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The Ostrogradsky series and related Cantor-like sets

by

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Introduction. There are many different methods of expansion and encoding (representation) of real numbers by using a finite or an infinite alphabet A. The s-adic expansions, continued fractions, f-expansions, the Lüroth expansions etc. are widely used in mathematics (see, e.g., [17]). Each representation has its own features and generates its own "geometry" and metric theory. To each representation there is associated a system of cylindrical sets, partitioning the unit interval (or the real line). From the ratios of the lengths of cylindrical sets the basic metric relations follow (in the form of equalities and inequalities) which are crucial for the development of the corresponding metric theory, i.e., a theory about measure (e.g., Jordan, Lebesgue, Hausdorff, Hausdorff–Billingsley, ...) of sets of real numbers defined by characteristic properties of their digits in the corresponding representation (see, e.g., [1, 2, 6, 9, 10, 15, 17]).

The present paper is devoted to the investigation of the expansion of real numbers in the first Ostrogradsky series (introduced by M. V. Ostrogradsky (1801–1862), a well known Ukrainian mathematician). In this case the alphabet A coincides with the set $\mathbb{N}$ of positive integers.

The expansion

$$x = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{n-1}}{q_1 q_2 \cdots q_n} + \cdots, \tag{1}$$

where the q_n are positive integers and $q_{n+1} > q_n$ for all n, is said to be the expansion of x in the *first Ostrogradsky series*. The expansion

$$x = \frac{1}{q_1} - \frac{1}{q_2} + \cdots + \frac{(-1)^{n-1}}{q_n} + \cdots, \tag{2}$$

where the q_n are positive integers and $q_{n+1} \geq q_n(q_n + 1)$ for all n, is said

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to be the expansion of x in the *second Ostrogradsky series*. Each irrational number has a unique expansion of the form (1) or (2). Rational numbers have two different finite representations of the above forms (see, e.g., [16]).

Equality (1) can be rewritten as

$$x = \frac{1}{g_1} - \frac{1}{g_1(g_1+g_2)} + \cdots + \frac{(-1)^{n-1}}{g_1(g_1+g_2)\ldots(g_1+\cdots+g_n)} + \cdots, \tag{3}$$

where $g_1 = q_1$, $g_{n+1} = q_{n+1} - q_n$ for any $n \in \mathbb{N}$. The expression (3) is said to be the $\overline{\mathrm{O}}^1$-*representation* and the number $g_n = g_n(x)$ is the nth $\overline{\mathrm{O}}^1$-*symbol* of x.

Shortly before his death, M. V. Ostrogradsky found two algorithms for the representation of real numbers via alternating series of the form (1) and (2), but he did not publish them. Short unpublished remarks of Ostrogradsky concerning the above representations have been found by E. Ya. Remez [16]. Some similarities between the Ostrogradsky series and continued fractions have been pointed out in the same paper. E. Ya. Remez also dealt with applications of the Ostrogradsky series to numerical solution of algebraic equations. In the editorial comments to the book [6] B. V. Gnedenko has pointed out that there are no fundamental investigations of properties of the above mentioned representations. Analogous problems were studied by W. Sierpiński [18] and T. A. Pierce [11] independently. Some algorithms for representation of real numbers by means of positive and alternating series were proposed in [18]. Two of these algorithms lead to the Ostrogradsky series (1) and (2). An algorithm leading to the representation of irrational numbers in the form (1) has also been considered in [11]. An algorithm for a general alternating series expansion for real numbers in terms of rationals has been considered in [5], where the so-called alternating Lüroth and modified Engel-type expansions were also studied. This algorithm also leads to expansions of real numbers in Ostrogradsky series.

Let us mention some papers devoted to applications of the Ostrogradsky series. Connections between the Ostrogradsky algorithms and the algorithm for continued fractions have been established in [4]. This book also contains generalizations of the above algorithms. In [7] different types of p-adic continued fractions have been constructed on the basis of p-adic analogs of the Euclid and Ostrogradsky algorithms. Combining the algorithms of Engel and Ostrogradsky in a special way, the same author [8] has constructed an algorithm for representation of real numbers via series which converge faster than the corresponding Engel and Ostrogradsky series. The paper [19] is devoted to the investigation of the first Ostrogradsky algorithm and to the determination of the expectation of the random variables $(q_j+1)^{\nu}$, $\nu \geq 0$, and $r_n = \sum_{j=n+1}^{\infty} (-1)^{j+1}/q_1 \ldots q_j$, where the $q_j = q_j(\alpha)$ are random variables depending on the random variable α, uniformly distributed on the unit interval. In the same paper a generalization of the Ostrogradsky algorithm to approximations in Banach spaces has been proposed.

In this paper we continue to study the "geometry" of the representation generated by the first Ostrogradsky series [3, 13, 14]. In Section 1 we prove basic metric relations for the $\overline{\mathrm{O}}^1$-representation and compare them with the corresponding relations for continued fractions. Section 2 is devoted to the study of the set $C[\overline{\mathrm{O}}^1, \{V_n\}]$, consisting of the real numbers whose nth $\overline{\mathrm{O}}^1$-symbols take values from the set $V_n \subset \mathbb{N}$, for each $n \in \mathbb{N}$. Conditions for the set $C[\overline{\mathrm{O}}^1, \{V_n\}]$ to be of zero resp. positive Lebesgue measure λ are found. In particular, we prove that $\lambda(C[\overline{\mathrm{O}}^1, \{V_n\}]) > 0$ if $V_n = V = \{m+1, m+2, \dots\}$, where m is an arbitrary positive integer. This marks an essential difference between the metric theories of continued fractions and $\overline{\mathrm{O}}^1$-representations.

1. Representations of real numbers by the Ostrogradsky series

Definition 1. A finite or an infinite expression

$$\sum_n \frac{(-1)^{n-1}}{q_1 \cdots q_n} = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots , \tag{4}$$

where the q_n are positive integers and $q_{n+1} > q_n$ for all n, is called the *first Ostrogradsky series* (hereafter, the *Ostrogradsky series*). The numbers q_n are called the *elements of the Ostrogradsky series* (4).

We denote the expression (4) briefly by $\mathrm{O}^1(q_1, \dots, q_n)$ if it contains a finite number of terms, and we speak in this case of a finite Ostrogradsky series. We write $\mathrm{O}^1(q_1, q_2, \dots)$ in the case of infinitely many terms.

It is known (see, e.g., [16]) that every Ostrogradsky series is convergent and its sum belongs to $[0, 1]$, and any real number $x \in (0, 1)$ can be represented in the form (4). If x is irrational then the expression (4) is unique and it has an infinite number of terms. If x is rational then it can be represented in the form (4) in two different ways:

$$x = \mathrm{O}^1(q_1, \dots, q_{n-1}, q_n, q_n + 1) = \mathrm{O}^1(q_1, \dots, q_{n-1}, q_n + 1).$$

We can find the elements of the Ostrogradsky series for a given number x using the following algorithm:

$$\begin{array}{ll}
1 = q_1 x + \alpha_1 & (0 \le \alpha_1 < x), \\
1 = q_2 \alpha_1 + \alpha_2 & (0 \le \alpha_2 < \alpha_1), \\
\dots\dots\dots\dots\dots\dots & \\
1 = q_n \alpha_{n-1} + \alpha_n & (0 \le \alpha_n < \alpha_{n-1}), \\
\dots\dots\dots\dots\dots\dots & .
\end{array}$$

Let

$$g_1 = q_1, \qquad g_{n+1} = q_{n+1} - q_n \quad \text{for any } n \in \mathbb{N}.$$

Then one can rewrite (4) in the form

$$\sum_n \frac{(-1)^{n-1}}{g_1(g_1+g_2)\dots(g_1+\dots+g_n)} = \frac{1}{g_1} - \frac{1}{g_1(g_1+g_2)} + \cdots. \tag{5}$$

We denote the expression (5) by $\overline{\mathrm{O}}^1(g_1, g_2, \dots)$. A representation of a number $x \in (0,1)$ by (5) is called the $\overline{\mathrm{O}}^1$-*representation* and $g_n = g_n(x)$ is the nth $\overline{\mathrm{O}}^1$-*symbol* of x.

Let $c_1, \dots, c_m$ be a fixed sequence of positive integers.

Definition 2. The set $\overline{\mathrm{O}}^1_{[c_1\dots c_m]}$, which is the closure of the set of all $x \in (0,1)$ whose first m $\overline{\mathrm{O}}^1$-symbols are $c_1, \dots, c_m$ is said to be the *cylindrical set* (*cylinder*) *of rank* m *with base* $(c_1, \dots, c_m)$, i.e.,

$$\overline{\mathrm{O}}^1_{[c_1\dots c_m]} = (\{x : x = \overline{\mathrm{O}}^1(g_1(x), \dots),\ g_k(x) = c_k, 1 \le k \le m\})^{\mathrm{cl}}.$$

It is not hard to prove that $\overline{\mathrm{O}}^1_{[c_1\dots c_m]}$ is a closed interval of length

$$|\overline{\mathrm{O}}^1_{[c_1\dots c_m]}| = \frac{1}{\sigma_1 \dots \sigma_m(\sigma_m + 1)}, \tag{6}$$

where $\sigma_k = \sum_{i=1}^k c_i$.

Remark. We shall denote by $\overline{\mathrm{O}}^1_{(c_1\dots c_m)}$ the interior of $\overline{\mathrm{O}}^1_{[c_1\dots c_m]}$.

Let us mention that the cylindrical set $\overline{\mathrm{O}}^1_{[\underbrace{11\dots1}_m]}$ has the largest length among all cylindrical sets of rank m, namely

$$|\overline{\mathrm{O}}^1_{[\underbrace{11\dots1}_m]}| = \frac{1}{(m+1)!},$$

and there exist cylindrical sets of different ranks with the same length. For instance,

$$|\overline{\mathrm{O}}^1_{[1c]}| = |\overline{\mathrm{O}}^1_{[c+1]}|, \qquad |\overline{\mathrm{O}}^1_{[1c_2c_3\dots c_m]}| = |\overline{\mathrm{O}}^1_{[(c_2+1)c_3\dots c_m]}|.$$

Lemma 1. *For any given* $s \in \mathbb{N}$, *the ratio of the lengths of the cylindrical sets* $\overline{\mathrm{O}}^1_{[c_1\dots c_m s]}$ *and* $\overline{\mathrm{O}}^1_{[c_1\dots c_m]}$ *is*

$$\frac{|\overline{\mathrm{O}}^1_{[c_1\dots c_m s]}|}{|\overline{\mathrm{O}}^1_{[c_1\dots c_m]}|} = \frac{a}{(a+s-1)(a+s)} = f_s(a), \tag{7}$$

where $a = 1 + \sigma_m$. *Moreover*,

$$f_s(a) \le \frac{1}{2(2s-1)} \tag{8}$$

and for $m \geq s-1$,

$$\frac{|\overline{\mathrm{O}}^1_{[c_1\ldots c_m s]}|}{|\overline{\mathrm{O}}^1_{[c_1\ldots c_m]}|} \leq \frac{m+1}{(m+s)(m+s+1)}. \tag{9}$$

Proof. Equality (7) follows directly from (6). Consider

$$f_s(x) = \frac{x}{(x+s-1)(x+s)}$$

as a function of $x \geq 1$. This function increases on $[1, \sqrt{(s-1)s}]$ and decreases on $[\sqrt{(s-1)s}, \infty)$. Since a takes only positive integer values, we have

$$\max_{a\in\mathbb{N}} f_s(a) = f_s(s-1) = f_s(s) = \frac{1}{2(2s-1)}.$$

So, inequality (8) holds.

As $f_s(x)$ decreases on (s,∞), we have $f_s(a) \leq f_s(m+1)$, so inequality (9) holds. ∎

Corollary. *If* $c_1 + \cdots + c_m = s_1 + \cdots + s_k$ *then*

$$\frac{|\overline{\mathrm{O}}^1_{[c_1\ldots c_m s]}|}{|\overline{\mathrm{O}}^1_{[c_1\ldots c_m]}|} = \frac{|\overline{\mathrm{O}}^1_{[s_1\ldots s_k s]}|}{|\overline{\mathrm{O}}^1_{[s_1\ldots s_k]}|}.$$

Remark. Let $\Delta^{\text{c.f.}}_{c_1\ldots c_m}$ be a cylindrical set generated by the continued fraction representation of real numbers. It is well known (see, e.g., [6]) that

$$\frac{|\Delta^{\text{c.f.}}_{c_1\ldots c_m s}|}{|\Delta^{\text{c.f.}}_{c_1\ldots c_m}|} = \frac{1}{s^2}\cdot\frac{1+\frac{Q_{m-1}}{Q_m}}{\left(1+\frac{Q_{m-1}}{sQ_m}\right)\left(1+\frac{1}{s}+\frac{Q_{m-1}}{sQ_m}\right)}, \tag{10}$$

where Q_k is the denominator of the kth convergent of the continued fraction $[c_1, c_2, \ldots]$, i.e.,

$$Q_k = c_k Q_{k-1} + Q_{k-2} \quad \text{with} \quad Q_0 = 1,\, Q_1 = c_1.$$

From (10) it follows that

$$\frac{1}{3s^2} < \frac{|\Delta^{\text{c.f.}}_{c_1\ldots c_m s}|}{|\Delta^{\text{c.f.}}_{c_1\ldots c_m}|} < \frac{2}{s^2}$$

for any sequence $(c_1,\ldots,c_m)$ and for any $s \in \mathbb{N}$. For the $\overline{\mathrm{O}}^1$-representation we have $f_s(a) \to 0$ as $a \to \infty$, and Lemma 1 shows the fundamental difference between metric relations in the representation of numbers by the first Ostrogradsky series and by continued fractions.

Lemma 2. *Let* $\overline{\mathrm{O}}^1_{[c_1\ldots c_m]}$ *be a fixed cylindrical set. Then*

$$\lambda\Big(\bigcup_{s=1}^{k} \overline{\mathrm{O}}^1_{[c_1\ldots c_m s]}\Big) = \frac{k}{\sigma_m + k + 1}\,|\overline{\mathrm{O}}^1_{[c_1\ldots c_m]}|.$$

Proof. From (6) it follows that

$$\begin{aligned}\lambda\Big(\bigcup_{s=1}^{k}\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m s]}\Big) &= \sum_{s=1}^{k}|\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m s]}|\\ &= \frac{1}{\sigma_1\ldots\sigma_m}\sum_{s=1}^{k}\frac{1}{(\sigma_m+s)(\sigma_m+s+1)}\\ &= \frac{1}{\sigma_1\ldots\sigma_m}\Big(\frac{1}{\sigma_m+1}-\frac{1}{\sigma_m+k+1}\Big)\\ &= \frac{1}{\sigma_1\ldots\sigma_m(\sigma_m+1)}\cdot\frac{k}{\sigma_m+k+1}\\ &= |\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m]}|\,\frac{k}{\sigma_m+k+1},\end{aligned}$$

which proves Lemma 2. ■

COROLLARY 1. *For any $k\in\mathbb{N}$ and for any sequence $(c_1,\ldots,c_m)$,*

$$\frac{1}{\sigma_m+2}\,|\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m]}| \le \lambda\Big(\bigcup_{s=1}^{k}\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m s]}\Big) \le \frac{k}{m+k+1}\,|\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m]}|.$$

REMARK. If $V\subset\mathbb{N}$, then it is evident that

$$\sum_{s\in V}|\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m s]}| = |\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m]}| - \sum_{s\in\mathbb{N}\setminus V}|\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m s]}|.$$

COROLLARY 2. *Let $\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m]}$ be a cylindrical set. Then*

$$\lambda\Big(\bigcup_{c=k+1}^{\infty}\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m c]}\Big) = \frac{\sigma_m+1}{\sigma_m+k+1}\,|\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m]}|.$$

COROLLARY 3. *For any $k\in\mathbb{N}$ and for any sequence $(c_1,\ldots,c_m)$*

$$\frac{m+1}{m+k+1}\,|\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m]}| \le \lambda\Big(\bigcup_{c=k+1}^{\infty}\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m c]}\Big) \le \frac{\sigma_m+1}{\sigma_m+2}\,|\overline{\mathrm{O}}^{1}_{[c_1\ldots c_m]}|.$$

2. The set $C[\overline{\mathrm{O}}^1,\{V_n\}]$. In this section we shall study the metric properties of the set $C[\overline{\mathrm{O}}^1,\{V_n\}]$, which is the closure of the set $\{x : g_n(x)\in V_n,\ n\in\mathbb{N}\}$, consisting of the real numbers $x\in[0,1]$ whose $\overline{\mathrm{O}}^1$-symbols satisfy the condition $g_n(x)\in V_n$, where $\{V_n\}$ is a fixed sequence of nonempty subsets of $\mathbb{N}$.

It is evident that

(1) if $V_n=\mathbb{N}$ for all $n\in\mathbb{N}$, then $C[\overline{\mathrm{O}}^1,\{V_n\}]=[0,1]$,
(2) if $V_n=\mathbb{N}$ for all $n>n_0$, then $C[\overline{\mathrm{O}}^1,\{V_n\}]$ is a union of segments.

We are interested only in the case where $V_n \neq \mathbb{N}$ for infinitely many n. Let

$$F_k = \Big(\bigcup_{c_1\in V_1}\cdots\bigcup_{c_k\in V_k} \overline{\mathrm{O}}^1_{[c_1\ldots c_k]}\Big)^{\mathrm{cl}},$$

where cl stands for closure, let $F_0 = [0,1]$ and let $\overline{F}_{k+1} = F_k \setminus F_{k+1}$. It is not hard to prove that

$$C[\overline{\mathrm{O}}^1, \{V_n\}] = \bigcap_{k=1}^{\infty} F_k.$$

It is a perfect set (that is, a closed set without isolated points). If $V_n \neq \mathbb{N}$ for infinitely many n, then it is a nowhere dense set. Then

$$\lambda(F_k) = \sum_{c_1\in V_1}\cdots\sum_{c_k\in V_k}\frac{1}{\sigma_1\ldots\sigma_k(\sigma_k+1)},$$

$$\begin{aligned}\lambda(\overline{F}_{k+1}) &= \sum_{c_1\in V_1}\cdots\sum_{c_k\in V_k}\sum_{s\notin V_{k+1}}\frac{1}{\sigma_1\ldots\sigma_k(\sigma_k+s)(\sigma_k+s+1)}\\ &= \sum_{c_1\in V_1}\cdots\sum_{c_k\in V_k}\bigg(\frac{1}{\sigma_1\ldots\sigma_k}\sum_{s\notin V_{k+1}}\frac{1}{(\sigma_k+s)(\sigma_k+s+1)}\bigg),\end{aligned}$$

and from the continuity of Lebesgue measure it follows that

$$\lambda(C[\overline{\mathrm{O}}^1, \{V_n\}]) = \lim_{k\to\infty}\lambda(F_k).$$

Lemma 3. *The Lebesgue measure of* $C[\overline{\mathrm{O}}^1, \{V_n\}]$ *is* 0 *if and only if*

$$\sum_{k=1}^{\infty}\frac{\lambda(\overline{F}_{k+1})}{\lambda(F_k)} = \infty.$$

Proof. We have

$$\begin{aligned}\lambda(C[\overline{\mathrm{O}}^1, \{V_n\}]) &= \lim_{k\to\infty}\lambda(F_{k+1}) = \lim_{k\to\infty}\frac{\lambda(F_{k+1})}{\lambda(F_k)}\cdot\frac{\lambda(F_k)}{\lambda(F_{k-1})}\cdot\ldots\cdot\frac{\lambda(F_1)}{\lambda(F_0)}\\ &= \prod_{k=0}^{\infty}\frac{\lambda(F_{k+1})}{\lambda(F_k)} = \prod_{k=0}^{\infty}\frac{\lambda(F_k)-\lambda(\overline{F}_{k+1})}{\lambda(F_k)}\\ &= \prod_{k=0}^{\infty}\bigg(1-\frac{\lambda(\overline{F}_{k+1})}{\lambda(F_k)}\bigg) = 0\end{aligned}$$

if and only if

$$\sum_{k=1}^{\infty}\frac{\lambda(\overline{F}_{k+1})}{\lambda(F_k)} = \infty,$$

since $0 \le \lambda(\overline{F}_{k+1})/\lambda(F_k) < 1$. ∎

First of all we shall study the problem of determining the Lebesgue measure of $C[\overline{\mathrm{O}}^1, V] = C[\overline{\mathrm{O}}^1, \{V_n\}]$ with $V_n = V$, a fixed proper subset of positive integers. The sets $C[\overline{\mathrm{O}}^1, V]$ with

(1) $V = \{1, \dots, m\}$,
(2) $V = \{m+1, m+2, \dots\}$,
(3) $V = \{1, 3, 5, \dots\}$

are the simplest among $C[\overline{\mathrm{O}}^1, V]$.

Let us solve the first problem in a more general setting.

Theorem 1. *If V_k contains n_k symbols $(k \in \mathbb{N})$ and*

$$\varliminf_{k\to\infty} \frac{n_1 \dots n_k}{(k+1)!} = 0$$

then $\lambda(C[\overline{\mathrm{O}}^1, \{V_k\}]) = 0$.

Proof. From the properties of cylindrical sets it follows that

$$\lambda(F_k) = \sum_{\substack{v_i \in V_i \\ i=\overline{1,k}}} |\overline{\mathrm{O}}^1_{[v_1 \dots v_k]}| \le \frac{n_1 \dots n_k}{(k+1)!}$$

and

$$\lambda(C[\overline{\mathrm{O}}^1, \{V_k\}]) = \lim_{k\to\infty} \lambda(F_k) \le \varliminf_{k\to\infty} \frac{n_1 \dots n_k}{(k+1)!} = 0. \ \blacksquare$$

Corollary. *If $n_k \le m$ (for any $k \in \mathbb{N}$) for some fixed m, then we have $\lambda(C[\overline{\mathrm{O}}^1, \{V_k\}]) = 0$.*

Theorem 2. *Let $V_k = \{1, \dots, m_k\}$, $m_k \in \mathbb{N}$. If $\sum_{k=1}^{\infty} 1/m_k = \infty$, then $\lambda(C[\overline{\mathrm{O}}^1, \{V_n\}]) = 0$.*

Proof. Let $\overline{\mathrm{O}}^1_{[c_1 \dots c_k]}$ be a fixed cylindrical set of rank k. Then

$$\sum_{c \notin V_{k+1}} |\overline{\mathrm{O}}^1_{(c_1 \dots c_k c)}| = \frac{1}{\sigma_1 \dots \sigma_k} \sum_{c=m_{k+1}+1}^{\infty} \frac{1}{(\sigma_k + c)(\sigma_k + c + 1)}$$
$$= \frac{1}{\sigma_1 \dots \sigma_k (\sigma_k + m_{k+1} + 1)}.$$

Since

$$\frac{1}{\sigma_k + m_{k+1} + 1} > \frac{1}{(m_{k+1}+1)(\sigma_k + 1)},$$

we have

$$\sum_{c \notin V_{k+1}} |\overline{\mathrm{O}}^1_{(c_1 \dots c_k c)}| > \frac{1}{m_{k+1}+1} |\overline{\mathrm{O}}^1_{[c_1 \dots c_k]}|.$$

Summing over all $c_1 \in V_1, \ldots, c_k \in V_k$, we have

$$\lambda(\overline{F}_{k+1}) > \frac{1}{m_{k+1}+1}\,\lambda(F_k), \quad \text{i.e.,} \quad \frac{\lambda(\overline{F}_{k+1})}{\lambda(F_k)} > \frac{1}{m_{k+1}+1}$$

for any $k \in \mathbb{N}$, and the statement follows directly from Lemma 3. ∎

Let E be the set of all real numbers with bounded $\overline{\mathrm{O}}^1$-symbols, i.e., $x \in E$ iff there exists a constant K_x such that $g_k(x) \le K_x$ for all $k \in \mathbb{N}$.

THEOREM 3. *The Lebesgue measure of E is* 0.

Proof. For $m \in \mathbb{N}$, consider the set $E_m = \{x : g_k(x) \le m, \forall k \in \mathbb{N}\}$ of uniformly m-bounded symbols. It is not hard to see that $E_m = C[\overline{\mathrm{O}}^1, \{V_k\}]$ with $V_k = \{1, \ldots, m\}$. From Theorem 2 it follows that $\lambda(C[\overline{\mathrm{O}}^1, \{V_k\}]) = 0$.
Since $E = \bigcup_{m=1}^{\infty} E_m$ and $\lambda(E_m) = 0$, we have the desired conclusion. ∎

COROLLARY. *For Lebesgue almost all $x \in [0,1]$,*

$$\varlimsup_{k\to\infty} g_k(x) = \infty.$$

Now consider the case where $V_k = \{v_k+1, v_k+2, \ldots\}$ and $\{v_k\}$ is a fixed sequence of positive integers.

LEMMA 4. *Let $\overline{\mathrm{O}}^1_{[c_1\ldots c_n]}$ be a fixed cylindrical set or, if $n = 0$, the unit interval $[0,1]$; let $\{v_k\}$ be a fixed sequence of positive integers, let $V_k = \{v_k + 1, v_k + 2, \ldots\}$, and let*

$$F_k^{c_1\ldots c_n} := F_{n+k} \cap \overline{\mathrm{O}}^1_{[c_1\ldots c_n]} = \bigcup_{c_{n+1}>v_{n+1}} \cdots \bigcup_{c_{n+k}>v_{n+k}} \overline{\mathrm{O}}^1_{[c_1\ldots c_n c_{n+1}\ldots c_{n+k}]},$$

$$\overline{F}_{k+1}^{c_1\ldots c_n} := F_k^{c_1\ldots c_n} \setminus F_{k+1}^{c_1\ldots c_n} = \bigcup_{c_{n+1}>v_{n+1}} \cdots \bigcup_{c_{n+k}>v_{n+k}} \bigcup_{s=1}^{v_{n+k+1}} \overline{\mathrm{O}}^1_{(c_1\ldots c_n c_{n+1}\ldots c_{n+k} s)}.$$

Then

$$\frac{\lambda(\overline{F}_{k+1}^{c_1\ldots c_n})}{\lambda(\overline{F}_k^{c_1\ldots c_n})} < \frac{1}{2}\cdot\frac{v_{n+k+1}}{v_{n+k}}. \tag{11}$$

Proof. Let $\overline{\mathrm{O}}^1_{(c_1\ldots c_{n+k-1})}$ be a cylindrical interval of rank $n+k-1$. Then

$$\begin{aligned}
\sum_{s\notin V_{n+k}} |\overline{\mathrm{O}}^1_{(c_1\ldots c_{n+k-1}s)}| &= \sum_{s=1}^{v_{n+k}} \frac{1}{\sigma_1\ldots\sigma_{n+k-1}(\sigma_{n+k-1}+s)(\sigma_{n+k-1}+s+1)} \\
&= \frac{1}{\sigma_1\ldots\sigma_{n+k-1}}\left(\frac{1}{\sigma_{n+k-1}+1} - \frac{1}{\sigma_{n+k-1}+v_{n+k}+1}\right) \\
&= \frac{v_{n+k}}{\sigma_1\ldots\sigma_{n+k-1}(\sigma_{n+k-1}+1)(\sigma_{n+k-1}+v_{n+k}+1)}.
\end{aligned}$$

For the same cylindrical interval we have

$$\sum_{l\in V_{n+k}}\sum_{s\notin V_{n+k+1}}|\overline{\mathrm{O}}^1_{(c_1\ldots c_{n+k-1}ls)}|$$

$$\begin{aligned}
&= \sum_{l=v_{n+k}+1}^{\infty}\frac{1}{\sigma_1\ldots\sigma_{n+k-1}(\sigma_{n+k-1}+l)}\bigg(\frac{1}{\sigma_{n+k-1}+l+1}\\
&\quad -\frac{1}{\sigma_{n+k-1}+l+v_{n+k+1}+1}\bigg)\\
&= \frac{1}{\sigma_1\ldots\sigma_{n+k-1}}\sum_{l=v_{n+k}+1}^{\infty}\bigg(\frac{1}{(\sigma_{n+k-1}+l)(\sigma_{n+k-1}+l+1)}\\
&\quad -\frac{1}{(\sigma_{n+k-1}+l)(\sigma_{n+k-1}+l+v_{n+k+1}+1)}\bigg)\\
&= \frac{1}{\sigma_1\ldots\sigma_{n+k-1}}\bigg(\frac{1}{\sigma_{n+k-1}+v_{n+k}+1}\\
&\quad -\frac{1}{1+v_{n+k+1}}\sum_{i=1}^{v_{n+k+1}+1}\frac{1}{\sigma_{n+k-1}+v_{n+k}+i}\bigg)\\
&= \frac{v_{n+k}}{\sigma_1\ldots\sigma_{n+k-1}(\sigma_{n+k-1}+1)(\sigma_{n+k-1}+v_{n+k}+1)}\,X_k.
\end{aligned}$$

Let us estimate the expression

$$\begin{aligned}
X_k &= \frac{(\sigma_{n+k-1}+1)(\sigma_{n+k-1}+v_{n+k}+1)}{v_{n+k}}\bigg(\frac{1}{\sigma_{n+k-1}+v_{n+k}+1}\\
&\quad -\frac{1}{1+v_{n+k+1}}\sum_{i=1}^{1+v_{n+k+1}}\frac{1}{\sigma_{n+k-1}+v_{n+k}+i}\bigg)\\
&= \frac{\sigma_{n+k-1}+1}{v_{n+k}}\bigg(1-\frac{1}{1+v_{n+k+1}}\sum_{i=1}^{1+v_{n+k+1}}\frac{\sigma_{n+k-1}+v_{n+k}+1}{\sigma_{n+k-1}+v_{n+k}+i}\bigg)\\
&= \frac{\sigma_{n+k-1}+1}{v_{n+k}}\bigg(1-\frac{1}{1+v_{n+k+1}}\sum_{i=1}^{1+v_{n+k+1}}\bigg(1-\frac{i-1}{\sigma_{n+k-1}+v_{n+k}+i}\bigg)\bigg)\\
&= \frac{\sigma_{n+k-1}+1}{v_{n+k}}\cdot\frac{1}{1+v_{n+k+1}}\sum_{i=2}^{1+v_{n+k+1}}\frac{i-1}{\sigma_{n+k-1}+v_{n+k}+i}.
\end{aligned}$$

Now let us estimate the sum

$$\frac{1}{n_0+1}+\frac{2}{n_0+2}+\cdots+\frac{m_k}{n_0+m_k},$$

where n_0 and $m_k > 1$ are positive integers. Let

$$C_k := \frac{1}{n_0+1} + \frac{1}{n_0+2} + \cdots + \frac{1}{n_0+m_k}$$

and consider the matrix

$$\begin{bmatrix} \frac{1}{n_0+1} & \frac{1}{n_0+2} & \frac{1}{n_0+3} & \cdots & \frac{1}{n_0+m_k} \\ \frac{1}{n_0+1} & \frac{1}{n_0+2} & \frac{1}{n_0+3} & \cdots & \frac{1}{n_0+m_k} \\ \frac{1}{n_0+1} & \frac{1}{n_0+2} & \frac{1}{n_0+3} & \cdots & \frac{1}{n_0+m_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n_0+1} & \frac{1}{n_0+2} & \frac{1}{n_0+3} & \cdots & \frac{1}{n_0+m_k} \end{bmatrix}.$$

The sum of its elements is $m_k C_k$. The sum of all elements on the main diagonal is C_k. The sum of all elements above the diagonal is less than the sum of those below the diagonal (for any element above the diagonal, the symmetrical element is greater).

The sum of all off-diagonal elements is $(m_k - 1)C_k$. So, the sum of all elements above the diagonal is less than $(m_k - 1)C_k/2$, and the sum of all elements above or on the diagonal is equal to

$$\frac{1}{n_0+1} + \frac{2}{n_0+2} + \cdots + \frac{m_k}{n_0+m_k} < \frac{m_k-1}{2}\,C_k + C_k = \frac{m_k+1}{2}\,C_k.$$

So,

$$\frac{1}{n_0+1} + \frac{2}{n_0+2} + \cdots + \frac{m_k}{n_0+m_k} < \frac{m_k+1}{2}\left(\frac{1}{n_0+1} + \frac{1}{n_0+2} + \cdots + \frac{1}{n_0+m_k}\right).$$

Therefore,

$$\begin{aligned} X_k &= \frac{\sigma_{n+k-1}+1}{v_{n+k}} \cdot \frac{1}{1+v_{n+k+1}} \sum_{i=1}^{v_{n+k+1}} \frac{i}{(\sigma_{n+k-1}+v_{n+k}+1)+i} \\ &< \frac{\sigma_{n+k-1}+1}{v_{n+k}} \cdot \frac{1}{1+v_{n+k+1}} \cdot \frac{v_{n+k+1}+1}{2} \sum_{i=1}^{v_{n+k+1}} \frac{1}{\sigma_{n+k-1}+v_{n+k}+i+1} \\ &= \frac{1}{2v_{n+k}} \sum_{i=1}^{v_{n+k+1}} \frac{\sigma_{n+k-1}+1}{\sigma_{n+k-1}+v_{n+k}+i+1} < \frac{1}{2} \cdot \frac{v_{n+k+1}}{v_{n+k}}. \end{aligned}$$

So,

$$\sum_{l \in V_{n+k}} \sum_{s \notin V_{n+k+1}} |\overline{\mathrm{O}}^1_{(c_1 \ldots c_{n+k-1} l s)}| < \frac{1}{2} \cdot \frac{v_{n+k+1}}{v_{n+k}} \sum_{l \notin V_{n+k}} |\overline{\mathrm{O}}^1_{(c_1 \ldots c_{n+k-1} l)}|.$$

Summing over all $c_{n+1} \in V_{n+1}, \dots, c_{n+k-1} \in V_{n+k-1}$, we have

$$\lambda(\overline{F}_{k+1}^{c_1 \dots c_n}) < \frac{1}{2} \cdot \frac{v_{n+k+1}}{v_{n+k}} \lambda(\overline{F}_k^{c_1 \dots c_n}),$$

which proves the lemma. ∎

Corollary 1. *Let* $V_k = \{v_k + 1, v_k + 2, \dots\}$, $v_k \in \mathbb{N}$. *Then*

$$\lambda(\overline{F}_{k+1}) < \frac{1}{2} \cdot \frac{v_{k+1}}{v_k} \lambda(\overline{F}_k).$$

Corollary 2. *Let* $V_k = V = \{m + 1, m + 2, \dots\}$, $m \in \mathbb{N}$. *Then*

$$\lambda(\overline{F}_{k+1}^{c_1 \dots c_n}) < \frac{1}{2} \lambda(\overline{F}_k^{c_1 \dots c_n})$$

for any positive integer k *and any* $c_1 \in V$, $\dots$, $c_n \in V$, *and therefore*,

$$\lambda(\overline{F}_{k+1}) < \frac{1}{2} \lambda(\overline{F}_k).$$

Theorem 4. *Let* $\{v_k\}$ *be a fixed sequence of positive integers*, *and let*

$$V_k = \{v_k + 1, v_k + 2, \dots\}.$$

If there exists $k_0 \in \mathbb{N}$ *such that*

$$v_{k+1}/v_k \le C_0 < 2 \quad \textit{for any } k > k_0,$$

then $\lambda(C[\overline{\mathrm{O}}^1, \{V_k\}]) > 0$.

Proof. Fix $\overline{\mathrm{O}}^1_{[c_1 \dots c_n]}$ with $n > k_0$ and $c_i \in V_i$. We shall prove that the set

$$\Delta_{c_1 \dots c_n} = C[\overline{\mathrm{O}}^1, \{V_k\}] \cap \overline{\mathrm{O}}^1_{[c_1 \dots c_n]}$$

has positive Lebesgue measure. To this end, consider $\overline{\mathrm{O}}^1_{[c_1 \dots c_{n+1}]}$, $c_{n+1} > v_{n+1}$, and the corresponding subset

$$\Delta_{c_1 \dots c_{n+1}} = C[\overline{\mathrm{O}}^1, \{V_k\}] \cap \overline{\mathrm{O}}^1_{[c_1 \dots c_{n+1}]}.$$

From Lemma 4 it follows that

$$\begin{aligned} \lambda(\overline{F}_{k+1}^{c_1 \dots c_{n+1}}) &< \frac{1}{2} \cdot \frac{v_{n+k+1}}{v_{n+k}} \lambda(\overline{F}_k^{c_1 \dots c_{n+1}}) \le \frac{1}{2} C_0 \lambda(\overline{F}_k^{c_1 \dots c_{n+1}}) \\ &< \frac{1}{2} C_0 \cdot \frac{1}{2} \cdot \frac{v_{n+k}}{v_{n+k-1}} \lambda(\overline{F}_{k-1}^{c_1 \dots c_{n+1}}) \le \left(\frac{C_0}{2}\right)^2 \lambda(\overline{F}_{k-1}^{c_1 \dots c_{n+1}}) \cdots \\ &\le (C_0/2)^k \lambda(\overline{F}_1^{c_1 \dots c_{n+1}}) \end{aligned}$$

for any $k \in \mathbb{N}$. Using Lemma 2, we have

$$\lambda(\overline{F}_1^{c_1 \dots c_{n+1}}) = \sum_{s=1}^{v_{n+2}} |\overline{\mathrm{O}}^1_{(c_1 \dots c_{n+1} s)}| = \frac{v_{n+2}}{\sigma_{n+1} + v_{n+2} + 1} \cdot |\overline{\mathrm{O}}^1_{[c_1 \dots c_{n+1}]}|.$$

So,

$$\begin{aligned}\lambda(\varDelta_{c_1\dots c_{n+1}}) &= |\overline{\mathrm{O}}^1_{[c_1\dots c_{n+1}]}| - \sum_{k=1}^{\infty}\lambda(\overline{F}^{c_1\dots c_{n+1}}_k)\\ &> |\overline{\mathrm{O}}^1_{[c_1\dots c_{n+1}]}| - \sum_{k=1}^{\infty}(C_0/2)^{k-1}\lambda(\overline{F}^{c_1\dots c_{n+1}}_1)\\ &= |\overline{\mathrm{O}}^1_{[c_1\dots c_{n+1}]}|\cdot\left(1-\frac{2}{2-C_0}\cdot\frac{v_{n+2}}{\sigma_{n+1}+v_{n+2}+1}\right).\end{aligned}$$

Since the numbers $c_1,\dots,c_n$, v_{n+2}, C_0 are fixed, and $c_{n+1} > v_{n+1}$, there exists $c^* \in \mathbb{N}$ such that

$$1-\frac{2}{2-C_0}\cdot\frac{v_{n+2}}{\sigma_{n+1}+v_{n+2}+1} > 0$$

for any $c_{n+1} > c^*$. Hence, $\lambda(\varDelta_{c_1\dots c_{n+1}}) > 0$ for any $c_{n+1} > c^*$, and therefore,

$$\lambda(C[\overline{\mathrm{O}}^1,\{V_k\}]) > \lambda(\varDelta_{c_1\dots c_n}) > \lambda(\varDelta_{c_1\dots c_{n+1}}) > 0. \ \blacksquare$$

Corollary 1. *Let $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x^1 + a_0$ with $n \in \mathbb{N}$, $a_i \in \mathbb{Z}$ and $P_n(x) > 0$ for any $x \in \mathbb{N}$. If $v_k = P_n(k)$, then $\lambda(C[\overline{\mathrm{O}}^1,\{V_k\}]) > 0$.*

Corollary 2. *If the sequence $\{v_k\}$ is bounded, then $\lambda(C[\overline{\mathrm{O}}^1,\{V_k\}]) > 0$.*

Remark. Let us compare Theorem 4 with the corresponding proposition from the theory of continued fractions. Let $C[\text{c.f.},\{V_n\}]$ be the closure of the set of all real numbers

$$x = [a_1(x), a_2(x), \dots],$$

whose continued fraction's elements $a_n(x)$ satisfy $a_n(x) \in V_n$ for any $n \in \mathbb{N}$ (here $\{V_n\}$ is a fixed sequence of nonempty subsets of $\mathbb{N}$ as above). For example, if $V_n = V = \mathbb{N}\setminus\{1\}$ for any $n \in \mathbb{N}$, then $\lambda(C[\text{c.f.},\{V_n\}]) = 0$ (see, e.g., [6, 12]), but $\lambda(C[\overline{\mathrm{O}}^1,\{V_n\}]) > 0$. So, Theorem 4 indicates an essential difference between the metric theories of continued fractions and $\overline{\mathrm{O}}^1$-representations.

Theorem 5. *Let $m \in \mathbb{N}$ and $V = \mathbb{N}\setminus\{1,\dots,m\}$. Then*

$$\lambda(C[\overline{\mathrm{O}}^1, V]) > \frac{1}{(m+1)^2}. \tag{12}$$

Proof. Consider an arbitrary cylindrical set $\overline{\mathrm{O}}^1_{[c_1]}$ such that $c_1 \in V$. From Corollary 2 to Lemma 4 it follows that

$$\lambda(\overline{F}^{c_1}_{k+1}) < \frac{1}{2^k}\,\lambda(\overline{F}^{c_1}_1).$$

So, we have

$$\lambda(\Delta_{c_1}) = |\overline{\mathrm{O}}^1_{[c_1]}| - \sum_{k=1}^{\infty} \lambda(\overline{F}^{c_1}_k) > |\overline{\mathrm{O}}^1_{[c_1]}| - \lambda(\overline{F}^{c_1}_1) \sum_{k=0}^{\infty} \frac{1}{2^k} = |\overline{\mathrm{O}}^1_{[c_1]}| - 2\lambda(\overline{F}^{c_1}_1).$$

Since

$$\lambda(\overline{F}^{c_1}_1) = \sum_{c=1}^{m} |\overline{\mathrm{O}}^1_{(c_1 c)}| = \frac{m}{c_1+m+1}\,|\overline{\mathrm{O}}^1_{[c_1]}| \le \frac{m}{2m+2}\,|\overline{\mathrm{O}}^1_{[c_1]}|,$$

it follows that

$$\lambda(\Delta_{c_1}) > \frac{1}{m+1}\,|\overline{\mathrm{O}}^1_{[c_1]}|.$$

So,

$$\lambda(C[\overline{\mathrm{O}}^1, V]) = \sum_{c_1=m+1}^{\infty} \lambda(\Delta_{c_1}) > \frac{1}{m+1} \sum_{c_1=m+1}^{\infty} |\overline{\mathrm{O}}^1_{[c_1]}| = \frac{1}{(m+1)^2}. \;\blacksquare$$

Finally, consider a more general case where $V_k = V = \mathbb{N} \setminus \{a_1, a_2, \dots\}$ and $\{a_n\}$ is an arbitrary increasing sequence of positive integers.

THEOREM 6. *Let $\{a_n\}$ be an increasing sequence of positive integers with $a_{n+1} - a_n \le d$ for some fixed positive integer $d \ge 2$, and any $n \in \mathbb{N}$. If $V_k = V = \mathbb{N} \setminus \{a_1, a_2, \dots\}$, then $\lambda(C[\overline{\mathrm{O}}^1, V]) = 0$.*

Proof. Fix a cylindrical set $\overline{\mathrm{O}}^1_{[c_1 \dots c_k]}$. Then

$$\sum_{c \notin V} |\overline{\mathrm{O}}^1_{(c_1 \dots c_k c)}| = \frac{1}{\sigma_1 \dots \sigma_k} \sum_{n=1}^{\infty} \frac{1}{(\sigma_k + a_n)(\sigma_k + a_n + 1)}$$

$$> \frac{1}{\sigma_1 \dots \sigma_k} \sum_{n=1}^{\infty} \frac{1}{(\sigma_k + a'_n)(\sigma_k + a'_n + d)} = \frac{1}{d} \cdot \frac{1}{\sigma_1 \dots \sigma_k (\sigma_k + a_1)},$$

where $a'_1 = a_1$ and $a'_{n+1} = a'_n + d \ge a_{n+1}$ for any positive integer n. Since

$$\frac{1}{\sigma_k + a_1} \ge \frac{1}{a_1(\sigma_k + 1)},$$

we have

$$\sum_{c \notin V} |\overline{\mathrm{O}}^1_{(c_1 \dots c_k c)}| > \frac{1}{a_1 d}\,|\overline{\mathrm{O}}^1_{[c_1 \dots c_k]}|.$$

Summing over all $c_1 \in V, \dots, c_k \in V$, we have

$$\lambda(\overline{F}_{k+1}) > \frac{1}{a_1 d}\,\lambda(F_k), \quad \text{i.e.,} \quad \frac{\lambda(\overline{F}_{k+1})}{\lambda(F_k)} > \frac{1}{a_1 d}$$

for any $k \in \mathbb{N}$, and the statement follows directly from Lemma 3. $\blacksquare$

COROLLARY 1. *If $V_k = V = \{b_1, b_2, \dots\}$ with $b_{n+1} - b_n \ge 2$, then $\lambda(C[\overline{\mathrm{O}}^1, V]) = 0$.*

COROLLARY 2. *If $V = \{1, 3, 5, \dots\}$ or $V = \{2, 4, 6, \dots\}$ then $\lambda(C[\overline{\mathrm{O}}^1, V]) = 0$.*

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References

[1] S. Albeverio, M. Pratsiovytyi, and G. Torbin, *Singular probability distributions and fractal properties of sets of real numbers defined by the asymptotic frequencies of their s-adic digits*, Ukrain. Math. J. 57 (2005), 1361–1370.

[2] —, —, —, *Topological and fractal properties of real numbers which are not normal*, Bull. Sci. Math. 129 (2005), 615–630.

[3] O. M. Baranovskyi, *On some problems of metric theory of numbers defined by the first Ostrogradsky series*, Trans. National Pedagog. Univ. of Ukraine Phys.-Math. Sci. 3 (2002), 391–402 (in Ukrainian).

[4] P. I. Bodnarchuk and V. Ya. Skorobogat'ko, *Branching Continued Fractions and Their Applications*, Naukova Dumka, Kyiv, 1974 (in Ukrainian).

[5] S. Kalpazidou, A. Knopfmacher, and J. Knopfmacher, *Lüroth-type alternating series representations for real numbers*, Acta Arith. 55 (1990), 311–322.

[6] A. Ya. Khintchine, *Continued Fractions*, Noordhoff, Groningen, 1963.

[7] Yu. V. Melnichuk, *p-adic continued fractions, generated by algorithms of Euclid and Ostrogradsky*, in: Computational Mathematics in Modern Scientific and Technical Progress (Kanev, 1974), 259–265 (in Russian).

[8] —, *On representation of real numbers by fast converging series*, in: Continued Fractions and Their Applications, Inst. Mat., Akad. Nauk Ukrain. SSR, Kiev, 1976, 77–78 (in Russian).

[9] L. Olsen, *Applications of multifractal divergence points to sets of numbers defined by their N-adic expansion*, Math. Proc. Cambridge Philos. Soc. 136 (2004), 139–165.

[10] —, *Applications of multifractal divergence points to some sets of d-tuples of numbers defined by their N-adic expansion*, Bull. Sci. Math. 128 (2004), 265–289.

[11] T. A. Pierce, *On an algorithm and its use in approximating roots of algebraic equations*, Amer. Math. Monthly 36 (1929), 523–525.

[12] M. V. Pratsiovytyi, *Fractal approach to investigations of singular probability distributions*, National Pedagogical Univ., Kyiv, 1998 (in Ukrainian).

[13] M. V. Pratsiovytyi and O. M. Baranovskyi, *Applications of the Ostrogradsky series for the analytical description of distributions of random variables and mappings*, in: Dynamical Systems: Proc. Ukrainian Mathematical Congress–2001, Inst. Math. NASU, Kyiv, 2003, 59–76 (in Ukrainian).

[14] —, —, *Properties of distributions of random variables with independent differences of consecutive elements of the Ostrogradskiĭ series*, Teor. Ĭmovīr. Mat. Stat. 2004, no. 70, 131–143 (in Ukrainian); English transl.: Theory Probab. Math. Statist. 2005, no. 70, 147–160.

[15] M. Pratsiovytyi and G. Torbin, *Superfractality of the set of numbers having no frequency of n-adic digits, and fractal probability distributions*, Ukrain. Math. J. 47 (1995), 1113–1118.

[16] E. Ya. Remez, *On series with alternating signs which may be connected with two algorithms of M. V. Ostrogradskiĭ for approximation of irrational numbers*, Uspekhi Mat. Nauk 6 (1951), no. 5, 33–42 (in Russian).

[17] F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford Sci. Publ., Oxford Univ. Press, New York, 1995.
[18] W. Sierpiński, *Sur quelques algorithmes pour développer les nombres réels en séries*, in: Oeuvres choisies, t. I, PWN, Warszawa, 1974, 236–254.
[19] K. G. Valēēv and E. D. Zlēbov, *The metric theory of the Ostrogradskiĭ algorithm*, Ukrain. Math. J. 27 (1975), 47–51.

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