ACTA ARITHMETICA
130.3 (2007)

A congruence involving the quotients of Euler and its applications (II)

by

T. X. Cai, X. D. Fu and X. Zhou (Hangzhou)

1. Introduction. In [1], the first author showed that for any odd $n > 1$,

$$\sum_{\substack{r=1\\(r,n)=1}}^{(n-1)/2} \frac{1}{r} \equiv -2q_2(n) + nq_2^2(n) \pmod{n^2}, \tag{1}$$

where

$$q_i(n) = \frac{i^{\phi(n)} - 1}{n}, \qquad (i,n) = 1,$$

is Euler's quotient of n with base i. In particular, if n is prime, (1) becomes Lehmer's famous congruence (cf. [2] or [3])

$$\sum_{r=1}^{(p-1)/2} \frac{1}{r} \equiv -2q_2(p) + pq_2^2(p) \pmod{p^2},$$

which, along with some other congruences, plays an important role in studying the first case of Fermat's last theorem. In this paper, we wish to generalize some other congruences of Lehmer to arbitrary positive integer moduli.

Theorem 1. *If n is odd, then*

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/2\rfloor} \frac{1}{n-2r} \equiv q_2(n) - \frac{1}{2}\,nq_2^2(n) \pmod{n^2}, \qquad (3,n) = 1, \tag{2}$$

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/3\rfloor} \frac{1}{n-3r} \equiv \frac{1}{2}\,q_3(n) - \frac{1}{4}\,nq_3^2(n) \pmod{n^2}, \qquad (3,n) = 1, \tag{3}$$

2000 *Mathematics Subject Classification*: 11A25, 11B65, 11B68.

Key words and phrases: quotients of Euler, Bernoulli numbers, binomial coefficients.
Research partly supported by the project NNSFC 10371107.

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/4\rfloor} \frac{1}{n-4r} \equiv \frac{3}{4}\,q_2(n) - \frac{3}{8}\,nq_2^2(n) \pmod{n^2}, \qquad (3,n)=1, \tag{4}$$

$$\sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/6\rfloor} \frac{1}{n-6r} \equiv \frac{1}{3}\,q_2(n) + \frac{1}{4}\,q_3(n) - \frac{1}{6}\,nq_2^2(n) - \frac{1}{8}\,nq_3^2(n) \pmod{n^2}, \quad (15,n)=1. \tag{5}$$

As an application of Theorem 1, we have the following result.

THEOREM 2. *If n is odd, then*

$$\prod_{d\mid n} \binom{d-1}{\lfloor d/3\rfloor}^{\mu(n/d)} \equiv (-1)^{\phi_3(n)}(3^{\phi(n)+1}-1)/2 \pmod{n^2}, \qquad (3,n)=1, \tag{6}$$

$$\prod_{d\mid n} \binom{d-1}{\lfloor d/4\rfloor}^{\mu(n/d)} \equiv (-1)^{\phi_4(n)}(3\cdot 2^{\phi(n)}-2) \pmod{n^2}, \qquad (3,n)=1, \tag{7}$$

$$\prod_{d\mid n} \binom{d-1}{\lfloor d/6\rfloor}^{\mu(n/d)} \equiv (-1)^{\phi_6(n)}(2^{\phi(n)+2}+3^{\phi(n)+1}-5)/2 \pmod{n^2}, \quad (15,n)=1, \tag{8}$$

where

$$\phi_e(n) = \sum_{d\mid n} \mu\left(\frac{n}{d}\right)\left\lfloor \frac{d}{e}\right\rfloor$$

is the generalized Euler totient function.

REMARK. We shall prove in Lemma 4 below that $(-1)^{\phi_e(n)} = -1$ only when n is a prime power p^α with $p \equiv 2 \pmod 3$ for $e=3$, $p\equiv 5$ or $7 \pmod 8$ for $e=4$, and $p \equiv 7$ or $11 \pmod{12}$ for $e=6$.

COROLLARY 1. *If p,q are distinct odd primes, then*

$$\binom{pq-1}{\lfloor pq/3\rfloor} \equiv \frac{1}{2}\,(3^{\phi(pq)+1}-1)\binom{p-1}{\lfloor p/3\rfloor}\binom{q-1}{\lfloor q/3\rfloor} \pmod{p^2q^2},$$

$$\binom{pq-1}{\lfloor pq/4\rfloor} \equiv (3\cdot 2^{\phi(pq)}-2)\binom{p-1}{\lfloor p/4\rfloor}\binom{q-1}{\lfloor q/4\rfloor} \pmod{p^2q^2},$$

$$\binom{pq-1}{\lfloor pq/6\rfloor} \equiv \frac{1}{2}\,(2^{\phi(pq)+2}+3^{\phi(pq)+1}-5)\binom{p-1}{\lfloor p/6\rfloor}\binom{q-1}{\lfloor q/6\rfloor} \pmod{p^2q^2}.$$

However, we have not been able to generalize the following formula of Lehmer to arbitrary positive integer moduli:

$$\sum_{r=1}^{\lfloor p/4\rfloor} \frac{1}{r^2} \equiv (-1)^{(p-1)/2}4E_{p-3} \pmod p$$

for any prime $p \geq 5$, where E_{2n} is the $2n$th Euler number which can be defined by the generating function

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \quad \text{for } |x| < \frac{\pi}{2}.$$

What we prove is the following result.

THEOREM 3. *If p is an odd prime and $l \geq 1$, then*

$$\sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/4 \rfloor} \frac{1}{r^2} \equiv (-1)^{(p^l-1)/2} 4E_{\phi(p^l)-2} \begin{cases} \pmod{p^l} & \text{for } p \geq 5, \\ \pmod{3^{l-1}} & \text{for } p = 3. \end{cases}$$

2. Auxiliary results

LEMMA 1. *If $p \geq 5$ is a prime and $k \geq 2$, l, t are positive integers, and s is the least positive residue of p^l modulo t, then*

$$\sum_{r=1}^{\lfloor p^l/t \rfloor} (p^l - tr)^{2k-1} \equiv \frac{t^{2k-1}}{2k} \left\{ B_{2k} - B_{2k}\left(\frac{s}{t}\right) + \frac{p^{2l}}{t^2} \binom{2k}{2} B_{2k-2} \right\} \pmod{p^{2l}}, \tag{9}$$

$$\sum_{r=1}^{\lfloor p^l/t \rfloor} (p^l - tr)^{2k} \equiv \frac{t^{2k}}{2k+1} \left\{ \frac{2k+1}{t} p^l B_{2k} - B_{2k+1}\left(\frac{s}{t}\right) \right\} \pmod{p^{3l-1}}, \tag{10}$$

$$\frac{pB_{\phi(p^{2l})}}{p-1} \equiv 1 \pmod{p^{2l}}, \tag{11}$$

where B_n is the nth Bernoulli number.

Proof. It is well-known that

$$B_{v+1}(x+1) - B_{v+1}(x) = (v+1)x^v.$$

We let $x = (p^l - rt)/t$ $(r = 1, \ldots, \lfloor p^l/t \rfloor)$, and add all the resulting equations; after cancellation we obtain

$$\sum_{r=1}^{\lfloor p^l/t \rfloor} (p^l - tr)^v = \frac{t^v}{v+1} \left\{ B_{v+1}\left(\frac{p^l}{t}\right) - B_{v+1}\left(\frac{s}{t}\right) \right\}, \tag{12}$$

where we have written s for the least positive residue of p^l modulo t. From

$$B_v(x) = \sum_{r=0}^{v} \binom{v}{r} B_r x^{v-r}$$

and the von Staudt–Clausen theorem, for $k \geq 2$ we have

$$B_{2k}\left(\frac{p^l}{t}\right) \equiv B_{2k} + \frac{p^{2l}}{t^2}\binom{2k}{2}B_{2k-2} \pmod{p^{2l}},$$

$$B_{2k+1}\left(\frac{p^l}{t}\right) \equiv B_{2k}\,\frac{2k+1}{t}\,p^l \pmod{p^{3l-1}}.$$

Taking $v = 2k$ and $v = 2k+1$ in (12), from the above congruences we get (9) and (10) respectively.

Now we prove (11). Taking $2k = \phi(p^{2l})$ and $t = 1$ in (10), noting that $B_{2k+1}(1) = 0$, we have

$$\sum_{\substack{r=1\\(r,p)=1}}^{p^l-1} r^{\phi(p^{2l})} \equiv \sum_{r=1}^{p^l}(p^l - r)^{\phi(p^{2l})} \equiv p^l B_{\phi(p^{2l})} \pmod{p^{3l-1}}. \tag{13}$$

By using the fundamental congruence

$$q_a(p^{2l}) + q_b(p^{2l}) \equiv q_{ab}(p^{2l}) \pmod{p^{2l}},$$

we have

$$\sum_{\substack{r=1\\(r,p)=1}}^{p^l-1} q_r(p^{2l}) \equiv \frac{(\prod_{1\leq r\leq p^l,\, p\nmid r} r)^{\phi(p^{2l})} - 1}{p^{2l}} = \frac{(w_{p^l}p^l - 1)^{\phi(p^{2l})} - 1}{p^{2l}} \tag{14}$$

$$\equiv -\phi(p^l)w_{p^l} \pmod{p^l},$$

where

$$w_{p^l} = \Big(\prod_{1\leq i\leq p^l,\, p\nmid i} i + 1\Big)/p^l$$

is the generalized Wilson quotient. Note that

$$\sum_{\substack{r=1\\(r,p)=1}}^{p^l-1} r^{\phi(p^{2l})} = \sum_{\substack{r=1\\(r,p)=1}}^{p^l-1} (1 + q_r(p^{2l})p^{2l}). \tag{15}$$

Combining (13)–(15), we have

$$p^l B_{\phi(p^{2l})} \equiv \sum_{\substack{r=1\\(r,p)=1}}^{p^l-1} 1 = \phi(p^l) \pmod{p^{3l-1}},$$

and (11) is obtained.

LEMMA 2 ([2]). *If v is even, $e = 2, 3, 4,$ or 6, then*

$$B_v\left(\frac{1}{e}\right) = B_v\left(\frac{e-1}{e}\right) = \prod_{q\mid e}(1 - q^{v-1})\,\frac{B_v}{\phi(e^v)},$$

where q is a prime factor of e. If v is odd and $e=4$, then

$$B_v\left(\frac{1}{4}\right) = -B_v\left(\frac{3}{4}\right) = -\frac{vE_{v-1}}{4^v}.$$

LEMMA 3. *If $p \geq 5$ is a prime, $l, d \geq 1$, $e \geq 2$, $p \nmid e$, $d \equiv \pm 1 \pmod e$, then*

$$\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{dp^l - er} \equiv \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{p^l - er} \pmod{p^{2l}}.$$

Proof. Let

$$D = \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{dp^l - er} - \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{p^l - er}.$$

By expanding the summands as geometric series, we have

$$\begin{aligned}
D &= \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{dp^l - er} - \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{p^l - er} + \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{p^l - er} - \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor} \frac{1}{p^l - er}\\
&\equiv -\frac{(d-1)p^l}{e^2} \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{r^2} + \sum_{\substack{r=[p^l/e]+1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{p^l - er}\\
&\equiv -\frac{(d-1)p^l}{e^2} \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{r^2} - \frac{p^l}{e^2} \sum_{\substack{r=[p^l/e]+1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{r^2} - \frac{1}{e} \sum_{\substack{r=[p^l/e]+1\\ p\nmid r}}^{\lfloor dp^l/e\rfloor} \frac{1}{r} \pmod{p^{2l}}.
\end{aligned}$$

For any prime $p \geq 5$, let t be the least positive residue of d modulo e; $t=1$ or $e-1$. Noting that $p \nmid e$ implies $p^l - (\lfloor (e-1)p^l/e\rfloor + 1) = \lfloor p^l/e\rfloor$, we derive that

$$\begin{aligned}
D \equiv &-\frac{(d-1)p^l}{e^2} \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor tp^l/e\rfloor} \frac{1}{r^2} - \frac{p^l}{e^2} \sum_{\substack{r=\lfloor p^l/e\rfloor+1\\ p\nmid r}}^{\lfloor tp^l/e\rfloor} \frac{1}{r^2}\\
&- \frac{1}{e}\left(\sum_{\substack{r=\lfloor p^l/e\rfloor+1\\ p\nmid r}}^{p^l} \frac{1}{r} + \sum_{\substack{r=1\\ p\nmid r}}^{\lfloor tp^l/e\rfloor} \frac{1}{r + d_0p^l}\right) \pmod{p^{2l}},
\end{aligned}$$

where $d_0 = (d-t)/e$. Recalling the well-known congruences

$$\sum_{\substack{r=1\\ p\nmid r}}^{p^l}\frac{1}{r}\equiv 0 \pmod{p^{2l}},\qquad \sum_{\substack{r=1\\ p\nmid r}}^{p^l}\frac{1}{r^2}\equiv 0 \pmod{p^{l}},$$

if $t=1$ we get

$$D\equiv\left\{-\frac{(d-1)}{e^2}+\frac{d_0}{e}\right\}p^l\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor}\frac{1}{r^2}\equiv 0 \pmod{p^{2l}},$$

and if $t=e-1$ we have

$$p^l\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (e-1)p^l/e\rfloor}\frac{1}{r^2}\equiv -p^l\sum_{\substack{r=\lfloor (e-1)p^l/e+1\rfloor\\ p\nmid r}}^{p^l}\frac{1}{r^2}\equiv -p^l\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor}\frac{1}{r^2} \pmod{p^{2l}},$$

$$p^l\sum_{\substack{r=\lfloor p^l/e\rfloor+1\\ p\nmid r}}^{\lfloor (e-1)p^l/e\rfloor}\frac{1}{r^2}\equiv -p^l\Bigg(\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor}\frac{1}{r^2}+\sum_{\substack{r=\lfloor (e-1)p^l/e\rfloor+1\\ p\nmid r}}^{p^l}\frac{1}{r^2}\Bigg)$$

$$\equiv -2p^l\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor}\frac{1}{r^2} \pmod{p^{2l}},$$

$$\sum_{\substack{r=\lfloor p^l/e\rfloor+1\\ p\nmid r}}^{p^l}\frac{1}{r}+\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (e-1)p^l/e\rfloor}\frac{1}{r+d_0p^l}\equiv\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (e-1)p^l/e\rfloor}\left(\frac{1}{p^l-r}+\frac{1}{r+d_0p^l}\right)$$

$$\equiv -(1+d_0)p^l\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (e-1)p^l/e\rfloor}\frac{1}{r^2}\equiv (1+d_0)p^l\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor}\frac{1}{r^2} \pmod{p^{2l}},$$

therefore,

$$D\equiv\left\{\frac{d-1}{e^2}+\frac{2}{e^2}-\frac{1}{e}(1+d_0)\right\}p^l\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/e\rfloor}\frac{1}{r^2}\equiv 0 \pmod{p^{2l}}. \ \blacksquare$$

LEMMA 4. *Let $e=2,3,4$ or 6, n be odd, $3\nmid n$ for $e=3$ or 6. Then*

$$(-1)^{\phi_e(n)}=-1$$

only when n is a prime power p^α with $p\equiv 3 \pmod 4$ for $e=2$, $p\equiv 2 \pmod 3$ for $e=3$, $p\equiv 3$ or $5 \pmod 8$ for $e=4$, and $p\equiv 7$ or $11 \pmod{12}$ for $e=6$.

Proof. Let $p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ be the standard factorization of n. If $e=2$, we assume that $p_i\equiv t_i \pmod 4$, $t_i=\pm 1$. Then

$$\phi_2(n) \equiv \sum_{\substack{\alpha_i-1\le\beta_i\le\alpha_i\\1\le i\le s}} \left\lfloor \frac{p_1^{\beta_1}\cdots p_s^{\beta_s}}{2} \right\rfloor \equiv \sum_{\substack{\alpha_i-1\le\beta_i\le\alpha_i\\1\le i\le s}} \frac{1-t_1^{\beta_1}\cdots t_s^{\beta_s}}{2}$$
$$\equiv \sum_{\substack{\alpha_i-1\le\beta_i\le\alpha_i\\1\le i\le s}} \left\{ \frac{1-t_1^{\beta_1}}{2} + \cdots + \frac{1-t_s^{\beta_s}}{2} \right\} \pmod 2.$$

Since for any $1 \le j \le s$,

$$\sum_{\substack{\alpha_i-1\le\beta_i\le\alpha_i\\1\le i\le s}} \frac{1-t_j^{\beta_j}}{2} = \Bigg(\sum_{\substack{\alpha_i-1\le\beta_i\le\alpha_i\\1\le i\le s,\, i\ne j}} 1 \Bigg) \cdot \sum_{\alpha_j-1\le\beta_j\le\alpha_j} \frac{1-t_j^{\beta_j}}{2}$$
$$= 2^{s-1} \sum_{\alpha_j-1\le\beta_j\le\alpha_j} \frac{1-t_j^{\beta_j}}{2},$$

we have

$$\phi_2(n) \equiv 2^{s-1} \Bigg\{ \sum_{\alpha_1-1\le\beta_1\le\alpha_1} \frac{1-t_1^{\beta_1}}{2} + \cdots + \sum_{\alpha_s-1\le\beta_s\le\alpha_s} \frac{1-t_s^{\beta_s}}{2} \Bigg\}$$
$$\equiv 2^{s-1} \sum_{\substack{1\le i\le s\\ t_i=-1}} 1 \pmod 2,$$

where we use the fact that for any odd $a_1, \ldots, a_s$,

$$\frac{1-a_1\cdots a_s}{2} \equiv \frac{1-a_1}{2} + \cdots + \frac{1-a_s}{2} \pmod 2,$$

and $\phi_2(n)$ is odd only when n is a prime power p^l with $p \equiv 3 \pmod 4$.

If $e = 3$, we assume that $p_i \equiv t_i \pmod 6$, $t_i = \pm 1$. Then

$$\phi_3(n) \equiv \sum_{\substack{\alpha_i-1\le\beta_i\le\alpha_i\\1\le i\le s}} \left\lfloor \frac{p_1^{\beta_1}\cdots p_s^{\beta_s}}{3} \right\rfloor \equiv \sum_{\substack{\alpha_i-1\le\beta_i\le\alpha_i\\1\le i\le s}} \frac{1-t_1^{\beta_1}\cdots t_s^{\beta_s}}{2}$$
$$\equiv \sum_{\substack{\alpha_i-1\le\beta_i\le\alpha_i\\1\le i\le s}} \left\{ \frac{1-t_1^{\beta_1}}{2} + \cdots + \frac{1-t_s^{\beta_s}}{2} \right\}$$
$$\equiv 2^{s-1} \Bigg\{ \sum_{\alpha_1-1\le\beta_1\le\alpha_1} \frac{1-t_1^{\beta_1}}{2} + \cdots + \sum_{\alpha_s-1\le\beta_s\le\alpha_s} \frac{1-t_s^{\beta_s}}{2} \Bigg\}$$
$$\equiv 2^{s-1} \sum_{\substack{1\le i\le s\\ t_i=-1}} 1 \pmod 2,$$

and $\phi_3(n)$ is odd only when n is a prime power p^l with $p \equiv 2 \pmod 3$.

If $e = 4$, we assume that $p_i \equiv t_i \pmod 8$, $t_i = \pm 1$ or ± 3. Then

$$\begin{aligned}
\phi_4(n) &\equiv \sum_{\substack{\alpha_i - 1 \le \beta_i \le \alpha_i \\ 1 \le i \le s}} \left\lfloor \frac{p_1^{\beta_1} \cdots p_s^{\beta_s}}{4} \right\rfloor \\
&\equiv \sum_{\substack{\alpha_i - 1 \le \beta_i \le \alpha_i \\ 1 \le i \le s}} \frac{(1 - t_1^{\beta_1} \cdots t_s^{\beta_s})(3 - t_1^{\beta_1} \cdots t_s^{\beta_s})}{8} \\
&\equiv \sum_{\substack{\alpha_i - 1 \le \beta_i \le \alpha_i \\ 1 \le i \le s}} \left\{ \frac{(1 - t_1^{\beta_1})(3 - t_1^{\beta_1})}{8} + \cdots + \frac{(1 - t_s^{\beta_s})(3 - t_s^{\beta_s})}{8} \right\} \\
&\equiv 2^{s-1} \sum_{1 \le i \le s} \sum_{\alpha_i - 1 \le \beta_i \le \alpha_i} \frac{(1 - t_i^{\beta_i})(3 - t_i^{\beta_i})}{8} \equiv 2^{s-1} \sum_{\substack{1 \le i \le s \\ t_i = -1 \text{ or } -3}} 1 \pmod 2,
\end{aligned}$$

where we use the fact that for any odd $a_1, \ldots, a_s$,

$$\frac{(1 - \prod_{i=1}^s a_i)(3 - \prod_{i=1}^s a_i)}{8} \equiv \sum_{1 \le i \le s} \frac{(1 - a_i)(3 - a_i)}{8} \pmod 2,$$

and $\phi_4(n)$ is odd only when n is a prime power p^l with $p \equiv 5$ or $7 \pmod 8$.

If $e = 6$, we assume that $p_i \equiv t_i \pmod{12}$, $t_i = \pm 1$, or ± 5. Then

$$\begin{aligned}
\phi_6(n) &\equiv \sum_{\substack{\alpha_i - 1 \le \beta_i \le \alpha_i \\ 1 \le i \le s}} \left\lfloor \frac{p_1^{\beta_1} \cdots p_s^{\beta_s}}{6} \right\rfloor \\
&\equiv \sum_{\substack{\alpha_i - 1 \le \beta_i \le \alpha_i \\ 1 \le i \le s}} \frac{(1 - t_1^{\beta_1} \cdots t_s^{\beta_s})(5 - t_1^{\beta_1} \cdots t_s^{\beta_s})}{12} \\
&\equiv \sum_{\substack{\alpha_i - 1 \le \beta_i \le \alpha_i \\ 1 \le i \le s}} \left\{ \frac{(1 - t_1^{\beta_1})(5 - t_1^{\beta_1})}{12} + \cdots + \frac{(1 - t_s^{\beta_s})(5 - t_s^{\beta_s})}{12} \right\} \\
&\equiv 2^{s-1} \sum_{1 \le i \le s} \sum_{\alpha_i - 1 \le \beta_i \le \alpha_i} \frac{(1 - t_i^{\beta_i})(5 - t_i^{\beta_i})}{12} \equiv 2^{s-1} \sum_{\substack{1 \le i \le s \\ t_i = -1 \text{ or } -5}} 1 \pmod 2,
\end{aligned}$$

where we use the fact that for any $a_i \equiv \pm 1$ or $\pm 5 \pmod{12}$, $1 \le i \le s$,

$$\frac{(1 - \prod_{i=1}^s a_i)(5 - \prod_{i=1}^s a_i)}{12} \equiv \sum_{1 \le i \le s} \frac{(1 - a_i)(5 - a_i)}{12} \pmod 2,$$

and $\phi_6(n)$ is odd only when n is a prime power p^l with $p \equiv 7$ or $11 \pmod{12}$. This completes the proof of Lemma 4. ∎

3. Proof of Theorem 1

Proof of (2). We first assume that $n = p^l$ with prime $p \geq 5$. Taking $t = 2$ and $2k = \phi(p^{2l})$ in (9), by using Lemma 2 with $e = 2$ and (11), we have

$$\tag{16} \begin{aligned} \sum_{\substack{r=1 \\ p\nmid r}}^{(p^l-1)/2} \frac{1}{p^l - 2r} &\equiv \sum_{r=1}^{(p^l-1)/2} (p^l - 2r)^{\phi(p^{2l})-1} \\ &\equiv q_2(p^{2l}) \equiv q_2(p^l) - \frac{1}{2}\, q_2^2(p^l)p^l \pmod{p^{2l}}. \end{aligned}$$

Now, let $n = p^l m$, $p \geq 3$, $m > 1$, $p \nmid m$. By Lemma 3 and (16),

$$\tag{17} \begin{aligned} \sum_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{1}{n - 2r} &\equiv \sum_{\substack{r=1 \\ p\nmid r}}^{(n-1)/2} \frac{1}{n - 2r} - \sum_{\substack{r=1 \\ p\nmid r \\ q_1 \mid r}}^{(n-1)/2} \frac{1}{n - 2r} \\ &\quad + \cdots + (-1)^s \sum_{\substack{r=1 \\ p\nmid r \\ q_1\cdots q_g \mid r}}^{(n-1)/2} \frac{1}{n - 2r} \\ &\equiv \prod_{q\mid m}\left(1 - \frac{1}{q}\right) \sum_{\substack{r=1 \\ p\nmid r}}^{(p^l-1)/2} \frac{1}{p^l - 2r} \\ &\equiv \frac{\phi(m)}{m}\left(q_2(p^l) - \frac{1}{2}\, p^l q_2^2(p^l)\right) \pmod{p^{2l}}, \end{aligned}$$

where q_i $(1 \leq i \leq g)$ are the prime factors of m.

On the other hand,

$$\begin{aligned} q_2(n) &= \frac{(1 + p^l q_2(p^l))^{\phi(m)} - 1}{p^l m} \\ &\equiv \frac{\phi(m)}{m}\, q_2(p^l) + \frac{\phi(m)(\phi(m) - 1)}{2m}\, p^l q_2^2(p^l) \pmod{p^{2l}}, \\ n q_2^2(n) &\equiv n\left(\frac{\phi(m)}{m}\, q_2(p^l)\right)^2 \equiv \frac{\phi^2(m)}{m}\, p^l q_2^2(p^l) \pmod{p^{2l}}, \end{aligned}$$

hence

$$\tag{18} q_2(n) - \frac{1}{2}\, n q_2^2(n) \equiv \frac{\phi(m)}{m}\left(q_2(p^l) - \frac{1}{2}\, p^l q_2^2(p^l)\right) \pmod{p^{2l}}.$$

Combining (17) with (18) yields (2).

Proof of (3). Similarly, we first assume that $n = p^l$ with prime $p \geq 5$. Taking $t = 3$ and $2k = \phi(p^{2l})$ in (9), by using Lemma 2 with $e = 3$ and (11), we have

$$\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (p^l-1)/3\rfloor} \frac{1}{p^l-3r} \equiv \sum_{r=1}^{\lfloor (p^l-1)/3\rfloor} (p^l-3r)^{\phi(p^{2l})-1} \equiv \frac{1}{2}\,q_3(p^{2l}) \equiv \frac{1}{2}\,q_3(p^l) - \frac{1}{4}\,q_3^2(p^l)p^l \pmod{p^{2l}}. \tag{19}$$

If $n = p^l m$, $p \ge 5$, $m > 1$, $p\nmid m$, $3\nmid m$, noting that $m \equiv \pm 1 \pmod 6$, by using Lemma 3 and (19), we can prove (3) in a way similar to (2).

Proof of (4). Taking $t = 4$ and $2k = \phi(p^{2l})$ in (9), by using Lemma 2 with $e = 4$ and (11), we have

$$\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (p^l-1)/4\rfloor} \frac{1}{p^l-4r} \equiv \sum_{r=1}^{\lfloor (p^l-1)/4\rfloor} (p^l-4r)^{\phi(p^{2l})-1} \equiv \frac{1}{4}\,(q_4(p^{2l}) + q_2(p^{2l})) \equiv \frac{3}{4}\,q_2(p^l) - \frac{3}{8}\,q_2^2(p^l)p^l \pmod{p^{2l}}. \tag{20}$$

If $n = p^l m$, $p \ge 5$, $m > 1$, $p\nmid m$, noting that $m \equiv \pm 1 \pmod 4$, by using Lemma 3 and (20), we can prove (4) in a way similar to (2).

Proof of (5). Taking $t = 6$ and $2k = \phi(p^{2l})$ in (9), by using Lemma 2 with $e = 6$ and (11), we have

$$\begin{aligned}\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor (p^l-1)/6\rfloor} \frac{1}{p^l-6r} &\equiv \sum_{r=1}^{\lfloor (p^l-1)/6\rfloor} (p^l-6r)^{\phi(p^{2l})-1}\\ &\equiv \frac{1}{12}\,(q_6(p^{2l}) + 2q_3(p^{2l}) + 3q_2(p^{2l}))\\ &\equiv \frac{1}{3}\,q_2(p^l) + \frac{1}{4}\,q_3(p^l) - \frac{1}{6}\,p^l q_2^2(p^l) - \frac{1}{8}\,p^l q_3^2(p^l) \pmod{p^{2l}}.\end{aligned} \tag{21}$$

If $n = p^l m$, $p \ge 7$, $m > 1$, $p\nmid m$, noting that $2\nmid n$, $3\nmid n$, $m \equiv \pm 1 \pmod 6$, by using Lemma 3 and (21), we can prove (5) once again as above.

4. Proof of Theorem 2. Define

$$A_n = \binom{n-1}{\lfloor n/e\rfloor}.$$

Then

$$A_n = \prod_{r=1}^{\lfloor n/e\rfloor} \frac{n-r}{r} = \prod_{d\mid n} \prod_{\substack{r=1\\ (r,n)=d}}^{\lfloor n/e\rfloor} \frac{n-r}{r} = \prod_{d\mid n} T_{n/d} = \prod_{d\mid n} T_d,$$

where

$$T_d = \prod_{\substack{r=1\\(r,d)=1}}^{\lfloor d/e \rfloor} \frac{d-r}{r}.$$

Using the Möbius inversion formula, we have

$$T_n = \prod_{d|n} A_d^{\mu(n/d)} = \prod_{d|n} \binom{d-1}{\lfloor d/e \rfloor}^{\mu(n/d)}.$$

On the other hand,

$$\begin{aligned}(22)\qquad T_n &= \prod_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e \rfloor} \frac{n-r}{r} = (-1)^{\phi_e(n)} \prod_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e \rfloor} \left(1-\frac{n}{r}\right)\\ &\equiv (-1)^{\phi_e(n)} \Bigg\{1 - n \sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r}\Bigg\}\\ &\equiv (-1)^{\phi_e(n)} \Bigg\{1 + en \sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{n-er}\Bigg\} \pmod{n^2},\end{aligned}$$

where

$$\phi_e(n) = \sum_{\substack{r=1\\(r,n)=1}}^{\lfloor n/e \rfloor} 1 = \sum_{r=1}^{\lfloor n/e \rfloor} \sum_{d|(r,n)} \mu(d) = \sum_{d|n} \mu(d) \left\lfloor \frac{n}{de} \right\rfloor = \sum_{d|n} \mu\left(\frac{n}{d}\right) \left\lfloor \frac{d}{e} \right\rfloor,$$

and applying Lemma 4 and (2)–(4) to (22), respectively, we complete the proof of Theorem 2.

If we set $d = pq$ in Theorem 2, then Corollary 1 can be derived easily with the help of Lemma 4.

5. Proof of Theorem 3. Taking $t = 4$ and $2k = \phi(p^l) - 2$ in (10), and using the von Staudt–Clausen theorem, we have

$$(23)\qquad \sum_{\substack{r=1\\p\nmid r}}^{\lfloor p^l/4 \rfloor} \frac{1}{r^2} \equiv \sum_{r=1}^{\lfloor p^l/4 \rfloor} r^{\phi(p^l)-2} \equiv -\frac{B_{\phi(p^l)-1}(s/4)}{\phi(p^l)-1} \begin{cases} \pmod{p^l} & \text{for } p \geq 5,\\ \pmod{3^{l-1}} & \text{for } p = 3,\end{cases}$$

where we have written s for the least positive residue of p^l modulo 4. By using Lemma 2, we have

$$B_{\phi(p^l)-1}\left(\frac{s}{4}\right) = -(-1)^{(p^l-1)/2}\,\frac{(\phi(p^l)-1)E_{\phi(p^l)-2}}{4^{\phi(p^l)-1}}. \tag{24}$$

Applying (23) to (24) yields

$$\sum_{\substack{r=1\\ p\nmid r}}^{\lfloor p^l/4\rfloor} \frac{1}{r^2} \equiv (-1)^{(p^l-1)/2} 4E_{\phi(p^l)-2} \begin{cases} (\bmod\, p^l) & \text{for } p \geq 5,\\ (\bmod\, 3^{l-1}) & \text{for } p = 3,\end{cases}$$

which completes the proof of Theorem 3.

Acknowledgements. The authors are grateful to the referee for his valuable suggestions and constructive report which made this paper more readable.

References

[1] T. X. Cai, *A congruence involving the quotients of Euler and its applications* (I), Acta Arith. 103 (2002), 313–320.
[2] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. of Math. 39 (1938), 350–359.
[3] P. Ribenboim, *13 Lectures on Fermat's Last Theorem*, Springer, 1979.

Department of Mathematics
Zhejiang University
Hangzhou, 310027, P.R. China
E-mail: txcai@mail.hz.zj.cn
fuxudan@126.com
xiazhou0821@hotmail.com

Received on 3.5.2006
and in revised form on 4.4.2007 (5195)