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Power sums of Hecke eigenvalues and application

by

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1. Introduction. Let $k \geq 2$ be an even integer and $N \geq 1$ be squarefree. Denote by $\mathrm{H}_k^*(N)$ the set of all normalized Hecke primitive eigencuspforms of weight k for the congruence modular group

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Here the normalization is taken to have $\lambda_f(1) = 1$ in the Fourier series of $f \in \mathrm{H}_k^*(N)$ at the cusp ∞,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \qquad (\operatorname{Im} z > 0). \tag{1.1}$$

Inherited from the Hecke operators, the normalized Fourier coefficient $\lambda_f(n)$ satisfies the relation

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right) \tag{1.2}$$

for all integers $m, n \geq 1$. In particular, $\lambda_f(n)$ is multiplicative.

Following Deligne [4], for any prime number p there are two complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\begin{cases} \alpha_f(p) = \varepsilon_f(p) p^{-1/2}, \quad \beta_f(p) = 0 & \text{if } p \mid N, \\ |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1 & \text{if } p \nmid N, \end{cases} \tag{1.3}$$

and

$$\lambda_f(p^\nu) = \frac{\alpha_f(p)^{\nu+1} - \beta_f(p)^{\nu+1}}{\alpha_f(p) - \beta_f(p)} \tag{1.4}$$

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for all integers $\nu \geq 1$, where $\varepsilon_f(p) = \pm 1$. Hence $\lambda_f(n)$ is real and satisfies Deligne's inequality

$$|\lambda_f(n)| \leq d(n) \tag{1.5}$$

for all integers $n \geq 1$, where $d(n)$ is the divisor function. In particular, for each prime number $p \nmid N$ there is $\theta_f(p) \in [0, \pi]$ such that

$$\lambda_f(p) = 2\cos\theta_f(p). \tag{1.6}$$

See e.g. [9] for basic analytic facts about modular forms.

Positive real moments of Hecke eigenvalues were first studied by Rankin ([16], [17]). For $f \in \mathrm{H}_k^*(N)$ and $r \geq 0$, consider the sum of the $2r$th powers of $|\lambda_f(n)|$:

$$S_f^*(x; r) := \sum_{n \leq x} |\lambda_f(n)|^{2r}. \tag{1.7}$$

The method of Rankin [17] illustrates how to obtain optimal lower and upper bounds for $S_f^*(x; r)$ if we only know that the associated Dirichlet series

$$F_r(s) := \sum_{n \geq 1} |\lambda_f(n)|^{2r} n^{-s} \quad (\mathrm{Re}\, s > 1) \tag{1.8}$$

is invertible for $\mathrm{Re}\, s \geq 1$ (i.e. holomorphic and nonzero for $\mathrm{Re}\, s \geq 1$) when $r = 1, 2$. (The invertibility in these two cases is known by Moreno & Shahidi [15].) Rankin's result ([17, Theorem 1]) states that

$$x(\log x)^{\delta_r^{\mp}} \ll S_f^*(x; r) \ll x(\log x)^{\delta_r^{\pm}} \quad (r \in \mathcal{R}^{\mp}) \tag{1.9}$$

for $x \geq x_0(f, r)$, where

$$\mathcal{R}^- := [0, 1] \cup [2, \infty), \qquad \mathcal{R}^+ := [1, 2],$$

and

$$\delta_r^- := 2^{r-1} - 1, \qquad \delta_r^+ := \frac{2^{r-1}}{5}(2^r + 3^{2-r}) - 1.$$

The implied constants in (1.9) depend on f and r.

On the other hand, if the Sato–Tate conjecture holds for a newform f, then

$$S_f^*(x; r) \sim C_r(f) x (\log x)^{\theta_r} \quad (x \to \infty), \tag{1.10}$$

where $C_r(f)$ is a positive constant depending on f, r, and

$$\theta_r := \frac{4^r \Gamma(r + 1/2)}{\sqrt{\pi}\, \Gamma(r + 2)} - 1.$$

We remark that this conjecture has been proved for elliptic curves over $\mathbb{Q}$ with multiplicative reduction at some prime (cf. [1, 21, 7]).

Very recently, Tenenbaum [23] improved Rankin's exponent $\delta^+_{1/2} \approx -0.065$ to $\varrho^+_{1/2} \approx -0.118$ (see (1.13) below for the definition of ϱ^+_r), as an application of his general result on the mean values of multiplicative functions and the fact that $F_3(s)$ and $F_4(s)$ are invertible for $\operatorname{Re} s \geq 1$, proven in the remarkable work of Kim & Shahidi [11]. Although the result ([23, Corollary]) is stated only for Ramanujan's τ-function, it is apparent that Tenenbaum's method applies to establish the upper bound for $S^*_f(x;r)$ in (1.11) below. It should be pointed out that Tenenbaum's approach is different from that of Rankin and does not give a lower bound for $S^*_f(x;r)$.

The first aim of this paper is to improve the lower and upper bounds in (1.9), by generalizing Rankin's method to incorporate the aforementioned results of Kim & Shahidi on $F_3(s)$ and $F_4(s)$.

Theorem 1. *For any $f \in \mathrm{H}^*_k(N)$, we have*

$$x(\log x)^{\varrho^\mp_r} \ll S^*_f(x;r) \ll x(\log x)^{\varrho^\pm_r} \qquad (r \in \mathscr{R}^\mp) \tag{1.11}$$

for $x \geq x_0(f,r)$, where

$$\mathscr{R}^- := [0,1] \cup [2,3] \cup [4,\infty), \qquad \mathscr{R}^+ := [1,2] \cup [3,4], \tag{1.12}$$

and

$$\begin{cases} \varrho^-_r := \dfrac{3^{r-1}-1}{2}, \\ \varrho^+_r := \dfrac{102+7\sqrt{21}}{210}\left(\dfrac{6-\sqrt{21}}{5}\right)^r \\ \qquad + \dfrac{102-7\sqrt{21}}{210}\left(\dfrac{6+\sqrt{21}}{5}\right)^r + \dfrac{4^r}{35} - 1. \end{cases} \tag{1.13}$$

The implied constants in (1.11) *depend on f and r.*

The upper bound part in (1.11) is essentially due to Tenenbaum [23], since his method with a minor modification allows us to obtain this result. The lower bound part is new.

The following table illustrates progress on Rankin's (1.9) and the difference from the conjectured values (1.10).

r	0	0.5	1	1.5	2	2.5	3	3.5	4
δ^-_r	-0.5	-0.292	0	0.414	1	1.828	3	4.656	7
ϱ^-_r	-0.333	-0.211	0	0.366	1	2.098	4	7.294	13
θ_r	0	-0.151	0	0.358	1	2.104	4	7.278	13
ϱ^+_r	0	-0.118	0	0.350	1	2.111	4	7.257	13
δ^+_r	0	-0.065	0	0.289	1	2.526	5.666	12.017	24.777

In order to detect sign changes or cancellations among $\lambda_f(n)$, it is natural to study the summatory function

$$S_f(x) := \sum_{n\le x} \lambda_f(n) \tag{1.14}$$

and compare it with (1.11). Investigation of the upper estimate for $S_f(x)$ has a long history. In 1927, Hecke [8] showed

$$S_f(x) \ll_f x^{1/2}$$

for all $f \in \mathrm{H}_k^*(N)$ and $x \ge 1$. Subsequent improvements came with the use of the identity

$$\frac{1}{\Gamma(r+1)}\sum_{n\le x}(x-n)^r a_f(n) = \frac{1}{(2\pi)^3}\sum_{n\ge 1}\left(\frac{x}{n}\right)^{(k+3)/2} a_f(n) J_{k+3}(4\pi\sqrt{nx}),$$

where $a_f(n) := \lambda_f(n)n^{(k-1)/2}$ and $J_k(t)$ is the first kind Bessel function. Such an identity was first given by Wilton [26] for Ramanujan's τ-function, and later generalized by Walfisz [24] to other forms. Let ϑ be a constant satisfying

$$|\lambda_f(n)| \ll n^{\vartheta} \quad (n \ge 1).$$

Walfisz proved that

$$S_f(x) \ll_f x^{(1+\vartheta)/3} \quad (x \ge 1). \tag{1.15}$$

Inserting into (1.15) the values of ϑ from the historical record yields

$$S_f(x) \ll_{f,\varepsilon} \begin{cases} x^{11/24+\varepsilon} & \text{(Kloosterman [12])},\\ x^{4/9+\varepsilon} & \text{(Davenport [2], Salié [19])},\\ x^{5/12+\varepsilon} & \text{(Weil [25])},\\ x^{1/3+\varepsilon} & \text{(Deligne [4])},\end{cases}$$

for any $\varepsilon > 0$. Hafner & Ivić [6, Theorem 1] removed the factor x^{ε} of Deligne's result. On the other hand, by combining Walfisz' method with his idea in the study of (1.7), Rankin [18] showed that

$$S_f(x) \ll_{f,\varepsilon} x^{1/3}(\log x)^{\delta_{1/2}^+ +\varepsilon} \tag{1.16}$$

for any $\varepsilon > 0$ and $x \ge 2$.

Here we propose a better bound, by combining Walfisz' method [24] and Tenenbaum's approach [23]. It is worth pointing out that Tenenbaum's method is not only to improve $\delta_{1/2}^+$ to $\varrho_{1/2}^+$ but also remove the ε in (1.16).

Theorem 2. *For $f \in \mathrm{H}_k^*(N)$, we have*

$$S_f(x) \ll x^{1/3}(\log x)^{\varrho_{1/2}^+} \tag{1.17}$$

for $x \ge 2$, where the implied constant depends on f.

In the opposite direction, Hafner & Ivić [6, Theorem 2] proved that there is a positive constant D such that

$$S_f(x) = \Omega_{\pm}\bigg(x^{1/4}\exp\bigg\{\frac{D(\log_2 x)^{1/4}}{(\log_3 x)^{3/4}}\bigg\}\bigg),$$

where $\log_r$ denotes the r-fold iterated logarithm.

As an application of Theorems 1 and 2, we consider the quantities

$$\mathscr{N}_f^{\pm}(x) := \sum_{\substack{n\le x\\ \lambda_f(n)\gtrless 0}} 1. \tag{1.18}$$

Very recently Kohnen, Lau & Shparlinski [13, Theorem 1] proved

$$\mathscr{N}_f^{\pm}(x) \gg_f \frac{x}{(\log x)^{17}} \tag{1.19}$$

for $x \ge x_0(f)$ (1).

Here we propose a better bound.

Corollary 1. *For any $f \in \mathrm{H}_k^*(N)$, we have*

$$\mathscr{N}_f^{\pm}(x) \gg \frac{x}{(\log x)^{1-1/\sqrt{3}}}$$

for $x \ge x_0(f)$, where the implied constant depends on f. If we assume Sato–Tate's conjecture, then the exponent $1 - 1/\sqrt{3} \approx 0.422$ can be improved to $2 - 16/(3\pi) \approx 0.302$.

In a joint paper with Lau [14], we shall remove the logarithmic factor by a completely different method.

2. Method of Rankin.

Let $k \ge 2$ be an even integer, $N \ge 1$ be squarefree, $f \in \mathrm{H}_k^*(N)$ and $r > 0$. Following Rankin's idea [17], we shall find two optimal multiplicative functions $\lambda_{f,r}^{\pm}(n)$ such that

$$\lambda_{f,r}^{\mp}(p^\nu) \le |\lambda_f(p^\nu)|^{2r} \le \lambda_{f,r}^{\pm}(p^\nu) \quad (r \in \mathscr{R}^{\mp}) \tag{2.1}$$

for all primes p and integers $\nu \ge 1$; furthermore, their associated Dirichlet series $\Lambda_{f,r}^{\pm}(s)$ (see (2.8) below) in the half-plane $\operatorname{Re} s \ge 1$ will be controlled by $F_j(s)$ for $j = 1, \dots, 4$. Then we can apply Tauberian theorems to obtain the asymptotic behaviour of the summatory functions of $\lambda_{f,r}^{\pm}(n)$.

2.1. *Construction of $\lambda_{f,r}^{\pm}(n)$.* For $\boldsymbol{a} := (a_1, \dots, a_4) \in \mathbb{R}^4$ and $r > 0$, consider the function

$$h_r(t;\boldsymbol{a}) := t^r - a_1 t - a_2 t^2 - a_3 t^3 - a_4 t^4 \quad (0 \le t \le 1) \tag{2.2}$$

(1) It is worth indicating that they gave explicit values for the implied constant in $\ll$ and for $x_0(f)$.

and let

$$\kappa_- := \frac{1}{4}, \quad \eta_- := \frac{3}{4}, \quad \kappa_+ := \frac{6-\sqrt{21}}{20}, \quad \eta_+ := \frac{6+\sqrt{21}}{20}. \tag{2.3}$$

In Subsection 2.3, we shall explain the reason behind this choice.

LEMMA 2.1. *If the function $h_r(t;\boldsymbol{a})$ defined by* (2.2) *satisfies*

$$h_r'(\kappa_-;\boldsymbol{a}) = h_r'(\eta_-;\boldsymbol{a}) = h_r(\kappa_-;\boldsymbol{a}) = h_r(\eta_-;\boldsymbol{a}) = 0,$$

then

$$a_j = a_j^- := \frac{P_j^-(\kappa_-,\eta_-) - P_j^-(\eta_-,\kappa_-)}{(\kappa_- - \eta_-)^3} \tag{2.4}$$

for $1 \le j \le 4$, *where*

$$\begin{aligned}
P_1^-(\kappa,\eta) &:= \{(4-r)\kappa + (r-2)\eta\}\kappa^{r-1}\eta^2,\\
P_2^-(\kappa,\eta) &:= \{(2r-8)\kappa^2 + (1-r)\kappa\eta + (1-r)\eta^2\}\kappa^{r-2}\eta,\\
P_3^-(\kappa,\eta) &:= \{(4-r)\kappa^2 + (4-r)\kappa\eta + 2(r-1)\eta^2\}\kappa^{r-2},\\
P_4^-(\kappa,\eta) &:= \{(r-3)\kappa + (1-r)\eta\}\kappa^{r-2}.
\end{aligned}$$

Proof. This can be done by routine calculation. ∎

LEMMA 2.2. *If the function $h_r(t;\boldsymbol{a})$ defined by* (2.2) *is such that*

$$\begin{cases} h_r'(\kappa_+;\boldsymbol{a}) = h_r'(\eta_+;\boldsymbol{a}) = 0,\\ h_r(\kappa_+;\boldsymbol{a}) = h_r(\eta_+;\boldsymbol{a}) = h_r(1;\boldsymbol{a}), \end{cases}$$

then

$$a_j = a_j^+ := \frac{P_j^+(\kappa_+,\eta_+) - P_j^+(\eta_+,\kappa_+)}{(\kappa_+ - 1)^2(\eta_+ - 1)^2(\kappa_+ - \eta_+)^3} \tag{2.5}$$

for $1 \le j \le 4$, *where*

$$\begin{aligned}
P_1^+(\kappa,\eta) &:= r\kappa^{r-1}\eta(\kappa-1)(\eta-\kappa)(\kappa\eta+2\kappa+\eta)(\eta-1)^2\\
&\quad + 2(\kappa^r-1)\kappa\eta(\eta-1)^2(2\kappa\eta+4\kappa-\eta^2-2\eta-3),\\
P_2^+(\kappa,\eta) &:= r\kappa^{r-1}(\kappa-1)(\kappa-\eta)(\eta-1)^2(2\kappa\eta+\kappa+\eta^2+2\eta)\\
&\quad + (\eta^r-1)(\kappa-1)^2(8\kappa\eta^2+4\eta^2-\eta\kappa^2-2\kappa\eta-3\eta-\kappa^3-2\kappa^2-3\kappa),\\
P_3^+(\kappa,\eta) &:= r\kappa^{r-1}(\kappa-1)(\kappa+2\eta+1)(\eta-\kappa)(\eta-1)^2\\
&\quad + 2(\kappa^r-1)(2\kappa^2+2\kappa\eta-\eta^2-2\eta-1)(\eta-1)^2,\\
P_4^+(\kappa,\eta) &:= r\kappa^{r-1}(\kappa-1)(\kappa-\eta)(\eta-1)^2 + (\eta^r-1)(\kappa-1)^2(3\eta-\kappa-2).
\end{aligned}$$

Proof. This is done by routine calculation as well. ∎

LEMMA 2.3. *Let $\boldsymbol{a}^\pm := (a_1^\pm,\ldots,a_4^\pm)$, where the values of $a_i^\pm$ are given in Lemmas* 2.1 *and* 2.2. *Then for* $0 \le t \le 1$ *we have*

$$h_r(t;\boldsymbol{a}^-) \gtreqless 0 \quad \text{and} \quad h_r(t;\boldsymbol{a}^+) \lesseqgtr h_r(1;\boldsymbol{a}^+) \quad \text{for} \quad r \in \mathscr{R}^\mp.$$

Proof. We have

$$h_r^{(4)}(t;\boldsymbol{a}^-) = r(r-1)(r-2)(r-3)t^{r-4} - 24a_4^-,$$

so $h_r^{(4)}(t;\boldsymbol{a}^-)$ has at most one zero for $t>0$ and $h_r^{(i)}(t;\boldsymbol{a}^-)$ has at most $5-i$ zeros for $t>0$ $(i=3,2,1,0)$. Since $h_r(\kappa_-;\boldsymbol{a}^-) = h_r(\eta_-;\boldsymbol{a}^-) = h_r(0;\boldsymbol{a}^-)$, it follows that $h_r'(\xi_-;\boldsymbol{a}^-) = h_r'(\xi_-';\boldsymbol{a}^-) = 0$ for some $\xi_- \in (0,\kappa_-)$ and $\xi_-' \in (\kappa_-,\eta_-)$. Therefore ξ_-, κ_-, ξ_-' and η_- are the only zeros of $h_r'(t;\boldsymbol{a}^-)$ in $(0,1)$.

Now

$$\begin{aligned} h_r''(\kappa_-;\boldsymbol{a}^-) &= 8\cdot 4^{-r}(2r^2-2r+3+2r3^{r-2}-11\cdot 3^{r-2}),\\ h_r''(\eta_-;\boldsymbol{a}^-) &= 8\cdot 4^{-r}(2r^2-6r-3-2r3^{r}+43\cdot 3^{r-2}). \end{aligned}$$

From these, it is easy to verify that

$$h_r''(\kappa_-;\boldsymbol{a}^-), h_r''(\eta_-;\boldsymbol{a}^-) \begin{cases} \gtrless 0 & \text{if } r\in \mathring{\mathscr{R}}^{\mp},\\ =0 & \text{if } r=1,2,3,4, \end{cases}$$

where $\mathring{\mathscr{R}}^{\mp}$ denotes the interior of $\mathscr{R}^{\mp}$. Hence $h_r(t;\boldsymbol{a}^-)$ takes its minimum (maximum, respectively) values in $[0,1]$ at 0, κ_-, η_- when $r\in\mathring{\mathscr{R}}^-$ ($r\in\mathring{\mathscr{R}}^+$, respectively). Moreover, $h_r(t;\boldsymbol{a}^-)$ has local maxima (minima, respectively) at ξ_-, ξ_-' when $r\in\mathring{\mathscr{R}}^-$ ($r\in\mathring{\mathscr{R}}^+$, respectively). This proves the assertion about $h_r(t;\boldsymbol{a}^-)$.

Similarly we can prove the corresponding result on $h_r(t;\boldsymbol{a}^+)$. ■

Now we define the multiplicative function $\lambda_{f,r}^{\pm}(n)$ by

$$\lambda_{f,r}^{\mp}(p^\nu) := \begin{cases} \sum_{0\le j\le 4} 2^{2(r-j)}a_j^{\mp}\lambda_f(p)^{2j} & \text{if } \nu=1 \text{ and } r>0,\\ 0 & \text{if } \nu\ge 2 \text{ and } r\in\mathscr{R}^{\mp},\\ |\lambda_f(p^\nu)|^{2r} & \text{if } \nu\ge 2 \text{ and } r\in\mathscr{R}^{\pm}, \end{cases} \tag{2.6}$$

where

$$a_0^- := 0 \quad\text{and}\quad a_0^+ := 1-a_1^+-a_2^+-a_3^+-a_4^+. \tag{2.7}$$

In view of (1.6), we can apply Lemma 2.3 with $t=|\cos\theta_f(p)|$ to deduce that the inequality (2.1) holds for all primes p and integers $\nu\ge 1$. By multiplicativity, this inequality also holds for all integers $n\ge 1$ (in place of p^ν).

2.2. *Dirichlet series associated to* $\lambda_{f,r}^{\pm}(n)$. For $f\in \mathrm{H}_k^*(N)$, $r>0$ and $\operatorname{Re} s>1$, we define

$$\Lambda_{f,r}^{\pm}(s) := \sum_{n\ge 1}\lambda_{f,r}^{\pm}(n)n^{-s}. \tag{2.8}$$

Next we shall study their analytic properties in the half-plane $\operatorname{Re} s \geq 1$ by using the higher order symmetric power L-functions $L(s, \operatorname{sym}^m f)$ associated to $f \in \mathrm{H}_k^*(N)$, due to Gelbart & Jacquet [5] for $m = 2$, and Kim & Shahidi ([10], [11]) for $m = 3, 4, 5, 6, 7, 8$. Here the symmetric mth power associated to f is defined as

$$L(s, \operatorname{sym}^m f) := \prod_p \prod_{0 \leq j \leq m} (1 - \alpha_f(p)^{m-j} \beta_f(p)^j p^{-s})^{-1}$$

for $\operatorname{Re} s > 1$, where $\alpha_f(p)$ and $\beta_f(p)$ are given by (1.3) and (1.4). According to the references mentioned above, the function $L(s, \operatorname{sym}^m f)$ for $m = 2, 3, \ldots, 8$ is invertible for $\operatorname{Re} s \geq 1$.

We start by studying $F_1(s)$, $F_2(s)$, $F_3(s)$ and $F_4(s)$, where $F_r(s)$ is defined by (1.8).

Lemma 2.4. *Let $k \geq 2$ be an even integer, $N \geq 1$ be squarefree and $f \in \mathrm{H}_k^*(N)$. For $j = 1, 2, 3, 4$ and $\operatorname{Re} s > 1$, we have*

$$F_j(s) = \zeta(s)^{m_j} G_j(s) H_j(s), \tag{2.9}$$

where

$$m_1 := 1, \qquad m_2 := 2, \qquad m_3 := 5, \qquad m_4 := 14, \tag{2.10}$$

and

$$\begin{aligned} G_1(s) &:= L(s, \operatorname{sym}^2 f), \\ G_2(s) &:= L(s, \operatorname{sym}^2 f)^3 L(s, \operatorname{sym}^4 f), \\ G_3(s) &:= L(s, \operatorname{sym}^2 f)^9 L(s, \operatorname{sym}^4 f)^5 L(s, \operatorname{sym}^6 f), \\ G_4(s) &:= L(s, \operatorname{sym}^2 f)^{34} L(s, \operatorname{sym}^4 f)^{20} L(s, \operatorname{sym}^6 f)^7 L(s, \operatorname{sym}^8 f) \end{aligned}$$

are invertible for $\operatorname{Re} s \geq 1$. Here the function $H_j(s)$ admits a Dirichlet series convergent absolutely in $\operatorname{Re} s > 1/2$ and $H_j(s) \neq 0$ for $\operatorname{Re} s = 1$.

Proof. Write x for the trace of a local factor of $L(s, f)$ (i.e. $\alpha_f(p) + \beta_f(p)$), and denote by $T_n(x)$ the polynomial which is the trace of its nth symmetric power. Then

$$\begin{aligned} T_2 &= x^2 - 1, \\ T_4 &= x^4 - 3x^2 + 1, \\ T_6 &= x^6 - 5x^4 + 6x^2 - 1, \\ T_8 &= x^8 - 7x^6 + 15x^4 - 10x^2 + 1, \end{aligned}$$

from which we deduce

$$\begin{aligned} x^2 &= 1 + T_2, \\ x^4 &= 2 + 3T_2 + T_4, \\ x^6 &= 5 + 9T_2 + 5T_4 + T_6, \\ x^8 &= 14 + 34T_2 + 20T_4 + 7T_6 + T_8. \end{aligned}$$

This implies (2.9). By the results on $L(s, \mathrm{sym}^m f)$ mentioned above, $G_j(s)$ is invertible for $\mathrm{Re}\, s \geq 1$. ∎

LEMMA 2.5. *Let $k \geq 2$ be an even integer, $N \geq 1$ be squarefree and $f \in \mathrm{H}_k^*(N)$. For $r > 0$ and* $\mathrm{Re}\, s > 1$, *we have*

$$\Lambda_{f,r}^{\pm}(s) = \zeta(s)^{\varrho_r^{\pm}+1} H_{f,r}^{\pm}(s), \tag{2.11}$$

where

$$\varrho_r^{\pm} := 2^{2r-8}(2^8 a_0^{\pm} + 2^6 a_1^{\pm} + 2^4 \cdot 2a_2^{\pm} + 2^2 \cdot 5a_3^{\pm} + 14a_4^{\pm}) - 1 \tag{2.12}$$

and $H_{f,r}^{\pm}(s)$ is invertible for $\mathrm{Re}\, s \geq 1$.

Proof. By definition (2.6), for $\mathrm{Re}\, s > 1$ we can write

$$\Lambda_{f,r}^{-}(s) = \prod_p \Big(1 + \sum_{0 \leq j \leq 4} 2^{2(r-j)} a_j^{-} \lambda_f(p)^{2j} p^{-s}\Big) = \prod_{0 \leq j \leq 4} F_j(s)^{2^{2(r-j)} a_j^{-}} H_r^{-}(s)$$

for $r \in \mathscr{R}^{-}$, and

$$\begin{aligned} \Lambda_{f,r}^{-}(s) &= \prod_p \Big(1 + \sum_{0 \leq j \leq 4} 2^{2(r-j)} a_j^{-} \lambda_f(p)^{2j} p^{-s} + \sum_{\nu \geq 2} |\lambda_f(p^{\nu})|^{2r} p^{-\nu s}\Big) \\ &= \prod_{0 \leq j \leq 4} F_j(s)^{2^{2(r-j)} a_j^{-}} H_r^{-}(s) \end{aligned}$$

for $r \in \mathscr{R}^{+}$, where $F_0(s) = \zeta(s)$ is the Riemann zeta-function and $H_r^{-}(s)$ is a Dirichlet series absolutely convergent for $\mathrm{Re}\, s > 1/2$ such that $H_r^{-}(s) \neq 0$ for $\mathrm{Re}\, s = 1$. Now the desired result with the "−" sign follows from Lemma 2.4. The other part can be treated in the same way. ∎

2.3. *Optimization of $\lambda_{f,r}^{\pm}(p)$ and choice of $\kappa_{\pm}, \eta_{\pm}$.* If we regard $\kappa_{\pm}, \eta_{\pm}$ as parameters, the $\varrho_r^{\pm}$ given by (2.12) are functions of those parameters. We choose $(\kappa_{\pm}, \eta_{\pm})$ in $(0,1)^2$ optimally (they must be solutions of $\partial\varrho_r^{\pm}/\partial\kappa = 0$ and $\partial\varrho_r^{\pm}/\partial\eta = 0$), which can be done by using formal calculation via Maple. Their values are given by (2.3).

3. Proof of Theorem 1. In view of Lemma 2.5 and a classical fact on $\zeta(s)$, we can write

$$\Lambda_{f,r}^{\pm}(s) = \frac{H_{f,r}^{\pm}(1)}{(s-1)^{\varrho_r^{\pm}+1}} + g_{f,r}^{\pm}(s) \tag{3.1}$$

in some neighbourhood of $s = 1$ with $\operatorname{Re} s > 1$, where $H^{\pm}_{f,r}(1) \neq 1$ and $g^{\pm}_{f,r}(s)$ is holomorphic at $s = 1$. Since $\lambda^{\pm}_{f,r}(n) \geq 0$, we can apply Delange's tauberian theorem [3] to write

$$\sum_{n \leq x} \lambda^{\pm}_{f,r}(n) \sim H^{\pm}_{f,r}(1) x (\log x)^{\varrho^{\pm}_r} \quad (x \to \infty). \tag{3.2}$$

Now Theorem 1 follows from (2.1) and (3.2). ∎

4. Proof of Theorem 2. By (3.1), it follows that

$$\prod_p \left(1 + \sum_{\nu \geq 1} \frac{\lambda^{\pm}_{f,r}(p^\nu)}{p^{\nu\sigma}}\right) = \frac{H^{\pm}_{f,r}(1)}{(\sigma - 1)^{\varrho^{\pm}_r + 1}} + g^{\pm}_{f,r}(\sigma)$$

for $\sigma > 1$. From this, (2.6), (2.7) and Deligne's inequality, we deduce that

$$\sum_p \frac{\lambda^{\pm}_{f,r}(p)}{p^\sigma} = (\varrho^{\pm}_r + 1) \log(\sigma - 1)^{-1} + C^{\pm}_{f,r} + o(1) \quad (\sigma \to 1+),$$

where $C^{\pm}_{f,r}$ is some constant.

On the other hand, the prime number theorem implies, by partial integration, that

$$\sum_p p^{-\sigma} = \log(\sigma - 1)^{-1} + C + o(1) \quad (\sigma \to 1+),$$

where C is an absolute constant. Thus the preceding relation can be written as

$$\sum_p \frac{\lambda^{\pm}_{f,r}(p) - (\varrho^{\pm}_r + 1)}{p^\sigma} = C^{\pm}_{f,r} + (\varrho^{\pm}_r + 1) C + o(1) \quad (\sigma \to 1+). \tag{4.1}$$

According to Exercise II.7.8 of [22], the formula (4.1) implies

$$\sum_p \frac{\lambda^{\pm}_{f,r}(p) - (\varrho^{\pm}_r + 1)}{p} = C^{\pm}_{f,r} + (\varrho^{\pm}_r + 1) C.$$

Hence

$$\sum_{p \leq x} \frac{\lambda^{\pm}_{f,r}(p)}{p} = (\varrho^{\pm}_r + 1) \log_2 x + C^{\pm}_{f,r} + (\varrho^{\pm}_r + 1) C + o(1) \quad (x \to \infty).$$

Now we apply a well known result of Shiu [20] and (2.1) to write

$$\sum_{x\le n\le x+z} |\lambda_f(n)|^{2r} \ll \frac{z}{\log x}\exp\Big(\sum_{p\le x}\frac{|\lambda_f(p)|^{2r}}{p}\Big) \ll \frac{z}{\log x}\exp\Big(\sum_{p\le x}\frac{\lambda^+_{f,r}(p)}{p}\Big) \ll z(\log x)^{\varrho_r^+} \tag{4.2}$$

for $r\in\mathscr{R}^-$, any $\varepsilon>0$, $x\ge x_0(\varepsilon)$ and $x^{1/4}\le z\le x$. We use this with $r=1/2$ in (9) of [18]; then the first term on the right-hand side of (10) of [18] is replaced by $x^{1/2}z^{-1/2}(\log x)^{\varrho^+_{1/2}}$. Applying (4.2) with $r=1/2$ again to the second term on the right-hand side of (10) of [18] yields

$$S_f(x)\ll x^{1/2}z^{-1/2}(\log x)^{\varrho^+_{1/2}} + z(\log x)^{\varrho^+_{1/2}}.$$

Taking $z=x^{1/3}$, we obtain the required result when the level is $N=1$. The general case can be treated in much the same way as indicated in [18]. ∎

5. Proof of Corollary 1. By comparing (1.17) and the lower bound part in (1.11) with $r=1/2$, it is easy to deduce that

$$\sum_{\substack{n\le x\\ \lambda_f(n)\gtreqless 0}} |\lambda_f(n)| \gg_f x(\log x)^{\varrho^-_{1/2}}$$

for $x\ge x_0(f)$. Since $\varrho^-_{1/2}=-(1-1/\sqrt{3})/2$ and $\varrho^+_1=0$, a simple application of the Cauchy–Schwarz inequality yields the desired estimate.

The second assertion can be obtained by noticing that $\theta_{1/2}=8/(3\pi)-1$. ∎

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