

## Nineteen quaternary quadratic forms

by

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**1. Introduction.** Let  $\mathbb{N}$ ,  $\mathbb{N}_0$ ,  $\mathbb{Z}$  and  $\mathbb{C}$  denote the sets of positive integers, nonnegative integers, integers and complex numbers respectively. Throughout this paper  $q$  denotes a complex variable such that  $|q| < 1$ . For  $a, b, c, d \in \mathbb{N}$  and  $n \in \mathbb{N}_0$  we define

$$(1.1) \quad N(a, b, c, d; n) = \text{card}\{(x, y, z, t) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dt^2\}.$$

Clearly

$$(1.2) \quad N(a, b, c, d; 0) = 1.$$

In this paper we determine in a uniform manner a formula for  $N(a, b, c, d; n)$ , valid for all  $n \in \mathbb{N}$ , for each of the nineteen quaternary quadratic forms given by

$$(1.3) \quad (a, b, c, d) = (1, 1, 1, 1), (1, 1, 1, 4), (1, 1, 2, 2), (1, 1, 3, 3), (1, 1, 3, 12), \\ (1, 1, 4, 4), (1, 1, 6, 6), (1, 1, 12, 12), (1, 2, 2, 4), (1, 2, 3, 6), \\ (1, 3, 3, 4), (1, 3, 4, 12), (1, 4, 4, 4), (1, 4, 6, 6), (1, 4, 12, 12), \\ (2, 2, 3, 3), (2, 2, 3, 12), (3, 3, 4, 4), (3, 4, 4, 12).$$

We show that for each of the forms listed in (1.3), formulae for  $N(a, b, c, d; n)$  ( $n \in \mathbb{N}$ ) can be given in terms of the sum of divisors function

$$(1.4) \quad \sigma(n) = \begin{cases} \sum_{d|n} d, & n \in \mathbb{N}, \\ 0, & n \notin \mathbb{N}, \end{cases}$$

where  $d$  runs through the positive integers dividing  $n$ , and the integers  $a(n)$ , which are defined for  $n \in \mathbb{N}_0$  by

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$$(1.5) \quad \sum_{n=0}^{\infty} a(n)q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{4n})(1 - q^{6n})(1 - q^{12n}),$$

so that  $a(n) = 0$  for  $n \equiv 0 \pmod{2}$ . The formulae for  $N(a, b, c, d; n)$  for the seven quaternary forms given by

$$(a, b, c, d) = (1, 1, 1, 1), (1, 1, 1, 4), (1, 1, 2, 2), (1, 1, 3, 3), \\ (1, 1, 4, 4), (1, 2, 2, 4), (1, 4, 4, 4)$$

depend only on  $\sigma(n)$ , whereas those for the remaining twelve forms in (1.3) depend upon  $\sigma(n)$  and  $a(n)$ . Numerical evidence suggests that the nineteen forms listed in (1.3) are the only ones satisfying  $1 \leq a \leq b \leq c \leq d \leq 12$ ,  $\gcd(a, b, c, d) = 1$  and  $a, b, c, d \mid 12$  for which  $N(a, b, c, d; n)$  can be expressed in terms of  $\sigma(n)$  and  $a(n)$  for all  $n \in \mathbb{N}$ .

We remark that the  $q$ -series with coefficients  $a(n)$  is a cusp form of weight 2. In terms of Dedekind’s eta function  $\eta(z)$  we have

$$\sum_{n=1}^{\infty} a(n)q^n = \eta(2z)\eta(4z)\eta(6z)\eta(12z),$$

where  $q = e^{2\pi iz}$  and  $\operatorname{Re}(z) > 0$ . It is known that  $a(n)$  is a multiplicative function of  $n$  [25, p. 4853].

It is also convenient for our purposes to define the integers  $b(n)$  ( $n \in \mathbb{N}_0$ ) by

$$(1.6) \quad \sum_{n=0}^{\infty} b(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{10}(1 - q^{6n})^4}{(1 - q^n)^4(1 - q^{4n})^4(1 - q^{12n})^2},$$

and the integers  $c(n)$  ( $n \in \mathbb{N}_0$ ) by

$$(1.7) \quad \sum_{n=0}^{\infty} c(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^4(1 - q^{6n})^{10}}{(1 - q^{3n})^4(1 - q^{4n})^2(1 - q^{12n})^4}.$$

Jacobi’s theta function  $\varphi(q)$  is defined by

$$(1.8) \quad \varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Using Jacobi’s infinite product expansions

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^2(1 - q^{4n})^2}$$

and

$$\varphi(-q) = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^{2n})},$$

we see that

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{10}(1 - q^{6n})^4}{(1 - q^n)^4(1 - q^{4n})^4(1 - q^{12n})^2} = \varphi^2(q)\varphi^2(-q^6),$$

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^4(1 - q^{6n})^{10}}{(1 - q^{3n})^4(1 - q^{4n})^2(1 - q^{12n})^4} = \varphi^2(-q^2)\varphi^2(q^3),$$

so that for  $n \in \mathbb{N}_0$ ,

$$b(n) = \sum_{\substack{(x,y,z,t) \in \mathbb{Z}^4 \\ x^2+y^2+6z^2+6t^2=n}} (-1)^{z+t}, \quad c(n) = \sum_{\substack{(x,y,z,t) \in \mathbb{Z}^4 \\ 2x^2+2y^2+3z^2+3t^2=n}} (-1)^{x+y}.$$

Hence  $b(n)$  and  $c(n)$  can be viewed as “twists” of  $N(1, 1, 6, 6; n)$  and  $N(2, 2, 3, 3; n)$  respectively.

The values of  $a(n)$ ,  $b(n)$  and  $c(n)$  for  $n \in \{0, 1, 2, \dots, 20\}$  are given in Table 1.

**Table 1**

$n$	$a(n)$	$b(n)$	$c(n)$
0	0	1	1
1	1	4	0
2	0	4	-4
3	-1	0	4
4	0	4	4
5	-2	8	-16
6	0	-4	4
7	0	-16	16
8	0	-12	-12
9	1	4	0
10	0	-8	8
11	4	-32	16
12	0	4	4
13	-2	24	-32
14	0	0	0
15	2	-16	8
16	0	-12	-12
17	2	40	-32
18	0	4	-4
19	-4	-32	48
20	0	24	24

We now describe how the formulae for  $N(a, b, c, d; n)$  for the nineteen quaternary quadratic forms  $ax^2 + by^2 + cz^2 + dt^2$  listed in (1.3) are proved. From (1.8) we see that

$$(1.9) \quad \sum_{n=0}^{\infty} N(a, b, c, d; n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d).$$

Following [4, p. 178] we set

$$(1.10) \quad p = p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)},$$

$$(1.11) \quad k = k(q) = \frac{\varphi^3(q^3)}{\varphi(q)},$$

which, as functions in the variable  $z$ , are modular forms of weights 0 and 2. We begin by determining the series expansions of  $k^2$ ,  $pk^2$ ,  $p^2k^2$ ,  $p^3k^2$  and  $p^4k^2$  in powers of  $q$ . We prove the following theorem in Section 2.

**THEOREM 1.1.**

- (a)  $k^2 = 1 + \sum_{n=1}^{\infty} (-4\sigma(n) + 24\sigma(n/2) - 4\sigma(n/3) - 16\sigma(n/4) - 8\sigma(n/6) - 16\sigma(n/12))q^n.$
- (b)  $pk^2 = \sum_{n=1}^{\infty} (2\sigma(n) - 12\sigma(n/2) + 6\sigma(n/3) + 8\sigma(n/4) + 4\sigma(n/6) - 8\sigma(n/12))q^n.$
- (c)  $p^2k^2 = \sum_{n=1}^{\infty} (4\sigma(n/2) - 8\sigma(n/3) + 4\sigma(n/6))q^n.$
- (d)  $p^3k^2 = \sum_{n=1}^{\infty} (8\sigma(n/3) - 8\sigma(n/4) - 8\sigma(n/6) + 8\sigma(n/12))q^n.$
- (e)  $p^4k^2 = \sum_{n=1}^{\infty} (16\sigma(n/4) - 32\sigma(n/6) + 16\sigma(n/12))q^n.$

The following corollary follows immediately from Theorem 1.1.

**COROLLARY 1.1.**

- (a)  $(1 + 2p)^2k^2 = 1 + \sum_{n=1}^{\infty} (4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4) + 24\sigma(n/6) - 48\sigma(n/12))q^n.$
- (b)  $(1 + 2p)^3k^2 = 1 + \sum_{n=1}^{\infty} (8\sigma(n) - 32\sigma(n/4))q^n.$

$$(c) \quad (1 + 2p)(1 + p + p^2)k^2 = 1 + \sum_{n=1}^{\infty} (2\sigma(n) + 6\sigma(n/3) - 8\sigma(n/4) - 24\sigma(n/12))q^n.$$

$$(d) \quad (1 - p)(1 + p)(1 + 2p)k^2 = 1 + \sum_{n=1}^{\infty} (-4\sigma(n/2) + 16\sigma(n/4) + 12\sigma(n/6) - 48\sigma(n/12))q^n.$$

Then in Section 3 we determine the series expansions in powers of  $q$  of other functions of  $p$  and  $k$  for which the coefficients of the powers of  $q$  can also be expressed in terms of  $\sigma$ . Recall that the Legendre–Jacobi–Kronecker symbol for discriminant  $-4$  is defined for  $n \in \mathbb{N}$  by

$$(1.12) \quad \left(\frac{-4}{n}\right) = \begin{cases} (-1)^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

THEOREM 1.2.

$$(a) \quad (1 - p)^{3/2}(1 + p)^{1/2}(1 + 2p)^{3/2}k^2 = 1 + \sum_{n=1}^{\infty} (-8\sigma(n/2) + 48\sigma(n/4) - 64\sigma(n/8))q^n.$$

$$(b) \quad (1 - p)^{1/2}(1 + p)^{3/2}(1 + 2p)^{1/2}k^2 = 1 + \sum_{n=1}^{\infty} (-8\sigma(n/6) + 48\sigma(n/12) - 64\sigma(n/24))q^n.$$

$$(c) \quad (1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{9/4}k^2 = 1 + \sum_{n=1}^{\infty} \left(4\left(\frac{-4}{n}\right)\sigma(n) - 8\sigma(n/4) + 48\sigma(n/8) - 64\sigma(n/16)\right)q^n.$$

$$(d) \quad (1 - p)^{9/4}(1 + p)^{3/4}(1 + 2p)^{3/4}k^2 = 1 + \sum_{n=1}^{\infty} \left(-4\left(\frac{-4}{n}\right)\sigma(n) - 8\sigma(n/4) + 48\sigma(n/8) - 64\sigma(n/16)\right)q^n.$$

In Section 4 we determine each of the series  $\sum_{n=0}^{\infty} a(n)q^n$ ,  $\sum_{n=0}^{\infty} b(n)q^n$  and  $\sum_{n=0}^{\infty} c(n)q^n$  in terms of  $p$  and  $k$ .

THEOREM 1.3.

$$(a) \quad \frac{1}{4}p(2 + p)(1 - p)^{1/2}(1 + p)^{1/2}(1 + 2p)^{1/2}k^2 = \sum_{n=1}^{\infty} a(n)q^n.$$

$$(b) \quad (1 - p)^{1/4}(1 + p)^{3/4}(1 + 2p)^{7/4}k^2 = \sum_{n=0}^{\infty} b(n)q^n.$$

$$\begin{aligned}
 \text{(c)} \quad & (1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2 = \sum_{n=0}^{\infty} c(n)q^n. \\
 \text{(d)} \quad & (1-p)^{7/4}(1+p)^{5/4}(1+2p)^{1/4}k^2 = \sum_{n=0}^{\infty} b(n)(-1)^n q^n. \\
 \text{(e)} \quad & (1-p)^{5/4}(1+p)^{7/4}(1+2p)^{3/4}k^2 = \sum_{n=0}^{\infty} c(n)(-1)^n q^n.
 \end{aligned}$$

The next theorem shows that each of the quantities

$$p^j(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2, \quad j \in \{0, 1, 2\},$$

has a power series expansion in  $q$  with coefficients depending only on  $\sigma$  and  $a$ . This result is proved in Section 5.

**THEOREM 1.4.**

$$\begin{aligned}
 \text{(a)} \quad & (1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 \\
 & = 1 + \sum_{n=1}^{\infty} (2\sigma(n/2) - 12\sigma(n/4) - 10\sigma(n/6) + 16\sigma(n/8) \\
 & \quad + 60\sigma(n/12) - 80\sigma(n/24) - 2a(n))q^n. \\
 \text{(b)} \quad & p(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 \\
 & = \sum_{n=1}^{\infty} (-2\sigma(n/2) + 12\sigma(n/4) + 2\sigma(n/6) - 16\sigma(n/8) \\
 & \quad - 12\sigma(n/12) + 16\sigma(n/24) + 2a(n))q^n. \\
 \text{(c)} \quad & p^2(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 \\
 & = \sum_{n=1}^{\infty} (4\sigma(n/2) - 24\sigma(n/4) - 4\sigma(n/6) + 32\sigma(n/8) \\
 & \quad + 24\sigma(n/12) - 32\sigma(n/24))q^n.
 \end{aligned}$$

As

$$\begin{aligned}
 (1-p)^{1/2}(1+p)^{1/2}(1+2p)^{3/2} &= (1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2} \\
 & \quad + 2p(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}, \\
 (1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2} &= (1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2} \\
 & \quad + 3p(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2} \\
 & \quad + 2p^2(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2}, \\
 (1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2} &= (1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2} \\
 & \quad - p(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{1/2},
 \end{aligned}$$

we obtain the following result from Theorem 1.4.

COROLLARY 1.2.

- (a)  $(1 - p)^{1/2}(1 + p)^{3/2}(1 + 2p)^{3/2}k^2$   
 $= 1 + \sum_{n=1}^{\infty} (4\sigma(n/2) - 34\sigma(n/4) - 12\sigma(n/6) + 32\sigma(n/8)$   
 $\qquad\qquad\qquad + 72\sigma(n/12) - 96\sigma(n/24) + 4a(n))q^n.$
- (b)  $(1 - p)^{1/2}(1 + p)^{1/2}(1 + 2p)^{3/2}k^2$   
 $= 1 + \sum_{n=1}^{\infty} (-2\sigma(n/2) + 12\sigma(n/4) - 6\sigma(n/6) - 16\sigma(n/8)$   
 $\qquad\qquad\qquad + 36\sigma(n/12) - 48\sigma(n/24) + 2a(n))q^n.$
- (c)  $(1 - p)^{3/2}(1 + p)^{1/2}(1 + 2p)^{1/2}k^2$   
 $= 1 + \sum_{n=1}^{\infty} (4\sigma(n/2) - 24\sigma(n/4) - 12\sigma(n/6) + 32\sigma(n/8)$   
 $\qquad\qquad\qquad + 72\sigma(n/12) - 96\sigma(n/24) - 4a(n))q^n.$

Our next theorem, which is proved in [2, Theorem 2.4], expresses each of  $\varphi(q^k)$  ( $k \in \{1, 2, 3, 4, 6, 12\}$ ) in terms of  $p$  and  $k$ .

THEOREM 1.5.

- (a)  $\varphi(q) = (1 + 2p)^{3/4}k^{1/2},$
- (b)  $\varphi(q^2) = \frac{1}{\sqrt{2}} ((1 + 2p)^{3/2} + (1 - p)^{3/2}(1 + p)^{1/2})^{1/2}k^{1/2},$
- (c)  $\varphi(q^3) = (1 + 2p)^{1/4}k^{1/2},$
- (d)  $\varphi(q^4) = \frac{1}{2} ((1 + 2p)^{3/4} + (1 - p)^{3/4}(1 + p)^{1/4})k^{1/2},$
- (e)  $\varphi(q^6) = \frac{1}{\sqrt{2}} ((1 + 2p)^{1/2} + (1 - p)^{1/2}(1 + p)^{3/2})^{1/2}k^{1/2},$
- (f)  $\varphi(q^{12}) = \frac{1}{2} ((1 + 2p)^{1/4} + (1 - p)^{1/4}(1 + p)^{3/4})k^{1/2}.$

From Theorem 1.5(b)(e) we deduce

$$(1.13) \quad \varphi(q^2)\varphi(q^6) = \frac{1}{2} (1 + p + p^2 + (1 - p)^{1/2}(1 + p)^{1/2}(1 + 2p)^{1/2})k.$$

We are now in a position to determine a formula for  $N(a, b, c, d; n)$  ( $n \in \mathbb{N}$ ) for each of the nineteen forms given in (1.3). For each  $(a, b, c, d)$  in (1.3) we use (1.9) and Theorem 1.5 to express  $\sum_{n=0}^{\infty} N(a, b, c, d; n)q^n$  as a function of  $p$  and  $k$ . Then, using Theorems 1.1–1.4 and Corollaries 1.1 and 1.2, we express this function in terms of  $\sum_{n=0}^{\infty} \sigma(n)q^{kn}$  ( $k \in \{1, 2, 3, 4, 6, 12\}$ ),  $\sum_{n=0}^{\infty} a(n)q^n$ ,  $\sum_{n=0}^{\infty} b(n)q^n$  and  $\sum_{n=0}^{\infty} c(n)q^n$ . The formula for  $N(a, b, c, d; n)$  is then found by equating coefficients of  $q^n$  ( $n \in \mathbb{N}$ ). Each such formula depends at most on  $\sigma$ ,  $a$ ,  $b$  and  $c$ . The details are carried out

in Section 6. The formulae are given in Theorems 1.6–1.24. In the course of the proofs of these theorems, we obtain some relationships between  $a(n)$ ,  $b(n)$  and  $c(n)$ . These are stated in Theorem 1.25. Finally, we use Theorem 1.25 to give all the formulae of Theorems 1.6–1.24 in terms only of  $\sigma$  and  $a$ . The reader is advised that the final form of those theorems marked with an asterisk is given after Theorem 1.25.

**THEOREM 1.6.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha N$ , where  $\alpha \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 2) = 1$ . Then*

$$N(1, 1, 1, 1; n) = \begin{cases} 8\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

This is Jacobi’s famous formula for the number of representations of a positive integer as the sum of four squares.

**THEOREM 1.7.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha N$ , where  $\alpha \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 2) = 1$ . Then*

$$N(1, 1, 1, 4; n) = \begin{cases} (4 + 2(-1)^{(N-1)/2})\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 12\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

This formula was stated without proof by Liouville [20] and proved by Pepin [27].

**THEOREM 1.8.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha N$ , where  $\alpha \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 2) = 1$ . Then*

$$N(1, 1, 2, 2; n) = \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 8\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

This formula was stated without proof by Liouville [17] and proved by Pepin [26], [27], Bachmann [8], and Deutsch [10].

**THEOREM 1.9.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 1, 3, 3; n) = \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 4(2^{\alpha+1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

This formula was stated without proof by Liouville [16] (see also [22]) and proved by Pepin [27], Bachmann [8] and Kloosterman [14].

**THEOREM 1.10.\*** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*



$$N(1, 1, 3, 12; n) = \begin{cases} 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2(2^{\alpha+1} - 3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

Liouville did not consider this form.

**THEOREM 1.11.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha N$ , where  $\alpha \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 2) = 1$ . Then*

$$N(1, 1, 4, 4; n) = \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

This formula was stated without proof by Liouville [18] and proved by Pepin [26], [27] and Bachmann [8, p. 417].

**THEOREM 1.12.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 1, 6, 6; n) = \begin{cases} 2\sigma(N) + 2a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 4(2^\alpha - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

This form has been considered by Liouville [24], Kloosterman [14], Gongadze [11] and Köhler [15].

**THEOREM 1.13.\*** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 1, 12, 12; n) = \begin{cases} \sigma(N) + a(n) + \frac{1}{2}b(n) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \\ 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

This form was not considered by Liouville.

**THEOREM 1.14.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha N$ , where  $\alpha \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 2) = 1$ . Then*

$$N(1, 2, 2, 4; n) = \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

This form has been treated by Liouville [21], Pepin [26, p. 177], [27, pp. 31, 42] and Bachmann [8, p. 418].

**THEOREM 1.15.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 2, 3, 6; n) = \begin{cases} (3^{\beta+1} - 2)\sigma(N) + a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 6(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

This form has been considered by Liouville [23], Bachmann [8, p. 423], Griffiths [12], [13], Gongadze [11] and Köhler [15].

**THEOREM 1.16.\*** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 3, 3, 4; n) = \begin{cases} 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2(2^{\alpha+1} - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

This form was not treated by Liouville.

**THEOREM 1.17.\*** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 3, 4, 12; n) = \begin{cases} \sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 1 \pmod{2}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ 6(2^{\alpha-1} - 1)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Elementary remarks on this form were given by Liouville [19] and Pepin [27, p. 24].

**THEOREM 1.18.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha N$ , where  $\alpha \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 2) = 1$ . Then*

$$N(1, 4, 4, 4; n) = \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 2, 3 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

This result was stated by Liouville without proof in [20] and proved by Pepin [27, p. 43].

**THEOREM 1.19.\*** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 4, 6, 6; n) = \begin{cases} \sigma(N) + a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Liouville did not consider this form.

**THEOREM 1.20.\*** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 4, 12, 12; n) = \begin{cases} \frac{1}{2}\sigma(N) + \frac{1}{2}a(n) + \frac{1}{4}b(n) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 2, 3 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Liouville did not consider this form.

**THEOREM 1.21.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(2, 2, 3, 3; n) = \begin{cases} 2\sigma(N) - 2a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 4(2^\alpha - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Partial results were stated by Liouville [24] without proof.

**THEOREM 1.22.\*** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(2, 2, 3, 12; n) = \begin{cases} \sigma(N) - a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Liouville did not consider this form.

**THEOREM 1.23.\*** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(3, 3, 4, 4; n) = \begin{cases} \sigma(N) - a(n) + \frac{1}{2}c(n) & \text{if } n \equiv 3 \pmod{4}, \\ 0 & \text{if } n \equiv 1 \pmod{4}, \\ 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Liouville did not treat this form.

**THEOREM 1.24.\*** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(3, 4, 4, 12; n) = \begin{cases} \frac{1}{2}\sigma(N) - \frac{1}{2}a(n) + \frac{1}{4}c(n) & \text{if } n \equiv 3 \pmod{4}, \\ 0 & \text{if } n \equiv 1, 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

This form was not considered by Liouville.

In the course of proving Theorems 1.6–1.24 we establish the following result.

**THEOREM 1.25.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

- (a)  $b(n) = 2\sigma(N) + 2a(n)$  if  $n \equiv 1 \pmod{4}$ ,
- (b)  $b(n) = -2\sigma(N) - 2a(n)$  if  $n \equiv 3 \pmod{4}$ ,
- (c)  $c(n) = -2\sigma(N) + 2a(n)$  if  $n \equiv 1 \pmod{4}$ ,
- (d)  $c(n) = 2\sigma(N) - 2a(n)$ , if  $n \equiv 3 \pmod{4}$ ,
- (e)  $b(n) = -c(n) = 4a(n/2)$  if  $n \equiv 2 \pmod{4}$ ,
- (f)  $b(n) = c(n) = 4\sigma(N)$  if  $n \equiv 4 \pmod{8}$ ,
- (g)  $b(n) = c(n) = -12\sigma(N)$  if  $n \equiv 0 \pmod{8}$ .

Applying Theorem 1.25 to Theorems 1.10\*, 1.13\*, 1.16\*, 1.17\*, 1.19\*, 1.20\*, 1.22\*, 1.23\* and 1.24\*, we obtain formulae which do not depend upon  $b$  and  $c$ .

**THEOREM 1.10.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 1, 3, 12; n) = \begin{cases} 3\sigma(N) + a(n) & \text{if } n \equiv 1 \pmod{4}, \\ \sigma(N) - a(n) & \text{if } n \equiv 3 \pmod{4}, \\ 2\sigma(N) + 2a(n/2) & \text{if } n \equiv 2 \pmod{4}, \\ 12\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4(2^\alpha - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

**THEOREM 1.13.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 1, 12, 12; n) = \begin{cases} 2\sigma(N) + 2a(n) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \\ 2\sigma(N) + 2a(n/2) & \text{if } n \equiv 2 \pmod{4}, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

**THEOREM 1.16.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 3, 3, 4; n) = \begin{cases} \sigma(N) + a(n) & \text{if } n \equiv 1 \pmod{4}, \\ 3\sigma(N) - a(n) & \text{if } n \equiv 3 \pmod{4}, \\ 2\sigma(N) - 2a(n/2) & \text{if } n \equiv 2 \pmod{4}, \\ 12\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4(2^\alpha - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

**THEOREM 1.17.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 3, 4, 12; n) = \begin{cases} \sigma(N) + a(n) & \text{if } n \equiv 1 \pmod{4}, \\ \sigma(N) - a(n) & \text{if } n \equiv 3 \pmod{4}, \\ 0 & \text{if } n \equiv 2 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 12(2^{\alpha-2} - 1)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

**THEOREM 1.19.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 4, 6, 6; n) = \begin{cases} \sigma(N) + a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2\sigma(N) - 2a(n/2) & \text{if } n \equiv 2 \pmod{4}, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

**THEOREM 1.20.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(1, 4, 12, 12; n) = \begin{cases} \sigma(N) + a(n) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 2, 3 \pmod{4}, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

**THEOREM 1.22.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(2, 2, 3, 12; n) = \begin{cases} \sigma(N) - a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2\sigma(N) + 2a(n/2) & \text{if } n \equiv 2 \pmod{4}, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

**THEOREM 1.23.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(3, 3, 4, 4; n) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod{4}, \\ 2\sigma(N) - 2a(n) & \text{if } n \equiv 3 \pmod{4}, \\ 2\sigma(N) - 2a(n/2) & \text{if } n \equiv 2 \pmod{4}, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

**THEOREM 1.24.** *Let  $n \in \mathbb{N}$ . Set  $n = 2^\alpha 3^\beta N$ , where  $\alpha, \beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . Then*

$$N(3, 4, 4, 12; n) = \begin{cases} 0 & \text{if } n \equiv 1, 2 \pmod{4}, \\ \sigma(N) - a(n) & \text{if } n \equiv 3 \pmod{4}, \\ 4\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

The authors have treated other quaternary quadratic forms in [1]–[4], [6] and [28]. These forms require functions other than  $\sigma(n)$  and  $a(n)$  for their evaluation.

**2. Proof of Theorem 1.1.** We begin by recalling the duplication, triplication and change of sign principles for  $p$  and  $k$ , which were proved in [4, Theorems 9, 10, 11].

THEOREM 2.1 (Duplication principle).

$$p(q^2) = \frac{1 + p - p^2 - ((1 - p)(1 + p)(1 + 2p))^{1/2}}{p^2},$$

$$k(q^2) = \frac{(1 + p - p^2 + ((1 - p)(1 + p)(1 + 2p))^{1/2})k}{2}.$$

THEOREM 2.2 (Triplication principle).

$$p(q^3) = 3^{-1}((-4 - 3p + 6p^2 + 4p^3) + 2^{2/3}(1 - 2p - 2p^2)((1 - p)(1 + 2p)(2 + p))^{1/3} + 2^{1/3}(1 + 2p)((1 - p)(1 + 2p)(2 + p))^{2/3}),$$

$$k(q^3) = 3^{-2}(3 + 2^{2/3}(1 + 2p)((1 - p)(1 + 2p)(2 + p))^{1/3} + 2^{4/3}((1 - p)(1 + 2p)(2 + p))^{2/3})k.$$

THEOREM 2.3 (Change of sign principle).

$$p(-q) = \frac{-p}{1 + p}, \quad k(-q) = (1 + p)^2k.$$

The Eisenstein series  $L(q)$  is defined by

$$(2.1) \quad L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n.$$

We define

$$(2.2) \quad A(q) := 2L(q^2) - L(q) = 1 + 24 \sum_{n=1}^{\infty} (\sigma(n) - 2\sigma(n/2))q^n,$$

$$(2.3) \quad B(q) := 3L(q^3) - L(q) = 2 + 24 \sum_{n=1}^{\infty} (\sigma(n) - 3\sigma(n/3))q^n.$$

*Proof of Theorem 1.1.* It was shown in [7, eq. (3.84)] that

$$(2.4) \quad A(q) = (1 + 14p + 24p^2 + 14p^3 + p^4)k^2.$$

Applying the duplication principle (Theorem 2.1) to (2.4), we obtain

$$(2.5) \quad A(q^2) = \left(1 + 2p + 6p^2 + 5p^3 - \frac{1}{2}p^4\right)k^2.$$

From [7, eq. (3.87)] we have

$$(2.6) \quad B(q) = (2 + 16p + 36p^2 + 16p^3 + 2p^4)k^2.$$

Applying the duplication principle (Theorem 1.1) twice to (2.6), we deduce

$$(2.7) \quad B(q^2) = (2 + 4p + 6p^2 + 4p^3 + 2p^4)k^2,$$

$$(2.8) \quad B(q^4) = \left(2 + 4p - 2p^3 + \frac{1}{2}p^4\right)k^2.$$

From (2.4)–(2.8) we obtain

$$(2.9) \quad k^2 = -\frac{2}{9}A(q) + \frac{4}{9}A(q^2) + \frac{1}{18}B(q) + \frac{1}{9}B(q^2) + \frac{2}{9}B(q^4),$$

$$(2.10) \quad pk^2 = \frac{1}{6}A(q) - \frac{1}{9}A(q^2) - \frac{1}{12}B(q) - \frac{1}{18}B(q^2) + \frac{1}{9}B(q^4),$$

$$(2.11) \quad p^2k^2 = -\frac{1}{9}A(q) + \frac{1}{9}B(q) - \frac{1}{18}B(q^2),$$

$$(2.12) \quad p^3k^2 = \frac{1}{9}A(q) + \frac{1}{9}A(q^2) - \frac{1}{9}B(q) + \frac{1}{9}B(q^2) - \frac{1}{9}B(q^4),$$

$$(2.13) \quad p^4k^2 = -\frac{4}{9}A(q^2) + \frac{4}{9}B(q^2) - \frac{2}{9}B(q^4).$$

Appealing to (2.2) and (2.3) we obtain the assertions of Theorem 1.1.

**3. Proof of Theorem 1.2.** Applying the duplication principle to (2.5) we have

$$(3.1) \quad A(q^4) = \left(\frac{1}{4} + \frac{1}{2}p + \frac{3}{2}p^2 + \frac{5}{4}p^3 - \frac{1}{8}p^4\right)k^2 + \frac{3}{4}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2.$$

From (3.1), (2.5) and (2.2), we obtain

$$\begin{aligned} &(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 \\ &= \frac{4}{3}A(q^4) - \frac{1}{3}A(q^2) = \frac{8}{3}L(q^8) - 2L(q^4) + \frac{1}{3}L(q^2) \\ &= 1 + \sum_{n=1}^{\infty} (-8\sigma(n/2) + 48\sigma(n/4) - 64\sigma(n/8))q^n, \end{aligned}$$

which is part (a).

Next, applying the duplication principle to (3.1), we obtain

$$\begin{aligned}
 (3.2) \quad A(q^8) &= \left( \frac{1}{16} + \frac{1}{8}p + \frac{3}{8}p^2 + \frac{5}{16}p^3 - \frac{1}{32}p^4 \right) k^2 \\
 &\quad + \frac{3}{16} (1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 \\
 &\quad + \frac{3}{8} (1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 \\
 &\quad + \frac{3}{8} (1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2,
 \end{aligned}$$

and to (2.8) we obtain

$$\begin{aligned}
 (3.3) \quad B(q^8) &= \left( \frac{5}{4} + \frac{5}{2}p - \frac{3}{4}p^2 - 2p^3 + \frac{1}{8}p^4 \right) k^2 \\
 &\quad - \frac{3}{8} (1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 \\
 &\quad + \frac{9}{8} (1-p)^{1/2}(1+p)^{3/2}(1+2p)^{1/2}k^2.
 \end{aligned}$$

From (3.3), Theorem 1.1, part (a) and (2.3), we deduce

$$\begin{aligned}
 &(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{1/2}k^2 \\
 &= \frac{8}{9} B(q^8) - \frac{10}{9} k^2 - \frac{20}{9} pk^2 + \frac{2}{3} p^2k^2 + \frac{16}{9} p^3k^2 - \frac{1}{9} p^4k^2 \\
 &\quad + \frac{1}{3} (1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 \\
 &= 1 + \sum_{n=1}^{\infty} (-8\sigma(n/6) + 48\sigma(n/12) - 64\sigma(n/24))q^n,
 \end{aligned}$$

which is part (b).

Next, from (3.2), we obtain

$$\begin{aligned}
 &(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 + (1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2 \\
 &= \frac{8}{3} A(q^8) - \frac{1}{6} k^2 - \frac{1}{3} pk^2 - p^2k^2 - \frac{5}{6} p^3k^2 + \frac{1}{12} p^4k^2 \\
 &\quad - \frac{1}{2} (1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2.
 \end{aligned}$$

Then, appealing to (2.2), Theorem 1.1 and part (a), we deduce

$$\begin{aligned}
 (3.4) \quad &(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 + (1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2 \\
 &= 2 + \sum_{n=1}^{\infty} (-16\sigma(n/4) + 96\sigma(n/8) - 128\sigma(n/16))q^n.
 \end{aligned}$$



Our next objective is to determine

$$(1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{9/4}k^2 - (1 - p)^{9/4}(1 + p)^{3/4}(1 + 2p)^{3/4}k^2.$$

To do this we make use of the following two sums, which are closely related to  $L(q)$ :

$$(3.5) \quad L_{1,4}(q) = \sum_{\substack{n=1 \\ n \equiv 1 \pmod{4}}}^{\infty} \sigma(n)q^n, \quad L_{3,4}(q) = \sum_{\substack{n=1 \\ n \equiv 3 \pmod{4}}}^{\infty} \sigma(n)q^n.$$

These sums have been evaluated by Cheng [9]. From [9, Theorem 3.5.1] we have

$$(3.6) \quad L_{1,4}(q) = \frac{1}{32} (1 - g)(1 + g)^3w^2,$$

$$(3.7) \quad L_{3,4}(q) = \frac{1}{32} (1 - g)^3(1 + g)w^2,$$

where

$$(3.8) \quad g = (1 - x)^{1/4}, \quad 1 - x = \frac{(1 - p)^3(1 + p)}{(1 + 2p)^3}, \quad w = (1 + 2p)^{3/2}k$$

(see [7]). From (3.6) and (3.7) we deduce

$$(3.9) \quad L_{1,4}(q) - L_{3,4}(q) = \frac{1}{8} gw^2 - \frac{1}{8} g^3w^2.$$

By (3.8) we see that

$$(3.10) \quad (1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{9/4}k^2 = gw^2,$$

$$(3.11) \quad (1 - p)^{9/4}(1 + p)^{3/4}(1 + 2p)^{3/4}k^2 = g^3w^2.$$

From (3.5), (3.9), (3.10) and (3.11), we deduce

$$\begin{aligned} & (1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{9/4}k^2 - (1 - p)^{9/4}(1 + p)^{3/4}(1 + 2p)^{3/4}k^2 \\ &= gw^2 - g^3w^2 = 8L_{1,4}(q) - 8L_{3,4}(q) \\ &= 8 \sum_{\substack{n=1 \\ n \equiv 1 \pmod{4}}}^{\infty} \sigma(n)q^n - 8 \sum_{\substack{n=1 \\ n \equiv 3 \pmod{4}}}^{\infty} \sigma(n)q^n, \end{aligned}$$

that is,

$$(3.12) \quad (1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{9/4}k^2 - (1 - p)^{9/4}(1 + p)^{3/4}(1 + 2p)^{3/4}k^2 = 8 \sum_{\substack{n=1 \\ n \equiv 1 \pmod{2}}}^{\infty} (-1)^{(n-1)/2} \sigma(n)q^n.$$

Adding and subtracting (3.4) and (3.12), we obtain

$$\begin{aligned} & (1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{9/4}k^2 \\ &= 1 + \sum_{n=1}^{\infty} \left( 4 \left( \frac{-4}{n} \right) \sigma(n) - 8\sigma(n/4) + 48\sigma(n/8) - 64\sigma(n/16) \right) q^n \end{aligned}$$

and

$$\begin{aligned} & (1 - p)^{9/4}(1 + p)^{3/4}(1 + 2p)^{3/4}k^2 \\ &= 1 + \sum_{n=1}^{\infty} \left( -4 \left( \frac{-4}{n} \right) \sigma(n) - 8\sigma(n/4) + 48\sigma(n/8) - 64\sigma(n/16) \right) q^n, \end{aligned}$$

which are parts (c) and (d).

**4. Proof of Theorem 1.3.** Ramanujan’s discriminant function  $\Delta(q)$  is defined by

$$(4.1) \quad \Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Alaca, Alaca and Williams [5, eqns. (3.28)–(3.33)] have shown that

$$(4.2) \quad \Delta(q) = 2^{-4}p(1 + p)^4(1 - p)^{12}(1 + 2p)^3(2 + p)^3k^{12},$$

$$(4.3) \quad \Delta(q^2) = 2^{-8}p^2(1 + p)^2(1 - p)^6(1 + 2p)^6(2 + p)^6k^{12},$$

$$(4.4) \quad \Delta(q^3) = 2^{-4}p^3(1 + p)^{12}(1 - p)^4(1 + 2p)(2 + p)k^{12},$$

$$(4.5) \quad \Delta(q^4) = 2^{-16}p^4(1 + p)(1 - p)^3(1 + 2p)^3(2 + p)^{12}k^{12},$$

$$(4.6) \quad \Delta(q^6) = 2^{-8}p^6(1 + p)^6(1 - p)^2(1 + 2p)^2(2 + p)^2k^{12},$$

$$(4.7) \quad \Delta(q^{12}) = 2^{-16}p^{12}(1 + p)^3(1 - p)(1 + 2p)(2 + p)^4k^{12}.$$

Hence, by (1.5), (4.1), (4.3), (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} a(n)q^n &= q \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{4n})(1 - q^{6n})(1 - q^{12n}) \\ &= \Delta(q^2)^{1/24} \Delta(q^4)^{1/24} \Delta(q^6)^{1/24} \Delta(q^{12})^{1/24} \\ &= \frac{1}{4} p(2 + p)(1 - p)^{1/2}(1 + p)^{1/2}(1 + 2p)^{1/2}k^2, \end{aligned}$$

which is part (a).

Also, by (1.6), (4.1), (4.2), (4.3), (4.5), (4.6) and (4.7), we have

$$\begin{aligned} \sum_{n=0}^{\infty} b(n)q^n &= \prod_{n=1}^{\infty} (1 - q^n)^{-4}(1 - q^{2n})^{10}(1 - q^{4n})^{-4}(1 - q^{6n})^4(1 - q^{12n})^{-2} \\ &= \Delta(q)^{-1/6} \Delta(q^2)^{5/12} \Delta(q^4)^{-1/6} \Delta(q^6)^{1/6} \Delta(q^{12})^{-1/12} \\ &= (1 - p)^{1/4}(1 + p)^{3/4}(1 + 2p)^{7/4}k^2 \end{aligned}$$

and, by (1.7), (4.1), (4.3), (4.4), (4.5), (4.6) and (4.7),

$$\begin{aligned} \sum_{n=0}^{\infty} c(n)q^n &= \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{3n})^{-4} (1 - q^{4n})^{-2} (1 - q^{6n})^{10} (1 - q^{12n})^{-4} \\ &= \Delta(q^2)^{1/6} \Delta(q^3)^{-1/6} \Delta(q^4)^{-1/12} \Delta(q^6)^{5/12} \Delta(q^{12})^{-1/6} \\ &= (1 - p)^{3/4} (1 + p)^{1/4} (1 + 2p)^{5/4} k^2, \end{aligned}$$

which are parts (b) and (c).

Applying the change of sign principle (Theorem 2.3) to parts (b) and (c), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} b(n)(-q)^n &= (1 - p(-q))^{1/4} (1 + p(-q))^{3/4} (1 + 2p(-q))^{7/4} k(-q)^2 \\ &= \left(1 + \frac{p}{1 + p}\right)^{1/4} \left(1 - \frac{p}{1 + p}\right)^{3/4} \left(1 - \frac{2p}{1 + p}\right)^{7/4} (1 + p)^4 k^2 \\ &= (1 - p)^{7/4} (1 + p)^{5/4} (1 + 2p)^{1/4} k^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} c(n)(-q)^n &= (1 - p(-q))^{3/4} (1 + p(-q))^{1/4} (1 + 2p(-q))^{5/4} k(-q)^2 \\ &= \left(1 + \frac{p}{1 + p}\right)^{3/4} \left(1 - \frac{p}{1 + p}\right)^{1/4} \left(1 - \frac{2p}{1 + p}\right)^{5/4} (1 + p)^4 k^2 \\ &= (1 - p)^{5/4} (1 + p)^{7/4} (1 + 2p)^{3/4} k^2, \end{aligned}$$

proving parts (d) and (e).

**5. Proof of Theorem 1.4.** By Theorem 1.3(a) and Table 1 we have

$$(2p + p^2)(1 - p)^{1/2} (1 + p)^{1/2} (1 + 2p)^{1/2} k^2 = \sum_{n=1}^{\infty} 4a(n)q^n.$$

By Theorem 1.2(a) we have

$$\begin{aligned} (1 + p - 2p^2)(1 - p)^{1/2} (1 + p)^{1/2} (1 + 2p)^{1/2} k^2 \\ = 1 + \sum_{n=1}^{\infty} (-8\sigma(n/2) + 48\sigma(n/4) - 64\sigma(n/8))q^n. \end{aligned}$$

By Theorem 1.2(b) we have

$$\begin{aligned} (1 + p)(1 - p)^{1/2} (1 + p)^{1/2} (1 + 2p)^{1/2} k^2 \\ = 1 + \sum_{n=1}^{\infty} (-8\sigma(n/6) + 48\sigma(n/12) - 64\sigma(n/24))q^n. \end{aligned}$$

The three assertions of Theorem 1.4 now follow using

$$\begin{aligned}
 1 &= -\frac{1}{2}(2p + p^2) - \frac{1}{4}(1 + p - 2p^2) + \frac{5}{4}(1 + p), \\
 p &= \frac{1}{2}(2p + p^2) + \frac{1}{4}(1 + p - 2p^2) - \frac{1}{4}(1 + p), \\
 p^2 &= -\frac{1}{2}(1 + p - 2p^2) + \frac{1}{2}(1 + p).
 \end{aligned}$$

**6. Proofs of Theorems 1.6–1.25.** Let  $n \in \mathbb{N}$ . We note that in the proofs of Theorems 1.6–1.8, 1.11, 1.14 and 1.18 we set  $n = 2^\alpha N$ , where  $\alpha \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 2) = 1$ , whereas in Theorems 1.9, 1.10, 1.12, 1.13, 1.15–1.17 and 1.19–1.24 we set  $n = 2^\alpha 3^\beta N$ , where  $\alpha \in \mathbb{N}_0$ ,  $\beta \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$  and  $\gcd(N, 6) = 1$ . As  $\sigma(n)$  is a multiplicative function of  $n$ , we have

$$\begin{aligned}
 \sigma(2^\alpha N) &= \sigma(2^\alpha)\sigma(N) = (2^{\alpha+1} - 1)\sigma(N), \\
 \sigma(2^\alpha 3^\beta N) &= \sigma(2^\alpha)\sigma(3^\beta)\sigma(N) = \frac{1}{2}(2^{\alpha+1} - 1)(3^{\beta+1} - 1)\sigma(N).
 \end{aligned}$$

*Proof of Theorem 1.6.* We have

$$\begin{aligned}
 \sum_{n=0}^{\infty} N(1, 1, 1, 1; n)q^n &= \varphi(q)^4 && \text{(by (1.9))} \\
 &= (1 + 2p)^3 k^2 && \text{(by Theorem 1.5(a))} \\
 &= 1 + \sum_{n=1}^{\infty} (8\sigma(n) - 32\sigma(n/4)) && \text{(by Corollary 1.1(b)),}
 \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$N(1, 1, 1, 1; n) = 8\sigma(n) - 32\sigma(n/4) = \begin{cases} 8\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

*Proof of Theorem 1.7.* We have

$$\begin{aligned}
 \sum_{n=0}^{\infty} N(1, 1, 1, 4; n)q^n &= \varphi(q)^3 \varphi(q^4) && \text{(by (1.9))} \\
 &= \frac{1}{2}(1 + 2p)^3 k^2 + \frac{1}{2}(1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{9/4} k^2 && \\
 & && \text{(by Theorem 1.5(a)(d))} \\
 &= 1 + \sum_{n=1}^{\infty} \left( 2 \left( \frac{-4}{n} \right) \sigma(n) + 4\sigma(n) - 20\sigma(n/4) \right. && \\
 & && \left. + 24\sigma(n/8) - 32\sigma(n/16) \right) q^n && \\
 & && \text{(by Corollary 1.1(b) and Theorem 1.2(c)),}
 \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 N(1, 1, 1, 4; n) &= \left( 4 + 2 \left( \frac{-4}{n} \right) \right) \sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) \\
 &= \begin{cases} (4 + 2(-1)^{(N-1)/2})\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 12\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}
 \end{aligned}$$

*Proof of Theorem 1.8.* We have

$$\begin{aligned}
 \sum_{n=0}^{\infty} N(1, 1, 2, 2; n)q^n &= \varphi(q)^2\varphi(q^2)^2 \quad (\text{by (1.9)}) \\
 &= \frac{1}{2}(1 + 2p)^3k^2 + \frac{1}{2}(1 - p)^{3/2}(1 + p)^{1/2}(1 + 2p)^{3/2}k^2 \\
 &\quad (\text{by Theorem 1.5(a)(b)}) \\
 &= 1 + \sum_{n=1}^{\infty} (4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8))q^n \\
 &\quad (\text{by Corollary 1.1(b) and Theorem 1.2(a)}),
 \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 N(1, 1, 2, 2; n) &= 4\sigma(n) - 4\sigma(n/2) + 8\sigma(n/4) - 32\sigma(n/8) \\
 &= \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 8\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{4}. \end{cases}
 \end{aligned}$$

*Proof of Theorem 1.9.* We have

$$\begin{aligned}
 \sum_{n=0}^{\infty} N(1, 1, 3, 3; n)q^n &= \varphi(q)^2\varphi(q^3)^2 \quad (\text{by (1.9)}) \\
 &= (1 + 2p)^2k^2 \quad (\text{by Theorem 1.5(a)(c)}) \\
 &= 1 + \sum_{n=1}^{\infty} (4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4) \\
 &\quad + 24\sigma(n/6) - 48\sigma(n/12))q^n \\
 &\quad (\text{by Corollary 1.1(a)}),
 \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 N(1, 1, 3, 3; n) &= 4\sigma(n) - 8\sigma(n/2) - 12\sigma(n/3) + 16\sigma(n/4) \\
 &\quad + 24\sigma(n/6) - 48\sigma(n/12) \\
 &= \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 4(2^{\alpha+1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{2}. \end{cases}
 \end{aligned}$$

*Proof of Theorem 1.10.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 1, 3, 12; n)q^n &= \varphi(q)^2\varphi(q^3)\varphi(q^{12}) \quad (\text{by (1.9)}) \\ &= \frac{1}{2}(1+2p)^2k^2 + \frac{1}{2}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 \\ &\quad (\text{by Theorem 1.5(a)(c)(f)}) \\ &= 1 + \sum_{n=1}^{\infty} \left( 2\sigma(n) - 4\sigma(n/2) - 6\sigma(n/3) + 8\sigma(n/4) \right. \\ &\quad \left. + 12\sigma(n/6) - 24\sigma(n/12) + \frac{1}{2}b(n) \right) q^n \\ &\quad (\text{by Corollary 1.1(a) and Theorem 1.3(b)}), \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(1, 1, 3, 12; n) &= 2\sigma(n) - 4\sigma(n/2) - 6\sigma(n/3) + 8\sigma(n/4) \\ &\quad + 12\sigma(n/6) - 24\sigma(n/12) + \frac{1}{2}b(n) \\ &= \begin{cases} 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2(2^{\alpha+1} - 3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod{2}. \end{cases} \end{aligned}$$

*Proof of Theorem 1.11.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 1, 4, 4; n)q^n &= \varphi(q)^2\varphi(q^4)^2 \quad (\text{by (1.9)}) \\ &= \frac{1}{4}(1+2p)^3k^2 + \frac{1}{2}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 \\ &\quad + \frac{1}{4}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 \quad (\text{by Theorem 1.5(a)(d)}) \\ &= 1 + \sum_{n=1}^{\infty} \left( \left( 2 + 2\left(\frac{-4}{n}\right) \right) \sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16) \right) q^n \\ &\quad (\text{by Corollary 1.1(b) and Theorem 1.2(a)(c)}), \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(1, 1, 4, 4; n) &= \left( 2 + 2\left(\frac{-4}{n}\right) \right) \sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16) \\ &= \begin{cases} 4\sigma(N) & \text{if } n \equiv 1, 2 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases} \end{aligned}$$

*Proof of Theorem 1.12.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 1, 6, 6; n)q^n &= \varphi(q)^2\varphi(q^6)^2 \quad (\text{by (1.9)}) \\ &= \frac{1}{2}(1+2p)^2k^2 + \frac{1}{2}(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}k^2 \\ &\hspace{15em} (\text{by Theorem 1.5(a)(e)}) \\ &= 1 + \sum_{n=1}^{\infty} (2\sigma(n) - 2\sigma(n/2) - 6\sigma(n/3) - 4\sigma(n/4) \\ &\quad + 6\sigma(n/6) + 16\sigma(n/8) + 12\sigma(n/12) - 48\sigma(n/24) + 2a(n))q^n \\ &\hspace{15em} (\text{by Corollary 1.1(a) and Corollary 1.2(a)}), \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(1, 1, 6, 6; n) &= 2\sigma(n) - 2\sigma(n/2) - 6\sigma(n/3) - 4\sigma(n/4) \\ &\quad + 6\sigma(n/6) + 16\sigma(n/8) + 12\sigma(n/12) - 48\sigma(n/24) + 2a(n) \\ &= \begin{cases} 2\sigma(N) + 2a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 4(2^\alpha - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{4}, \end{cases} \end{aligned}$$

as  $a(n) = 0$  for  $n \equiv 0 \pmod{2}$ .

*Proof of Theorem 1.13 and Theorem 1.25(b).* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 1, 12, 12; n)q^n &= \varphi(q)^2\varphi(q^{12})^2 \quad (\text{by (1.9)}) \\ &= \frac{1}{4}(1+2p)^2k^2 + \frac{1}{2}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 \\ &\quad + \frac{1}{4}(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}k^2 \quad (\text{by Theorem 1.5(a)(f)}) \\ &= 1 + \sum_{n=1}^{\infty} \left( \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8) \right. \\ &\quad \left. + 6\sigma(n/12) - 24\sigma(n/24) + a(n) + \frac{1}{2}b(n) \right) q^n \\ &\hspace{15em} (\text{by Corollary 1.1(a), Theorem 1.3(b) and Corollary 1.2(a)}), \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(1, 1, 12, 12; n) &= \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) \\ &\quad + 3\sigma(n/6) + 8\sigma(n/8) + 6\sigma(n/12) \\ &\quad - 24\sigma(n/24) + a(n) + \frac{1}{2}b(n) \end{aligned}$$

$$= \begin{cases} \sigma(N) + a(n) + \frac{1}{2}b(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

As  $x^2 + y^2 + 12z^2 + 12t^2 \equiv x^2 + y^2 \equiv 0, 1, 2 \pmod{4}$ , we have

$$N(1, 1, 12, 12; n) = 0 \quad \text{if } n \equiv 3 \pmod{4}.$$

Thus

$$\sigma(n) + a(n) + \frac{1}{2}b(n) = 0 \quad \text{if } n \equiv 3 \pmod{4},$$

which is the assertion of Theorem 1.25(b).

*Proof of Theorem 1.14.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 2, 2, 4; n)q^n &= \varphi(q)\varphi(q^2)^2\varphi(q^4) \quad (\text{by (1.9)}) \\ &= \frac{1}{4}(1+2p)^3k^2 + \frac{1}{4}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 \\ &\quad + \frac{1}{4}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{9/4}k^2 \\ &\quad + \frac{1}{4}(1-p)^{9/4}(1+p)^{3/4}(1+2p)^{3/4}k^2 \quad (\text{by Theorem 1.5(a)(b)(d)}) \\ &= 1 + \sum_{n=1}^{\infty} (2\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16))q^n \\ &\quad (\text{by Corollary 1.1(b) and Theorem 1.2(a)(c)(d)}), \end{aligned}$$

so that for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(1, 2, 2, 4; n) &= 2\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16) \\ &= \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod{2}, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases} \end{aligned}$$

*Proof of Theorem 1.15.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 2, 3, 6; n)q^n &= \varphi(q)\varphi(q^2)\varphi(q^3)\varphi(q^6) \quad (\text{by (1.9)}) \\ &= \frac{1}{2}(1+2p)(1+p+p^2)k^2 + \frac{1}{2}(1-p)^{1/2}(1+p)^{1/2}(1+2p)^{3/2}k^2 \\ &\quad (\text{by Theorem 1.5(a)(c) and (1.13)}) \end{aligned}$$



$$\begin{aligned}
 &= 1 + \sum_{n=1}^{\infty} (\sigma(n) - \sigma(n/2) + 3\sigma(n/3) + 2\sigma(n/4) - 3\sigma(n/6) \\
 &\quad - 8\sigma(n/8) + 6\sigma(n/12) - 24\sigma(n/24) + a(n))q^n \\
 &\hspace{15em} \text{(by Corollary 1.1(c) and Corollary 1.2(b)),}
 \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 N(1, 2, 3, 6; n) &= \sigma(n) - \sigma(n/2) + 3\sigma(n/3) + 2\sigma(n/4) - 3\sigma(n/6) \\
 &\quad - 8\sigma(n/8) + 6\sigma(n/12) - 24\sigma(n/24) + a(n) \\
 &= \begin{cases} (3^{\beta+1} - 2)\sigma(N) + a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 6(3^{\beta+1} - 2)\sigma(N) & \text{if } n \equiv 0 \pmod{4}, \end{cases}
 \end{aligned}$$

as  $a(n) = 0$  when  $n$  is even.

*Proof of Theorem 1.16.* We have

$$\begin{aligned}
 \sum_{n=0}^{\infty} N(1, 3, 3, 4; n)q^n &= \varphi(q)\varphi(q^3)^2\varphi(q^4) \quad \text{(by (1.9))} \\
 &= \frac{1}{2}(1 + 2p)^2k^2 + \frac{1}{2}(1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{5/4}k^2 \\
 &\hspace{15em} \text{(by Theorem 1.5(a)(c)(d))} \\
 &= 1 + \sum_{n=1}^{\infty} \left( 2\sigma(n) - 4\sigma(n/2) - 6\sigma(n/3) + 8\sigma(n/4) + 12\sigma(n/6) \right. \\
 &\quad \left. - 24\sigma(n/12) + \frac{1}{2}c(n) \right) q^n \quad \text{(by Corollary 1.1(a) and Theorem 1.3(c)),}
 \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 N(1, 3, 3, 4; n) &= 2\sigma(n) - 4\sigma(n/2) - 6\sigma(n/3) + 8\sigma(n/4) + 12\sigma(n/6) \\
 &\quad - 24\sigma(n/12) + \frac{1}{2}c(n) \\
 &= \begin{cases} 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2(2^{\alpha+1} - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod{2}. \end{cases}
 \end{aligned}$$

*Proof of Theorem 1.17 and Theorem 1.25(e).* We have

$$\begin{aligned}
 \sum_{n=0}^{\infty} N(1, 3, 4, 12; n)q^n &= \varphi(q)\varphi(q^3)\varphi(q^4)\varphi(q^{12}) \quad \text{(by (1.9))} \\
 &= \frac{1}{4}(1 + 2p)^2k^2 + \frac{1}{4}(1 - p)^{1/4}(1 + p)^{3/4}(1 + 2p)^{7/4}k^2 \\
 &\quad + \frac{1}{4}(1 - p)^{3/4}(1 + p)^{1/4}(1 + 2p)^{5/4}k^2 + \frac{1}{4}(1 - p)(1 + p)(1 + 2p)k^2 \\
 &\hspace{15em} \text{(by Theorem 1.5(a)(c)(d)(f))}
 \end{aligned}$$

$$\begin{aligned}
 &= 1 + \sum_{n=1}^{\infty} \left( \sigma(n) - 3\sigma(n/2) - 3\sigma(n/3) + 8\sigma(n/4) + 9\sigma(n/6) \right. \\
 &\quad \left. - 24\sigma(n/12) + \frac{1}{4}b(n) + \frac{1}{4}c(n) \right) q^n \\
 &\qquad\qquad\qquad \text{(by Corollary 1.1(a)(d) and Theorem 1.3(b)(c)),}
 \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 N(1, 3, 4, 12; n) &= \sigma(n) - 3\sigma(n/2) - 3\sigma(n/3) + 8\sigma(n/4) + 9\sigma(n/6) \\
 &\quad - 24\sigma(n/12) + \frac{1}{4}b(n) + \frac{1}{4}c(n) \\
 &= \begin{cases} \sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 1 \pmod{2}, \\ \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 2 \pmod{4}, \\ 6(2^{\alpha-1} - 1)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases}
 \end{aligned}$$

Now  $x^2 + 3y^2 + 4z^2 + 12t^2 \equiv x^2 + 3y^2 \not\equiv 2 \pmod{4}$  so

$$N(1, 3, 4, 12; n) = 0 \quad \text{if } n \equiv 2 \pmod{4}.$$

Thus

$$b(n) = -c(n) \quad \text{if } n \equiv 2 \pmod{4}.$$

As

$$\begin{aligned}
 x^2 + y^2 + 12z^2 + 12t^2 &\equiv 2 \pmod{4} \Rightarrow x \equiv y \pmod{2}, \\
 x^2 + y^2 + 12z^2 + 12t^2 &= 2 \left( \left( \frac{x+y}{2} \right)^2 + \left( \frac{x-y}{2} \right)^2 + 6z^2 + 6t^2 \right), \\
 2(x^2 + y^2 + 6z^2 + 6t^2) &= (x+y)^2 + (x-y)^2 + 12z^2 + 12t^2,
 \end{aligned}$$

we have

$$N(1, 1, 12, 12; n) = N(1, 1, 6, 6; n/2) \quad \text{if } n \equiv 2 \pmod{4}.$$

Appealing to Theorems 1.13 and 1.12, we obtain

$$2\sigma(N) + \frac{1}{2}b(n) = 2\sigma(N) + 2a(n/2)$$

so that

$$b(n) = 4a(n/2) \quad \text{if } n \equiv 2 \pmod{4}.$$

This completes the proof of Theorem 1.25(e).

*Proof of Theorem 1.18.* We have

$$\sum_{n=0}^{\infty} N(1, 4, 4, 4; n)q^n = \varphi(q)\varphi(q^4)^3 \quad \text{(by (1.9))}$$

$$\begin{aligned}
 &= \frac{1}{8} (1 + 2p)^3 k^2 + \frac{3}{8} (1 - p)^{3/4} (1 + p)^{1/4} (1 + 2p)^{9/4} k^2 \\
 &\quad + \frac{3}{8} (1 - p)^{3/2} (1 + p)^{1/2} (1 + 2p)^{3/2} k^2 + \frac{1}{8} (1 - p)^{9/4} (1 + p)^{3/4} (1 + 2p)^{3/4} k^2 \\
 &\hspace{15em} \text{(by Theorem 1.5(a)(d))} \\
 &= 1 + \sum_{n=1}^{\infty} \left( \left( 1 + \left( \frac{-4}{n} \right) \right) \sigma(n) - 3\sigma(n/2) + 10\sigma(n/4) - 32\sigma(n/16) \right) q^n \\
 &\hspace{15em} \text{(by Corollary 1.1(b) and Theorem 1.2(a)(c)(d)),}
 \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 N(1, 4, 4, 4; n) &= \left( 1 + \left( \frac{-4}{n} \right) \right) \sigma(n) - 3\sigma(n/2) + 10\sigma(n/4) - 32\sigma(n/16) \\
 &= \begin{cases} 2\sigma(N) & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 2, 3 \pmod{4}, \\ 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 24\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}
 \end{aligned}$$

*Proof of Theorem 1.19.* We have

$$\begin{aligned}
 \sum_{n=0}^{\infty} N(1, 4, 6, 6; n) q^n &= \varphi(q)\varphi(q^4)\varphi(q^6)^2 \quad \text{(by (1.9))} \\
 &= \frac{1}{4} (1 + 2p)^2 k^2 + \frac{1}{4} (1 - p)^{3/4} (1 + p)^{1/4} (1 + 2p)^{5/4} k^2 \\
 &\quad + \frac{1}{4} (1 - p)^{1/2} (1 + p)^{3/2} (1 + 2p)^{3/2} k^2 + \frac{1}{4} (1 - p)^{5/4} (1 + p)^{7/4} (1 + 2p)^{3/4} k^2 \\
 &\hspace{15em} \text{(by Theorem 1.5(a)(d)(e))} \\
 &= 1 + \sum_{n=1}^{\infty} \left( \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8) \right. \\
 &\quad \left. + 6\sigma(n/12) - 24\sigma(n/24) + a(n) + \frac{1}{4} (1 + (-1)^n) c(n) \right) q^n \\
 &\hspace{15em} \text{(by Corollary 1.1(a), Theorem 1.3(c)(e) and Corollary 1.2(a)),}
 \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 N(1, 4, 6, 6; n) &= \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8) \\
 &\quad + 6\sigma(n/12) - 24\sigma(n/24) + a(n) + \frac{1}{4} (1 + (-1)^n) c(n) \\
 &= \begin{cases} \sigma(N) + a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod{4}, \end{cases}
 \end{aligned}$$

as  $a(n) = 0$  for  $n$  even.

*Proof of Theorem 1.20 and Theorem 1.25(a)(f)(g).* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 4, 12, 12; n)q^n &= \varphi(q)\varphi(q^4)\varphi(q^{12})^2 \quad (\text{by (1.9)}) \\ &= \frac{1}{8}(1+2p)^2k^2 + \frac{1}{8}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2 \\ &\quad + \frac{1}{8}(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}k^2 \\ &\quad + \frac{1}{8}(1-p)^{5/4}(1+p)^{7/4}(1+2p)^{3/4}k^2 \\ &\quad + \frac{1}{4}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 + \frac{1}{4}(1-p)(1+p)(1+2p)k^2 \\ &\hspace{15em} (\text{by Theorem 1.5(a)(d)(f)}) \\ &= 1 + \sum_{n=1}^{\infty} \left( \frac{1}{2}\sigma(n) - \frac{3}{2}\sigma(n/2) - \frac{3}{2}\sigma(n/3) + 3\sigma(n/4) + \frac{9}{2}\sigma(n/6) \right. \\ &\quad \left. + 4\sigma(n/8) - 9\sigma(n/12) - 12\sigma(n/24) \right. \\ &\quad \left. + \frac{1}{2}a(n) + \frac{1}{4}b(n) + \frac{1}{8}(1+(-1)^n)c(n) \right) q^n \end{aligned}$$

(by Corollary 1.1(a)(d), Corollary 1.2(a) and Theorem 1.3(b)(c)(e)),

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(1, 4, 12, 12; n) &= \frac{1}{2}\sigma(n) - \frac{3}{2}\sigma(n/2) - \frac{3}{2}\sigma(n/3) + 3\sigma(n/4) + \frac{9}{2}\sigma(n/6) \\ &\quad + 4\sigma(n/8) - 9\sigma(n/12) - 12\sigma(n/24) \\ &\quad + \frac{1}{2}a(n) + \frac{1}{4}b(n) + \frac{1}{8}(1+(-1)^n)c(n) \\ &= \begin{cases} \frac{1}{2}\sigma(N) + \frac{1}{2}a(n) + \frac{1}{4}b(n) & \text{if } n \equiv 1 \pmod{2}, \\ \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod{4}, \end{cases} \end{aligned}$$

as  $a(n) = 0$  when  $n$  is even.

As  $x^2 + 4y^2 + 6z^2 + 6t^2 \equiv 1 \pmod{4}$  implies  $z \equiv t \pmod{2}$ , and

$$x^2 + 4y^2 + 6z^2 + 6t^2 = x^2 + 4y^2 + 12\left(\frac{z+t}{2}\right)^2 + 12\left(\frac{z-t}{2}\right)^2,$$

$$x^2 + 4y^2 + 12z^2 + 12t^2 = x^2 + 4y^2 + 6(z+t)^2 + 6(z-t)^2,$$

we have

$$N(1, 4, 6, 6; n) = N(1, 4, 12, 12; n) \quad \text{if } n \equiv 1 \pmod{4}.$$

Appealing to Theorems 1.19 and 1.20, we deduce

$$\sigma(N) + a(n) = \frac{1}{2}\sigma(N) + \frac{1}{2}a(n) + \frac{1}{4}b(n) \quad \text{if } n \equiv 1 \pmod{4},$$

so that

$$b(n) = 2\sigma(N) + 2a(n) \quad \text{if } n \equiv 1 \pmod{4},$$

which is Theorem 1.25(a).

As  $x^2 + 4y^2 + 12z^2 + 12t^2 \equiv x^2 \not\equiv 2, 3 \pmod{4}$  we have

$$N(1, 4, 12, 12; n) = 0 \quad \text{if } n \equiv 2, 3 \pmod{4}.$$

As  $x^2 + 4y^2 + 12z^2 + 12t^2 \equiv 0 \pmod{4}$  implies  $x \equiv 0 \pmod{2}$  we have

$$N(1, 4, 12, 12; n) = N(1, 1, 3, 3; n/4) \quad \text{if } n \equiv 0 \pmod{4}.$$

Appealing to Theorems 1.20 and 1.9, we obtain

$$2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) = \begin{cases} 4\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ 4(2^{\alpha-1} - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{8}, \end{cases}$$

so that

$$b(n) + c(n) = \begin{cases} 8\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ -24\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{cases}$$

As  $x^2 + 4y^2 + 6z^2 + 6t^2 \equiv 0 \pmod{4}$  implies  $z \equiv t \pmod{2}$ , and

$$\begin{aligned} x^2 + 4y^2 + 6z^2 + 6t^2 &= x^2 + 4y^2 + 12\left(\frac{z+t}{2}\right)^2 + 12\left(\frac{z-t}{2}\right)^2, \\ x^2 + 4y^2 + 12z^2 + 12t^2 &= x^2 + 4y^2 + 6(z+t)^2 + 6(z-t)^2, \end{aligned}$$

we have

$$N(1, 4, 6, 6; n) = N(1, 4, 12, 12; n) \quad \text{if } n \equiv 0 \pmod{4}.$$

Hence, by Theorems 1.19 and 1.20, we have

$$2(2^\alpha - 3)\sigma(N) + \frac{1}{2}c(n) = 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) \quad \text{if } n \equiv 0 \pmod{4},$$

so that

$$b(n) = c(n) \quad \text{if } n \equiv 0 \pmod{4}.$$

Thus

$$\begin{aligned} b(n) = c(n) &= 4\sigma(N) & \text{if } n \equiv 4 \pmod{8}, \\ b(n) = c(n) &= -12\sigma(N) & \text{if } n \equiv 0 \pmod{8}. \end{aligned}$$

This completes the proof of Theorem 1.25(f)(g).

*Proof of Theorem 1.21.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(2, 2, 3, 3; n)q^n &= \varphi(q^2)^2\varphi(q^3)^2 \quad (\text{by (1.9)}) \\ &= \frac{1}{2}(1+2p)^2k^2 + \frac{1}{2}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 \\ &\hspace{15em} (\text{by Theorem 1.5(b)(c)}) \\ &= 1 + \sum_{n=1}^{\infty} (2\sigma(n) - 2\sigma(n/2) - 6\sigma(n/3) \\ &\quad - 4\sigma(n/4) + 6\sigma(n/6) + 16\sigma(n/8) + 12\sigma(n/12) \\ &\quad - 48\sigma(n/24) - 2a(n))q^n \\ &\hspace{15em} (\text{by Corollary 1.1(a) and Corollary 1.2(c)}), \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(2, 2, 3, 3; n) &= 2\sigma(n) - 2\sigma(n/2) - 6\sigma(n/3) - 4\sigma(n/4) + 6\sigma(n/6) \\ &\quad + 16\sigma(n/8) + 12\sigma(n/12) - 48\sigma(n/24) - 2a(n) \\ &= \begin{cases} 2\sigma(N) - 2a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 4\sigma(N) & \text{if } n \equiv 2 \pmod{4}, \\ 4(2^\alpha - 3)\sigma(N) & \text{if } n \equiv 0 \pmod{4}. \end{cases} \end{aligned}$$

*Proof of Theorem 1.22.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(2, 2, 3, 12; n)q^n &= \varphi(q^2)^2\varphi(q^3)\varphi(q^{12}) \quad (\text{by (1.9)}) \\ &= \frac{1}{4}(1+2p)^2k^2 + \frac{1}{4}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 \\ &\quad + \frac{1}{4}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 + \frac{1}{4}(1-p)^{7/4}(1+p)^{5/4}(1+2p)^{1/4}k^2 \\ &\hspace{15em} (\text{by Theorem 1.5(b)(c)(f)}) \\ &= 1 + \sum_{n=1}^{\infty} \left( \sigma(n) - \sigma(n/2) - 3\sigma(n/3) \right. \\ &\quad - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8) + 6\sigma(n/12) \\ &\quad \left. - 24\sigma(n/24) - a(n) + \frac{1}{4}(1+(-1)^n)b(n) \right) q^n \\ &\hspace{15em} (\text{by Corollary 1.1(a), Corollary 1.2(c) and Theorem 1.3(b)(d)}), \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(2, 2, 3, 12; n) &= \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) + 8\sigma(n/8) \\ &\quad + 6\sigma(n/12) - 24\sigma(n/24) - a(n) + \frac{1}{4}(1+(-1)^n)b(n) \end{aligned}$$

$$= \begin{cases} \sigma(N) - a(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}b(n) & \text{if } n \equiv 0 \pmod{4}, \end{cases}$$

as  $a(n) = 0$  when  $n$  is even.

*Proof of Theorem 1.23 and Theorem 1.25(c).* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(3, 3, 4, 4; n)q^n &= \varphi(q^3)^2\varphi(q^4)^2 \quad (\text{by (1.9)}) \\ &= \frac{1}{4}(1+2p)^2k^2 + \frac{1}{4}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 \\ &\quad + \frac{1}{2}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2 \quad (\text{by Theorem 1.5(c)(d)}) \\ &= 1 + \sum_{n=1}^{\infty} \left( \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) \right. \\ &\quad \left. + 8\sigma(n/8) + 6\sigma(n/12) - 24\sigma(n/24) - a(n) + \frac{1}{2}c(n) \right) q^n \\ &\quad (\text{by Corollary 1.1(a), Corollary 1.2(c) and Theorem 1.3(c)}), \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(3, 3, 4, 4; n) &= \sigma(n) - \sigma(n/2) - 3\sigma(n/3) - 2\sigma(n/4) + 3\sigma(n/6) \\ &\quad + 8\sigma(n/8) + 6\sigma(n/12) - 24\sigma(n/24) - a(n) + \frac{1}{2}c(n) \\ &= \begin{cases} \sigma(N) - a(n) + \frac{1}{2}c(n) & \text{if } n \equiv 1 \pmod{2}, \\ 2\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{2}c(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases} \end{aligned}$$

As  $3x^2 + 3y^2 + 4z^2 + 4t^2 \not\equiv 1 \pmod{4}$  we have

$$N(3, 3, 4, 4; n) = 0 \quad \text{if } n \equiv 1 \pmod{4}.$$

Hence

$$\sigma(N) - a(n) + \frac{1}{2}c(n) = 0 \quad \text{if } n \equiv 1 \pmod{4},$$

that is,

$$c(n) = -2\sigma(N) + 2a(n) \quad \text{if } n \equiv 1 \pmod{4},$$

which is Theorem 1.25(c).

*Proof of Theorem 1.24 and Theorem 1.25(d).* We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(3, 4, 4, 12; n)q^n &= \varphi(q^3)\varphi(q^4)^2\varphi(q^{12}) \quad (\text{by (1.9)}) \\ &= \frac{1}{8}(1+2p)^2k^2 + \frac{1}{4}(1-p)^{3/4}(1+p)^{1/4}(1+2p)^{5/4}k^2 \\ &\quad + \frac{1}{8}(1-p)^{3/2}(1+p)^{1/2}(1+2p)^{1/2}k^2 + \frac{1}{8}(1-p)^{1/4}(1+p)^{3/4}(1+2p)^{7/4}k^2 \\ &\quad + \frac{1}{4}(1-p)(1+p)(1+2p)k^2 + \frac{1}{8}(1-p)^{7/4}(1+p)^{5/4}(1+2p)^{1/4}k^2 \\ &\quad (\text{by Corollary 1.1(a)(d), Corollary 1.2(c) and Theorem 1.3(b)(c)(d)}) \\ &= 1 + \sum_{n=1}^{\infty} \left( \frac{1}{2}\sigma(n) - \frac{3}{2}\sigma(n/2) - \frac{3}{2}\sigma(n/3) + 3\sigma(n/4) + \frac{9}{2}\sigma(n/6) + 4\sigma(n/8) \right. \\ &\quad \left. - 9\sigma(n/12) - 12\sigma(n/24) - \frac{1}{2}a(n) + \frac{1}{8}(1+(-1)^n)b(n) + \frac{1}{4}c(n) \right) q^n, \end{aligned}$$

so for  $n \in \mathbb{N}$ ,

$$\begin{aligned} N(3, 4, 4, 12; n) &= \frac{1}{2}\sigma(n) - \frac{3}{2}\sigma(n/2) - \frac{3}{2}\sigma(n/3) + 3\sigma(n/4) + \frac{9}{2}\sigma(n/6) \\ &\quad + 4\sigma(n/8) - 9\sigma(n/12) - 12\sigma(n/24) \\ &\quad - \frac{1}{2}a(n) + \frac{1}{8}(1+(-1)^n)b(n) + \frac{1}{4}c(n) \\ &= \begin{cases} \frac{1}{2}\sigma(N) - \frac{1}{2}a(n) + \frac{1}{4}c(n) & \text{if } n \equiv 1 \pmod{2}, \\ \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 2 \pmod{4}, \\ 2(2^\alpha - 3)\sigma(N) + \frac{1}{4}b(n) + \frac{1}{4}c(n) & \text{if } n \equiv 0 \pmod{4}. \end{cases} \end{aligned}$$

As  $3x^2 + 4y^2 + 4z^2 + 12t^2 \equiv 3x^2 \equiv 0, 3 \pmod{4}$ , we have

$$N(3, 4, 4, 12; n) = 0 \quad \text{if } n \equiv 1, 2 \pmod{4}.$$

Finally, as  $2x^2 + 2y^2 + 3z^2 + 12t^2 \equiv 3 \pmod{4}$  implies  $x \equiv y \pmod{2}$ , and

$$2x^2 + 2y^2 + 3z^2 + 12t^2 = 3z^2 + 4\left(\frac{x+y}{2}\right)^2 + 4\left(\frac{x-y}{2}\right)^2 + 12t^2,$$

$$3x^2 + 4y^2 + 4z^2 + 12t^2 = 2(y+z)^2 + 2(y-z)^2 + 3x^2 + 12t^2,$$

we have

$$N(2, 2, 3, 12; n) = N(3, 4, 4, 12; n) \quad \text{if } n \equiv 3 \pmod{4}.$$

Appealing to Theorems 1.22 and 1.24, we obtain

$$\sigma(N) - a(n) = \frac{1}{2}\sigma(N) - \frac{1}{2}a(n) + \frac{1}{4}c(n) \quad \text{if } n \equiv 3 \pmod{4},$$



so that

$$c(n) = 2\sigma(N) - 2a(n) \quad \text{if } n \equiv 3 \pmod{4},$$

which is Theorem 1.25(d).

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