gamma_q(n + 1) = \frac{(q; q)_n}{(1 - q)^n} = p^{-n(n-1)/2} [n]_p!,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k \cdot (q;q)_{n-k}} = \frac{[n]_q!}{[k]_q! \, [n-k]_q!} = p^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_p,$$

where $k = 0, 1, \ldots, n$ and $n = 0, 1, 2, \ldots$

Since $p^n - 1 = \prod_{l|n} \Phi_l(p)$, where

$$(4) \qquad \Phi_l(p) := \prod_{\substack{k=1 \\ (k,l)=1}}^{l} (p - e^{2\pi ik/l}) \in \mathbb{Z}[p], \quad l = 1, 2, 3, \ldots,$$

are (irreducible over $\mathbb{Z}$) *cyclotomic* polynomials, we deduce the following claims:

- the polynomials (4) are the only irreducible divisors of the polynomial

$$[n]_p! = \prod_{\nu=1}^{n} \frac{p^\nu - 1}{p - 1} \in \mathbb{Z}[p];$$

moreover,

$$(5) \qquad \mathrm{ord}_{\Phi_1(p)} [n]_p! = 0, \quad \mathrm{ord}_{\Phi_l(p)} [n]_p! = \left\lfloor \frac{n}{l} \right\rfloor, \quad l = 2, 3, 4, \ldots,$$

where $\lfloor \cdot \rfloor$ denotes the integral part of a number;

- the polynomial $D_n(p) := \prod_{l=1}^{n} \Phi_l(p)$ is the least common multiple of the polynomials $p^k - 1$, $k = 1, \ldots, n$.

In the above notation, the q-arithmetic approach is characterized by the following assertions.

LEMMA 1 ([BV, Section 2]; [Ass, Lemma 2]). *We have*

$$\lim_{n \to \infty} \frac{\log |D_n(p)|}{n^2 \log |p|} = \frac{3}{\pi^2}.$$

LEMMA 2 ([Zu1, Lemma 1]). *For each demi-interval $[u, v) \subset (0, 1)$ with $u, v \in \mathbb{Q}$,*

$$\lim_{n \to \infty} \frac{1}{n^2 \log |p|} \sum_{l:\{n/l\} \in [u,v)} \log |\Phi_l(p)| = \frac{3}{\pi^2} (\psi'(u) - \psi'(v))$$

$$= \frac{3}{\pi^2} \int_u^v d(-\psi'(x)),$$

where $\{a\} = a - \lfloor a \rfloor$ and $\psi(x)$ is the logarithmic derivative of Euler's gamma function.

3. Linear forms involving q-harmonic series. Let a_0, a_1, a_2, and b be positive integers satisfying the condition

$$(6) \qquad a_1 + a_2 \leq b.$$

Consider the rational function

$$R(T) = \frac{(qT;q)_{a_1-1}}{(q;q)_{a_1-1}} \cdot \frac{(q;q)_{b-a_2-1}}{(q^{a_2}T;q)_{b-a_2-1}} \cdot T^{a_0},$$

so that

$$(7) \qquad R(q^t) = \frac{\Gamma_q(b-a_2)}{(1-q)\Gamma_q(a_1)} \cdot \frac{\Gamma_q(t+a_1)\,\Gamma_q(t+a_2)}{\Gamma_q(t+1)\,\Gamma_q(t+b)} \cdot q^{a_0 t}.$$

Denote by $a_1^* \le a_2^*$ the ordered version of the set a_1, a_2, i.e., $\{a_1^*, a_2^*\} = \{a_1, a_2\}$. Condition (6) implies that $R(T)T^{-a_0} = O(T^{-1})$ as $T \to \infty$, hence we can write the partial-fraction decomposition

$$R(T) = T^{a_0} \sum_{k=a_2^*}^{b-1} \frac{A_k}{1-q^k T},$$

where

$$(8) \quad A_k = (R(T)T^{-a_0}(1-q^k T))|_{T=q^{-k}}$$

$$= (-1)^{a_1+a_2+k+1} q^{(a_1-2k)(a_1-1)/2+(k-a_2)(k-a_2+1)/2}$$

$$\times \begin{bmatrix} k-1 \\ a_1-1 \end{bmatrix}_q \begin{bmatrix} b-a_2-1 \\ b-k-1 \end{bmatrix}_q$$

$$= (-1)^{a_1+a_2+k+1} p^{a_1(a_1-1)/2-(b-a_2)(b-a_2-1)/2+(b-k)(b-k-1)/2}$$

$$\times \begin{bmatrix} k-1 \\ a_1-1 \end{bmatrix}_p \begin{bmatrix} b-a_2-1 \\ b-k-1 \end{bmatrix}_p.$$

Furthermore, consider the series

$$(9) \quad F(\boldsymbol{a};b) = F(a_0, a_1, a_2; b) := \sum_{t=0}^{\infty} R(q^t) = \sum_{t=1-a_1^*}^{\infty} R(q^t)$$

$$= \sum_{k=a_2^*}^{b-1} A_k \sum_{t=1-a_1^*}^{\infty} \frac{q^{a_0 t}}{1-q^{k+t}} = \sum_{k=a_2^*}^{b-1} A_k q^{-a_0 k} \sum_{l=k-a_1^*+1}^{\infty} \frac{q^{a_0 l}}{1-q^l}$$

$$= \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k} \left(\sum_{l=1}^{\infty} \frac{q^l}{1-q^l} - \sum_{l=1}^{k-a_1^*} \frac{q^l}{1-q^l} - \sum_{l=k-a_1^*+1}^{\infty} \frac{q^l - q^{a_0 l}}{1-q^l} \right)$$

$$= A(p)h_p(1) - B_1(p) - B_2(p),$$

where

$$(10) \qquad A(p) = \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k}, \qquad B_1(p) = \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k} \sum_{l=1}^{k-a_1^*} \frac{1}{p^l - 1},$$

and

$$(11) \qquad B_2(p) = \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k} \sum_{l=k-a_1^*+1}^{\infty} \sum_{j=1}^{a_0-1} q^{jl}$$

$$= \sum_{k=a_2^*}^{b-1} A_k p^{a_0 k} \sum_{j=1}^{a_0-1} \frac{(q^j)^{k-a_1^*+1}}{1-q^j}$$

$$= \sum_{k=a_2^*}^{b-1} \sum_{j=1}^{a_0-1} \frac{A_k p^{a_0 k - j(k-a_1^*)}}{p^j - 1}.$$

Now, note that $a_0 k \geq a_0 a_2^*$ and $a_0 k - j(k - a_1^*) > a_0 k - a_0(k - a_1^*) = a_0 a_1^*$ for $a_2^* \leq k < b$, $1 \leq j < a_0$. Hence, taking

$$M_0 = a_0 a_1^* + a_1(a_1 - 1)/2 - (b - a_2)(b - a_2 - 1)/2,$$

from (8) we deduce the inclusions

$$p^{-M_0} A_k p^{a_0 k} \in \mathbb{Z}[p], \qquad p^{-M_0} A_k p^{a_0 k - j(k-a_1^*)} \in \mathbb{Z}[p].$$

Applying the results of Section 2 to formulae (10) and (11) we obtain

$$p^{-M_0} A(p) \in \mathbb{Z}[p],$$

$$p^{-M_0} D_{b-a_1^*-1}(p) B_1(p) \in \mathbb{Z}[p], \qquad p^{-M_0} D_{a_0-1}(p) B_2(p) \in \mathbb{Z}[p].$$

Therefore representation (9) yields the following assertion.

LEMMA 3. *With a suitable integer*

$$(12) \qquad M = M(\boldsymbol{a}; b) \geq a_0 a_1^* + \frac{a_1(a_1 - 1)}{2} - \frac{(b - a_2)(b - a_2 - 1)}{2}$$

we have the inclusion

$$p^{-M} D_{\max\{a_0-1, b-a_1^*-1\}}(p) F(\boldsymbol{a}; b) \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p].$$

Thanks to (7), the quantity $F(\boldsymbol{a}; b)$ can be identified with the Heine series:

$$(13) \qquad F(a_0, a_1, a_2; b) = \frac{\Gamma_q(a_2) \Gamma_q(b - a_2)}{(1 - q) \Gamma_q(b)} \cdot {}_2\phi_1\left(\begin{matrix} q^{a_1}, q^{a_2} \\ q^b \end{matrix} \, \middle| \, q, q^{a_0} \right).$$

The lower estimate in (12) for $M(\boldsymbol{a}; b)$ is rather rough and we require the following sharp form of it.

LEMMA 4. *Suppose that*

$$(14) \qquad a_1 \leq a_2, \qquad a_1 + a_2 \leq b \leq a_0 + a_2.$$

Then

$$(15) \qquad p^{-M} D_{\max\{a_0-1, b-a_1-1\}}(p) F(\boldsymbol{a}; b) \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p],$$

where

$$(16) \qquad M = M(\boldsymbol{a}; b) := \frac{a_1(a_1 - 1)}{2} + a_0 a_1 + (b - a_2)(a_2 - a_1).$$

In addition, for any $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, the estimates

(17) $\qquad |F(\boldsymbol{a}; b)| = |p|^{O(b)}, \qquad |A(p)| \leq |p|^{(a_0 + a_1 + a_2)b - (a_1^2 + a_2^2 + b^2)/2 + O(b)}$

hold with some absolute constant in $O(b)$.

Proof. The first condition in (14) allows us to write $a_1^* = a_1$, $a_2^* = a_2$ in (10), (11), and to apply Lemma 3 from [Zu1] to the quantity $B_2(p)$ after interchanging the summations in (11):

$$
B_2(p) = p^{a_1(a_1-1)/2 + a_0 a_1 + (b-a_2)(a_2-a_1)} \sum_{j=1}^{a_0-1} \frac{1}{p^j - 1}
$$
$$
\times \sum_{l=0}^{a_2-1} (-1)^l p^{(a_1-l)(a_1-l-1)/2}
$$
$$
\times \begin{bmatrix} b - a_2 + l - 1 \\ a_1 - 1 \end{bmatrix}_p \begin{bmatrix} a_2 - 1 \\ l \end{bmatrix}_p (p^{a_0 - j - 1}; p^{-1})_{b - a_1 - a_2 + l}.
$$

From this formula we deduce that

$$
p^{-M_2} D_{a_0-1}(p) B_2(p) \in \mathbb{Z}[p],
$$

where $M_2 = a_1(a_1 - 1)/2 + a_0 a_1 + (b - a_2)(a_2 - a_1)$, while formulae (8), (10) and the inequality

$$
\frac{(b-k)(b-k-1)}{2} + a_0 k \geq \frac{(b-a_2)(b-a_2-1)}{2} + a_0 a_2, \qquad k \geq a_2 \geq b - a_0
$$

yield

$$
p^{-M_1}(p) A(p) \in \mathbb{Z}[p], \qquad p^{-M_1} D_{b-a_1-1}(p) B_1(p) \in \mathbb{Z}[p],
$$

where $M_1 = a_1(a_1 - 1)/2 + a_0 a_2$. Thus using the fact that $\min\{M_1, M_2\} = M_2$ under the hypothesis (14), we arrive at (15).

To prove the second part of the lemma, we adopt the construction and results of [Zu1]. There we consider the family of series

(18) $\quad I(\boldsymbol{n}; m) = I(n_0, n_1, n_2; m)$

$$
= \frac{(q; q)_{n_2}}{(q; q)_{n_0}} q^{-n_0(n_1 - n_0)}
$$
$$
\times \sum_{s=n_1+1}^{\infty} \frac{(1 - q^{n_1 - n_0 + 1 - s}) \dots (1 - q^{n_1 - s})}{(q^{-s} - 1)(q^{-s} - q) \dots (q^{-s} - q^{n_2})} q^{(m-1)s}
$$
$$
= (-1)^{n_0} q^{n_0(n_0+1)/2 + (n_1+1)(n_2 - n_0 + m)}
$$
$$
\times \frac{(q; q)_{n_1}(q; q)_{n_2}}{(q; q)_{n_1 + n_2 + 1}} \cdot {}_2\phi_1 \left(\begin{matrix} q^{n_0+1}, q^{n_1+1} \\ q^{n_1 + n_2 + 2} \end{matrix} \, \middle| \, q, q^{n_2 - n_0 + m} \right),
$$

where n_0, n_1, n_2, and m are positive integers satisfying the conditions

(19) $\qquad\qquad n_1 \geq n_0, \qquad n_2 \geq n_0, \qquad m > n_0.$

Comparing representations (13) and (18) we conclude that

$$F(\boldsymbol{a}; b) = (-1)^{n_0} p^{n_0(n_0+1)/2+(n_1+1)(n_2-n_0+m)} I(\boldsymbol{n}; m),$$

where

$$n_0 = a_1 - 1, \quad n_1 = a_2 - 1, \quad n_2 = b - a_2 - 1, \quad m = a_0 + a_1 + a_2 - b,$$

and conditions (19) become (14). Therefore the estimates (17) are consequences of the corresponding results for the quantity $I(\boldsymbol{n}; m)$ (see [Zu1, Lemmas 6 and 7]), and the lemma follows.

REMARK. In [Zu1] we made a mistake in computing the exponent of p when applying the identity of [Zu1, Lemma 3]. The correct application of the identity leads to

$$p^{(n_1-n_0)(m-n_0+1)} D_{\max\{n_1+n_2-n_0, m\}} \cdot I \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p],$$

$$p^{(n_1-n_0)(m-n_0+1)} \widehat{D}_{n_1+n_2-n_0, m} \cdot \widehat{I} \in \mathbb{Z}[p] \ln_p(2) + \mathbb{Z}[p],$$

and to the corresponding changes in Propositions 1, 2 of [Zu1]. Fortunately, these changes do not influence the result of Theorem 2 of [Zu1] (concerning the irrationality exponent of $\ln_p(2)$), and Theorem 1 in the present work considerably improves the wrong result of Theorem 1 of [Zu1].

4. Permutation group for q-harmonic series. Heine's transform (1) yields the stability of the quantity

$$(20) \qquad \frac{F(a_0, a_1, a_2; b)}{\Gamma_q(a_0)\, \Gamma_q(a_2)\, \Gamma_q(b - a_2)}$$

$$= \frac{1}{(1-q)\, \Gamma_q(a_0)\, \Gamma_q(b)} \cdot {}_2\phi_1 \left(\begin{matrix} q^{a_1},\, q^{a_2} \\ q^b \end{matrix} \;\middle|\; q, q^{a_0} \right)$$

under the action of

$$\tau : (a_0, a_1, a_2; b) \mapsto (a_1, b - a_1, a_0; a_0 + a_2).$$

In addition, the quantity (20) is obviously stable under the action of the permutation

$$\sigma : (a_0, a_1, a_2; b) \mapsto (a_0, a_2, a_1; b)$$

interchanging the parameters a_1 and a_2. Let $\mathfrak{G}$ denote the group generated by τ, σ; the group $\mathfrak{G} = \langle \tau, \sigma : \tau^6 = \sigma^2 = (\tau\sigma)^2 = \mathrm{id} \rangle$ has order 12 (see [Fi, Section 20]).

To interpret $\mathfrak{G}$ as a permutation group, we now introduce the tuple $\boldsymbol{c}$ of the six additional parameters

$$c_{00} = a_0 + a_1 + a_2 - b - 1, \quad c_{01} = a_0 - 1, \quad c_{11} = a_1 - 1,$$

$$c_{21} = a_2 - 1, \quad c_{12} = b - a_1 - 1, \quad c_{22} = b - a_2 - 1,$$

and take $H(\boldsymbol{c}) := F(\boldsymbol{a}; b)$. Then

$$\tau = (c_{22}\ c_{21}\ c_{01}\ c_{11}\ c_{12}\ c_{00}), \qquad \sigma = (c_{11}\ c_{21})(c_{12}\ c_{22})$$

are permutations of the tuple $\boldsymbol{c}$ of orders 6 and 2, respectively, and the $\mathfrak{G}$-stability of the quantity (20) can be summarized by the following assertion.

LEMMA 5. *The quantity*

$$\frac{H(\boldsymbol{c})}{\Pi_q(\boldsymbol{c})}, \quad where \quad \Pi_q(\boldsymbol{c}) := [c_{01}]_q!\,[c_{21}]_q!\,[c_{22}]_q!,$$

is stable under the action of $\mathfrak{G}$.

By definition of the q-factorial coefficient, $\Pi_q(\boldsymbol{c}) = p^{-N(\boldsymbol{c})}\Pi_p(\boldsymbol{c})$, where

$$N(\boldsymbol{c}) := \frac{c_{01}(c_{01} + 1) + c_{21}(c_{21} + 1) + c_{22}(c_{22} + 1)}{2}.$$

We also require some additional characteristics: $M(\boldsymbol{c}) := M(\boldsymbol{a}; b)$ is the "suitable" integer of Lemma 3 if $a_1 + a_2 \le b$ or is determined by formula (16) if stronger conditions (14) hold, and

$$m(\boldsymbol{c}) := \max\{c_{00}, c_{01}, c_{11}, c_{21}, c_{12}, c_{22}\},$$
$$s_+(\boldsymbol{c}) := c_{01} + c_{21} + c_{22} = a_0 + b - 3,$$
$$s_-(\boldsymbol{c}) := c_{00} + c_{11} + c_{12} = a_0 + a_1 + a_2 - 3,$$
$$s(\boldsymbol{c}) := s_+(\boldsymbol{c}) - s_-(\boldsymbol{c}) = b - a_1 - a_2.$$

Then $m(\boldsymbol{c})$ is $\mathfrak{G}$-stable, while the quantities $s_\pm(\boldsymbol{c})$ and $s(\boldsymbol{c})$ obey the following rules.

LEMMA 6. *The following relations hold*:

$$s_+(\tau\boldsymbol{c}) = s_-(\boldsymbol{c}), \quad s_-(\tau\boldsymbol{c}) = s_+(\boldsymbol{c}), \quad s(\tau\boldsymbol{c}) = -s(\boldsymbol{c}),$$
$$s_+(\sigma\boldsymbol{c}) = s_+(\boldsymbol{c}), \quad s_-(\sigma\boldsymbol{c}) = s_-(\boldsymbol{c}), \quad s(\sigma\boldsymbol{c}) = s(\boldsymbol{c}),$$

where $\mathfrak{g}\boldsymbol{c}$ *denotes the image of* $\boldsymbol{c}$ *under the action of a permutation* $\mathfrak{g} \in \mathfrak{G}$.

Proof. By direct computation.

As is easily seen, condition (6) is equivalent to $s(\boldsymbol{c}) \ge 0$; hence we can write the conclusions of Lemmas 3 and 4 in the form

(21) $$p^{-M(\boldsymbol{c})}D_{m(\boldsymbol{c})}(p)H(\boldsymbol{c}) \in \mathbb{Z}[p]h_p(1) + \mathbb{Z}[p],$$

if (and only if) $s(\boldsymbol{c}) \ge 0$. This fact and Lemma 6 mean that we might use the group $\mathfrak{G}_+ = \langle \tau^2, \sigma \rangle \subset \mathfrak{G}$ of order 6 instead of the total group $\mathfrak{G}$ if $s(\boldsymbol{c}) > 0$. The case $s(\boldsymbol{c}) = 0$ is of no interest since it implies the relations $c_{00} = c_{01}$, $c_{11} = c_{22}$, $c_{21} = c_{12}$, and as a consequence

$$\Pi_p(\mathfrak{g}\boldsymbol{c}) = \Pi_p(\boldsymbol{c}) \quad \text{for all } \mathfrak{g} \in \mathfrak{G}.$$

If $s(\boldsymbol{c}) > 0$, we obtain at most three different values of $\Pi_p(\mathfrak{g}\boldsymbol{c})$, $\mathfrak{g} \in \mathfrak{G}_+$, namely,

$$\Pi_p = \Pi_p(\boldsymbol{c}) = \Pi_p(\sigma\tau^4\boldsymbol{c}) = [c_{01}]_p! \, [c_{21}]_p! \, [c_{22}]_p!,$$
$$\Pi'_p = \Pi_p(\tau^2\boldsymbol{c}) = \Pi_p(\sigma\boldsymbol{c}) = [c_{01}]_p! \, [c_{11}]_p! \, [c_{12}]_p!,$$
$$\Pi''_p = \Pi_p(\tau^4\boldsymbol{c}) = \Pi_p(\sigma\tau^2\boldsymbol{c}) = [c_{00}]_p! \, [c_{12}]_p! \, [c_{22}]_p!.$$

For each $l = 2, 3, \ldots, m(\boldsymbol{c})$, take

$$(22) \quad \nu_l := \max_{\mathfrak{g} \in \mathfrak{G}_+} \operatorname{ord}_{\Phi_l(p)} \frac{\Pi_p(\boldsymbol{c})}{\Pi_p(\mathfrak{g}\boldsymbol{c})}$$
$$= \max\left\{0, \left\lfloor\frac{c_{21}}{l}\right\rfloor + \left\lfloor\frac{c_{22}}{l}\right\rfloor - \left\lfloor\frac{c_{11}}{l}\right\rfloor - \left\lfloor\frac{c_{12}}{l}\right\rfloor,\right.$$
$$\left. \left\lfloor\frac{c_{01}}{l}\right\rfloor + \left\lfloor\frac{c_{21}}{l}\right\rfloor - \left\lfloor\frac{c_{00}}{l}\right\rfloor - \left\lfloor\frac{c_{12}}{l}\right\rfloor\right\},$$

and set

$$\Omega(p) := \prod_{l=2}^{m(\boldsymbol{c})} \Phi_l^{\nu_l}(p) \in \mathbb{Z}[p].$$

LEMMA 7. *We have*

$$(23) \qquad p^{-M(\boldsymbol{c})} D_{m(\boldsymbol{c})}(p) \Omega^{-1}(p) H(\boldsymbol{c}) \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p],$$

provided that $s(\boldsymbol{c}) > 0$.

Proof. We follow the lines of the proof of Proposition 2 in [Zu2]. The inclusion (21) and Lemma 5 yield

$$p^{-M(\mathfrak{g}\boldsymbol{c}) - N(\mathfrak{g}\boldsymbol{c}) + N(\boldsymbol{c})} D_{m(\boldsymbol{c})}(p) \cdot \frac{\Pi_p(\mathfrak{g}\boldsymbol{c})}{\Pi_p(\boldsymbol{c})} \cdot H(\boldsymbol{c})$$
$$= p^{-M(\mathfrak{g}\boldsymbol{c})} D_{m(\mathfrak{g}\boldsymbol{c})}(p) H(\mathfrak{g}\boldsymbol{c}) \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p]$$

for all $\mathfrak{g} \in \mathfrak{G}_+$. Since cyclotomic polynomials enter the p-factorial $[n]_p!$ in accordance with formula (5) and these polynomials are coprime to the polynomial $p \in \mathbb{Z}[p]$, we arrive at (23).

5. q-Conclusion. We now take a tuple of new positive integers (*directions*) α_0, α_1, α_2, and β satisfying the conditions

$$\alpha_1 \le \alpha_2, \quad \alpha_1 + \alpha_2 < \beta \le \alpha_0 + \alpha_2,$$

and to each integer $n = 1, 2, \ldots$ assign the old parameters $\boldsymbol{a}$ and b in accordance with the following rule:

$$a_j = \alpha_j n + 1, \quad j = 0, 1, 2, \quad b = \beta n + 2.$$

Then setting

$$c_{00} = \alpha_0 + \alpha_1 + \alpha_2 - \beta, \quad c_{j1} = \alpha_j, \; j = 0, 1, 2, \quad c_{j2} = \beta - \alpha_j, \; j = 1, 2,$$

and $m = m(\boldsymbol{c})$ we introduce the quantities

$$H_n := H(\boldsymbol{c} \cdot n) = F(\boldsymbol{a}; b) = A_n h_p(1) - B_n, \quad n = 1, 2, \ldots$$

Finally,

(24) $\quad p^{-M(\boldsymbol{a};b)} D_{mn}(p) \Omega^{-1}(p) H_n \in \mathbb{Z}[p] h_p(1) + \mathbb{Z}[p] \subset \mathbb{Z} h_p(1) + \mathbb{Z},$

$$n = 1, 2, \ldots,$$

by Lemma 7, and

(25)

$$\lim_{n \to \infty} \frac{\log |H_n|}{n^2 \log |p|} = 0,$$

$$\limsup_{n \to \infty} \frac{\log |A_n|}{n^2 \log |p|} \le (\alpha_0 + \alpha_1 + \alpha_2)\beta - \frac{\alpha_1^2 + \alpha_2^2 + \beta^2}{2} =: C_1$$

by Lemma 4. In addition, $\nu_l = \omega(n/l)$ in (22), where

$$\omega(x) := \max\{0, \lfloor c_{21} x \rfloor + \lfloor c_{22} x \rfloor - \lfloor c_{11} x \rfloor - \lfloor c_{12} x \rfloor,$$

$$\lfloor c_{01} x \rfloor + \lfloor c_{21} x \rfloor - \lfloor c_{00} x \rfloor - \lfloor c_{12} x \rfloor\}$$

is a 1-periodic integer-valued function; therefore,

(26) $\quad \lim_{n \to \infty} \dfrac{\log |p^{M(\boldsymbol{a};b)} D_{mn}^{-1}(p) \Omega(p)|}{n^2 \log |p|}$

$$= \frac{1}{2}\alpha_1^2 + \alpha_0 \alpha_1 + (\beta - \alpha_2)(\alpha_2 - \alpha_1)$$

$$- \frac{3}{\pi^2}\left(m^2 - \int_0^1 \omega(x)\, d(-\psi'(x))\right) =: C_0$$

by Lemmas 1, 2, and 4. By standard arguments, relations (24)–(26) yield the estimate $\mu(h_p(1)) \le C_1/C_0$, provided that $C_0 > 0$.

Take

$$\alpha_0 = 14, \quad \alpha_1 = 12, \quad \alpha_2 = 14, \quad \beta = 27$$

so that

$$c_{00} = 13, \quad c_{01} = 14, \quad c_{11} = 12, \quad c_{21} = 14, \quad c_{12} = 15, \quad c_{22} = 13.$$

Then

$$C_1 = 545.5, \quad C_0 = 221.30008816\ldots,$$

hence $\mu(h_p(1)) \le C_1/C_0 = 2.46497868\ldots$ Here $\omega(x) = 1$ for $x \in [0, 1)$ belonging to the set

$$\left[\tfrac{1}{14}, \tfrac{1}{12}\right) \cup \left[\tfrac{1}{7}, \tfrac{1}{6}\right) \cup \left[\tfrac{3}{14}, \tfrac{1}{4}\right) \cup \left[\tfrac{2}{7}, \tfrac{1}{3}\right) \cup \left[\tfrac{5}{14}, \tfrac{2}{5}\right) \cup \left[\tfrac{3}{7}, \tfrac{7}{15}\right) \cup \left[\tfrac{1}{2}, \tfrac{8}{15}\right)$$

$$\cup \left[\tfrac{4}{7}, \tfrac{3}{5}\right) \cup \left[\tfrac{9}{14}, \tfrac{2}{3}\right) \cup \left[\tfrac{5}{7}, \tfrac{11}{15}\right) \cup \left[\tfrac{11}{14}, \tfrac{4}{5}\right) \cup \left[\tfrac{6}{7}, \tfrac{13}{15}\right) \cup \left[\tfrac{13}{14}, \tfrac{14}{15}\right).$$

The proof of Theorem 1 is complete.

REMARK. The series (13) involving linear forms in 1 and the q-harmonic series can be represented as some q-integrals (see, e.g., [Ex, Section 2.5.1]). This q-integral representation is very similar to that used in [RV1] and [RV2] for describing the permutation groups for $\zeta(2)$ and $\zeta(3)$. In spite of this similarity, there exists no general pattern to change the variable of q-integration (see [Ask] and [Ex, Section 2.2.4]). Therefore the hypergeometric construction proposed in this paper as well as in our previous works [Zu1], [Zu2] looks a very natural way to extend the group-structure approach to solving new number-theoretic problems.

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$$1 + \frac{(q^\alpha - 1)(q^\beta - 1)}{(q-1)(q^\gamma - 1)}\, x + \frac{(q^\alpha - 1)(q^{\alpha+1} - 1)(q^\beta - 1)(q^{\beta+1} - 1)}{(q-1)(q^2 - 1)(q^\gamma - 1)(q^{\gamma+1} - 1)}\, x^2 + \ldots,$$

J. Reine Angew. Math. 32 (1846), 210–212.

[He2] —, *Untersuchungen über die Reihe*

$$1 + \frac{(q^\alpha - 1)(q^\beta - 1)}{(q-1)(q^\gamma - 1)}\, x + \frac{(q^\alpha - 1)(q^{\alpha+1} - 1)(q^\beta - 1)(q^{\beta+1} - 1)}{(q-1)(q^2 - 1)(q^\gamma - 1)(q^{\gamma+1} - 1)}\, x^2 + \ldots,$$

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