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A sum analogous to Dedekind sums and its hybrid mean value formula

by

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1. Introduction. For a positive integer k and an arbitrary integer h, the classical Dedekind sum $S(h,k)$ is defined by

$$S(h,k)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-1/2 & \text{if } x \text{ is not an integer,}\\ 0 & \text{if } x \text{ is an integer.}\end{cases}$$

The various properties of $S(h,k)$ were investigated by many authors. For example, T. M. Apostol [2] and L. Carlitz [3] obtained a reciprocity theorem of $S(h,k)$. J. B. Conrey *et al.* [5] studied the mean value distribution of $S(h,k)$, and first got an important asymptotic formula. The author [4] and [8] also studied some sums analogous to Dedekind sums, and proved several mean value theorems. In October, 2000, during his visit in Xi'an, Professor Todd Cochrane introduced a sum analogous to the Dedekind sum as follows:

$$C(h,k)=\sum_{a=1}^{k}{}'\left(\left(\frac{\overline{a}}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where $\overline{a}$ is defined by $a\overline{a}\equiv 1 \bmod k$ and $\sum_{a=1}^{\prime k}$ denotes the summation over all $1\le a\le k$ such that $(a,k)=1$. Then he suggested studying the arithmetical properties and mean value distribution properties of $C(h,k)$. Concerning these problems, we have not made any progress yet. But for a square-full number k (i.e. $p\,|\,k$ if and only if $p^2\,|\,k$), the author [7] found that there are some close relations between $C(h,k)$ and the classical Kloosterman sum

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$$K(m,n,k)=\sum_{b=1}^{k}{}' e\left(\frac{mb+n\bar{b}}{k}\right),$$

where $e(y)=e^{2\pi iy}$, and proved the following asymptotic formula:

$$\sum_{h=1}^{k}{}' K(h,1;k)C(h,k)=\frac{-1}{2\pi^2}\,k\phi(k)+O\left(k\exp\left(\frac{3\ln k}{\ln\ln k}\right)\right),$$

where $\exp(y)=e^y$.

In this paper, we shall discuss the hybrid mean value problem involving $C(h,k)$ and the general Kloosterman sum

$$K(m,n,r;k)=\sum_{b=1}^{k}{}' e\left(\frac{mb^r+n\bar{b}^r}{k}\right),$$

where r is any fixed positive integer. We shall use estimates for character sums and the mean value theorem of Dirichlet L-functions to prove the following:

THEOREM 1. *Let p be an odd prime. Then we have the asymptotic formula*

$$\sum_{h=1}^{p-1} K(h,1;p)C(h,p)=\frac{-1}{2\pi^2}\,p^2+O\left(p\exp\left(\frac{3\ln p}{\ln\ln p}\right)\right).$$

THEOREM 2. *Let p be an odd prime. Then for any positive integer $r\geq 2$,*

$$\sum_{h=1}^{p-1} K(h,1,r;p)C(h,p)=\frac{-1}{2\pi^2}\,p^2+O(rp^{3/2}\ln^2 p).$$

From Theorem 2 we may immediately deduce the following:

COROLLARY. *Let p be an odd prime. Then for any fixed $\varepsilon>0$, the asymptotic formula*

$$\sum_{h=1}^{p-1} K(h,1,r;p)C(h,p)\sim\frac{-1}{2\pi^2}\,p^2\qquad \text{as } p\to\infty$$

holds for all integer $2\leq r\leq p^{1/2-\varepsilon}$.

For general integer $k>2$, it is an unsolved problem whether there exists an asymptotic formula for $\sum_{h=1}^{\prime k} K(h,1,r;k)C(h,k)$. We conjecture that

$$\sum_{h=1}^{k}{}' K(h,1,r;k)C(h,k)\sim\frac{-1}{2\pi^2}\,k\phi(k)\qquad\text{as } k\to\infty$$

for all integer $k>2$ and any fixed positive integer r.

2. Some lemmas. We need the following lemmas:

LEMMA 1 (see [7]). *Let a, k be integers with $k \geq 3$ and $(a,k)=1$. Then*

$$C(a,k) = \frac{-1}{\pi^2\phi(k)} \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \overline{\chi}(a)\left(\sum_{n=1}^{\infty} \frac{G(\chi,n)}{n}\right)^2,$$

where χ runs through the Dirichlet characters modulo k with $\chi(-1)=-1$, and

$$G(\chi,n) = \sum_{b=1}^{k} \chi(b)e\left(\frac{bn}{k}\right)$$

denotes the Gauss sum corresponding to χ.

LEMMA 2 (see [7]). *Let $k > 2$ be any integer. Then*

$$\sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} L^2(1,\chi) = \frac{1}{2}\phi(k) + O\left(\exp\left(\frac{3\ln k}{\ln\ln k}\right)\right).$$

LEMMA 3. *Let $k > 2$ be any integer. Then*

$$\sum_{\substack{1\leq a\leq k \\ (a,k)=1}} \Bigg| \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \chi(a)L^2(1,\overline{\chi})\Bigg| \ll k\ln^2 k.$$

Proof. Let $\tau(n)$ be the Dirichlet divisor function. Then for any $N \geq k$ and non-principal character χ modulo k, applying Abel's identity we obtain

$$L^2(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)\tau(n)}{n} = \sum_{1\leq n\leq N} \frac{\chi(n)\tau(n)}{n} + \int_N^{\infty} \frac{A(y,\chi)}{y^2}\,dy, \tag{1}$$

where

$$A(y,\chi) = \sum_{N<n\leq y} \chi(n)\tau(n).$$

From (1) we have

$$\begin{aligned}
(2)\qquad & \sum_{\substack{1\leq a\leq k \\ (a,k)=1}} \Bigg| \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \chi(a)L^2(1,\overline{\chi})\Bigg| \\
&\leq \sum_{\substack{1\leq a\leq k \\ (a,k)=1}} \Bigg| \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \sum_{1\leq n\leq N} \frac{\tau(n)\chi(a)\overline{\chi}(n)}{n}\Bigg| \\
&\quad + \sum_{\substack{1\leq a\leq k \\ (a,k)=1}} \Bigg| \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \chi(a)\int_N^{\infty} \frac{A(y,\overline{\chi})}{y^2}\,dy\Bigg| \equiv M_1 + M_2.
\end{aligned}$$

Now we estimate M_1 and M_2 in (2) respectively. Note that for $(a,k)=1$, from the orthogonality relation for characters we have

$$\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} \chi(a) = \frac{1}{2}\sum_{\chi \bmod k}(1-\chi(-1))\chi(a) = \frac{1}{2}\sum_{\chi \bmod k}\chi(a) - \frac{1}{2}\sum_{\chi \bmod k}\chi(-a) = \begin{cases} \frac{1}{2}\phi(k) & \text{if } a \equiv 1 \pmod k, \\ -\frac{1}{2}\phi(k) & \text{if } a \equiv -1 \pmod k, \\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

Applying (3) we can get the estimate

$$M_1 = \sum_{\substack{1\le a\le k\\ (a,k)=1}} \left| \frac{1}{2}\phi(k) \sideset{}{'}\sum_{\substack{1\le n\le N\\ n\equiv a \,(\bmod k)}} \frac{\tau(n)}{n} - \frac{1}{2}\phi(k) \sideset{}{'}\sum_{\substack{1\le n\le N\\ n\equiv -a \,(\bmod k)}} \frac{\tau(n)}{n}\right| \le \frac{1}{2}\phi(k) \sum_{\substack{1\le a\le k\\ (a,k)=1}} \sideset{}{'}\sum_{\substack{1\le n\le N\\ n\equiv a \,(\bmod k)}} \frac{\tau(n)}{n} + \frac{1}{2}\phi(k) \sum_{\substack{1\le a\le k\\ (a,k)=1}} \sideset{}{'}\sum_{\substack{1\le n\le N\\ n\equiv -a \,(\bmod k)}} \frac{\tau(n)}{n} \le \phi(k) \sideset{}{'}\sum_{1\le n\le N} \frac{\tau(n)}{n} \ll \phi(k)\ln^2 N. \tag{4}$$

Applying Cauchy's inequality and estimates for character sums,

$$\sum_{\chi\ne\chi_0} \left| \sum_{N\le n\le M} \chi(n)\right|^2 = \sum_{\chi\ne\chi_0} \left| \sum_{N\le n\le M\le N+d} \chi(n)\right|^2 = \phi(d) \sum_{N\le n\le M\le N+d} \chi_0(n) - \left| \sum_{N\le n\le M\le N+d} \chi_0(n)\right|^2 \le \frac{\phi^2(d)}{4},$$

we have

$$\sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} |A(y,\chi)| \ll \sqrt{\phi(k)}\Bigg(\sqrt{y}\sum_{n\le\sqrt{y}} \sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} \left| \sum_{m\le y/n} \chi(m)\right|^2\Bigg)^{1/2} + \sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} \left| \sum_{n\le\sqrt{y}} \chi(n)\right|^2 + \sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} \left| \sum_{n\le\sqrt{N}} \chi(n)\right|^2 \ll \sqrt{y}\,\sqrt{\phi^3(k)}. \tag{5}$$

From (5) we can also get the estimate

$$M_2 = \sum_{\substack{1\le a\le k\\(a,k)=1}} \Bigg| \sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} \chi(a) \int_N^\infty \frac{A(y,\overline{\chi})}{y^2}\,dy \Bigg| \tag{6}$$

$$\le \sum_{\substack{1\le a\le k\\(a,k)=1}} \int_N^\infty \frac{1}{y^2} \sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} |A(y,\overline{\chi})|\,dy$$

$$\ll \frac{\phi^{5/2}(k)}{\sqrt{N}}.$$

Taking $N = k^3$, combining (4), (5) and (6) we immediately get the estimate

$$\sum_{\substack{1\le a\le k\\(a,k)=1}} \Bigg| \sum_{\substack{\chi \bmod k\\ \chi(-1)=-1}} \chi(a) L^2(1,\overline{\chi}) \Bigg| \ll k \ln^2 k.$$

This proves Lemma 3.

3. Proof of Theorems 1 and 2. In this section, we complete the proof of Theorems 1 and 2. Let p be an odd prime. Then from Lemma 1 and the properties of Gauss sums (see Theorem 8.19 of [1]) we can get the identity

$$C(a,p) = \frac{-1}{\pi^2\phi(p)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \overline{\chi}(a) \left(\sum_{n=1}^\infty \frac{G(\chi,n)}{n} \right)^2 \tag{7}$$

$$= \frac{-1}{\pi^2\phi(p)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \overline{\chi}(a) \tau^2(\chi) L^2(1,\overline{\chi}).$$

For any fixed positive integer r, applying (7) we deduce

$$\sum_{h=1}^{p-1} K(h,1,r;p) C(h,p) \tag{8}$$

$$= \frac{-1}{\pi^2\phi(p)} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left(\sum_{h=1}^{p-1} \overline{\chi}(h) K(h,1,r;p) \right) \tau^2(\chi) L^2(1,\overline{\chi}).$$

For any primitive character χ modulo p, from the properties of Gauss sums we have

$$\tau(\chi)\tau(\overline{\chi}) = -p \quad \text{if } \chi(-1) = -1$$

and

$$\sum_{h=1}^{p-1}\overline{\chi}(h)K(h,1,r;p)=\sum_{b=1}^{p-1}\sum_{h=1}^{p-1}\overline{\chi}(h)e\left(\frac{hb^r+\overline{b}^r}{p}\right)\tag{9}$$
$$=\sum_{b=1}^{p-1}\chi(b^r)e\left(\frac{\overline{b}^r}{p}\right)\sum_{h=1}^{p-1}\overline{\chi}(h)e\left(\frac{h}{p}\right)$$
$$=\tau(\overline{\chi})G(1,\overline{\chi}^r,r;p).$$

Noting $G(1,\chi,1;p)=\tau(\chi)$, from (8), (9), and Lemma 2 we immediately obtain

$$\sum_{h=1}^{p-1}K(h,1;p)C(h,p)=\frac{-1}{\pi^2\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\tau^2(\overline{\chi})\tau^2(\chi)L^2(1,\overline{\chi})$$
$$=\frac{-1}{\pi^2}\cdot\frac{p^2}{\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}L^2(1,\overline{\chi})$$
$$=\frac{-1}{2\pi^2}p^2+O\left(p\exp\left(\frac{3\ln p}{\ln\ln p}\right)\right).$$

This completes the proof of Theorem 1.

Now we prove Theorem 2. Note that

$$G(1,\overline{\chi}^r,r;p)\tau(\chi)=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\overline{\chi}^r(b)e\left(\frac{a+b^r}{p}\right)$$
$$=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)e\left(\frac{ab^r+b^r}{p}\right)$$
$$=\sum_{a=1}^{p-1}\chi(a)\left(\sum_{b=1}^{p-1}e\left(\frac{(a+1)b^r}{p}\right)\right)$$

and

$$\left|\sum_{b=1}^{p-1}e\left(\frac{nb^r}{p}\right)\right|\le r\sqrt{p}\quad\text{for all integer } n \text{ with } (p,n)=1.$$

(This result follows from Weil's general upper bound on exponential sums, see [6].)

From (8), (9), Lemma 2 and Lemma 3 we have

$$\sum_{h=1}^{p-1}K(h,1,r;p)C(h,p)=\frac{-1}{\pi^2\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\tau(\overline{\chi})G(1,\overline{\chi}^r,r;p)\tau^2(\chi)L^2(1,\overline{\chi})$$

$$= \frac{p}{\pi^2\phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} G(1, \overline{\chi}^r, r; p)\tau(\chi)L^2(1, \overline{\chi})$$

$$= \frac{p}{\pi^2\phi(p)} \sum_{a=1}^{p-1} \left(\sum_{b=1}^{p-1} e\left(\frac{(a+1)b^r}{p} \right) \right) \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a)L^2(1, \overline{\chi})$$

$$= \frac{-p(p-1)}{\pi^2\phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} L^2(1, \overline{\chi})$$

$$+ \frac{p}{\pi^2\phi(p)} \sum_{a=1}^{p-2} \left(\sum_{b=1}^{p-1} e\left(\frac{(a+1)b^r}{p} \right) \right) \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a)L^2(1, \overline{\chi})$$

$$= \frac{-1}{2\pi^2} p^2 + O\left(p \exp\left(\frac{3 \ln p}{\ln \ln p} \right) \right)$$

$$+ O\left(\sum_{a=1}^{p-2} \left| \sum_{b=1}^{p-1} e\left(\frac{(a+1)b^r}{p} \right) \right| \cdot \left| \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a)L^2(1, \overline{\chi}) \right| \right)$$

$$= \frac{-1}{2\pi^2} p^2 + O(rp^{3/2} \ln^2 p).$$

This proves Theorem 2.

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