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Multi-continued fraction algorithm on multi-formal Laurent series

by

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1. Introduction. Continued fractions [8, 14] are a useful tool in many number theoretical problems and in numerical computing. It is well known that the simple continued fraction expansion of a single real number gives the best solution to its rational approximation problem. Many people have contrived to construct multidimensional continued fractions in dealing with the rational approximation problem for multi-reals. One construction is the Jacobi–Perron algorithm (JPA) (see [1]). This algorithm and its modifications have been extensively studied [6, 7, 10, 13]. These algorithms are adapted to study the same problem for multi-formal Laurent series [2, 4, 11, 12]. But none of them guarantees the best rational approximation in general. In this paper, we deal with the multi-rational approximation problem over the formal Laurent series field $F((z^{-1}))$: given an element $\underline{r} \in F((z^{-1}))^m$, find $\underline{p} \in F[z]^m$ and $q \in F[z]$ such that $\underline{p}/q$ approximates $\underline{r}$ as close as possible while $\deg(q)$ is bounded.

We propose a new continued fraction algorithm for multi-formal Laurent series. It is proved that this algorithm guarantees best rational approximations for multi-formal Laurent series.

The paper is organized as follows: Section 2 deals with the indexed valuation of $F((z^{-1}))^m$. Section 3 contains the detailed definition of the problem of optimal rational approximation of multi-formal Laurent series. Section 4 proposes an algorithm called multidimensional continued fraction algorithm (m-CFA, for short), which produces a multi-continued fraction expansion $C(\underline{r})$ for any given multi-formal Laurent series $\underline{r}$. Section 5 shows that $C(\underline{r})$ satisfies three basic conditions. Section 6 states the main results of this pa-

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per: $C(\underline{r})$ provides optimal rational approximations to $\underline{r}$. In Section 7 we complete all proofs.

2. Indexed valuation over $F((z^{-1}))$. Denote by $\mathbb{Z}$ the ring of integers, by F an arbitrary field, by m a positive integer, and by Z_m the set $\{1,\ldots,m\}$. Let $F[z]$ be the polynomial ring over F, $F(z)$ the rational fraction field over F, and

$$F((z^{-1})) = \Big\{ \sum_{i\ge t} a_i z^{-i} \;\Big|\; t \in \mathbb{Z},\, a_i \in F \Big\}$$

the formal Laurent series field over F. By identifying $p(z)/q(z) \in F(z)$ with $p(z)q(z)^{-1} \in F((z^{-1}))$, where $p(z)$ and $q(z)$ ($\neq 0$) are polynomials, we view $F(z)$ as a subfield of $F((z^{-1}))$. We denote by F^m, $F[z]^m$ and $F((z^{-1}))^m$ the column vector space of dimension m over F, $F[z]$ and $F((z^{-1}))$ respectively.

Definition 1 (order over $Z_m \times \mathbb{Z}$). For any two elements (h,v) and (h',v') in $Z_m \times \mathbb{Z}$, we define $(h,v) < (h',v')$ if $v < v'$ or $v = v'$, $h < h'$.

The order defined above is linear [3]. It is clear that if $(j,n) < (j',n')$, then $n \le n'$ and $(j,n+x) < (j',n'+x)$ for any $x \in \mathbb{Z}$.

For $1 \le j \le m$, we write $\underline{e}_j = (\overset{1}{0},\ldots,0,\overset{j}{1},0,\ldots,\overset{m}{0})^\tau$, which is the jth standard base element in F^m, where τ means transpose; moreover, set

$$z^{-n}\underline{e}_j = (\overset{1}{0},\ldots,0,\overset{j}{z^{-n}},0,\ldots,\overset{m}{0})^\tau \in F((z^{-1}))^m, \qquad \forall (j,n) \in Z_m \times \mathbb{Z},$$

which is called the (j,n)th *monomial* in $F((z^{-1}))^m$, and define

$$rz^{-n}\underline{e}_j = z^{-n}\underline{e}_j r = (\overset{1}{0},\ldots,0,\overset{j}{rz^{-n}},0,\ldots,\overset{m}{0})^\tau, \qquad \forall (j,n) \in Z_m \times \mathbb{Z},\ r \in F((z^{-1})).$$

Definition 2. Any non-zero element $\underline{r} = (r_1,\ldots,r_m)^\tau$ in $F((z^{-1}))^m$, $r_j = \sum r_{j,n} z^{-n} \in F((z^{-1}))$, can be uniquely expressed as

$$\underline{r} = \sum_{(i,t)\le(j,n)} r_{j,n} z^{-n}\underline{e}_j, \quad r_{j,n} \in F, \tag{1}$$

for some $(i,t) \in Z_m \times \mathbb{Z}$, which is called its *monomial decomposition.* $r_{j,n}z^{-n}\underline{e}_j$ is called the (j,n)th *term* of $\underline{r}$; $r_{j,n}$ the (j,n)th *coefficient* of $\underline{r}$; $z^{-n}\underline{e}_j$ a *monomial* of $\underline{r}$ (written $z^{-n}\underline{e}_j \in \underline{r}$) if $r_{j,n} \neq 0$. For $\underline{0} \neq \underline{r} \in F((z^{-1}))^m$, define

$$\mathrm{Iv}(\underline{r}) = \min\{(j,n) \mid r_{j,n} \neq 0,\ (j,n) \in Z_m \times \mathbb{Z}\} \in Z_m \times \mathbb{Z}, \tag{2}$$

and $\mathrm{Iv}(\underline{0}) = (1,\infty)$.

The pair $\mathrm{Iv}(\underline{r})$ is called the *indexed valuation* of $\underline{r}$. If $\mathrm{Iv}(\underline{r}) = (h, v)$, then v is called the *valuation* of $\underline{r}$ and denoted by $v(\underline{r})$, and h the *index* of $\underline{r}$ and denoted by $I(\underline{r})$; and $r_{h,v} z^{-v} \underline{e}_h$ is the *leading term* of $\underline{r}$, denoted by $\mathrm{Ld}(\underline{r})$.

It is clear that $v(\cdot)$ is the discrete valuation on $F((z^{-1}))$ when $m = 1$. When $m > 1$, we have

$$v(\underline{r}) = \min\{v(r_j) \mid 1 \le j \le m\}, \quad I(\underline{r}) = \min\{j \mid v(r_j) = v(\underline{r}),\, 1 \le j \le m\}.$$

THEOREM 3. *Let* $\alpha, \beta \in F((z^{-1}))^m$.

(1) $\mathrm{Iv}(\alpha) \neq (1, \infty) \Leftrightarrow \alpha \neq 0$.
(2) *If* $\mathrm{Iv}(\alpha) = (h, v)$, *then* $\mathrm{Iv}(r\alpha) = (h, v + v(r))$ *for any* $0 \neq r \in F((z^{-1}))$. *In particular*, $\mathrm{Iv}(r\alpha) = \mathrm{Iv}(\alpha)$ *if* $0 \neq r \in F$.
(3) $\mathrm{Iv}(\alpha + \beta) \ge \min\{\mathrm{Iv}(\alpha), \mathrm{Iv}(\beta)\}$, *and equality holds if and only if* $\mathrm{Ld}(\alpha) + \mathrm{Ld}(\beta) \neq \underline{0}$. *In particular*, $\mathrm{Iv}(\alpha+\beta) = \mathrm{Iv}(\alpha)$ *if* $\mathrm{Iv}(\alpha) < \mathrm{Iv}(\beta)$.

In studying the rational approximation problem of multi-formal Laurent series, we need the concept of limit with respect to the indexed valuation [15]. We say that a sequence $\{\underline{x}_k\}_{k\ge0}$ in $F((z^{-1}))^m$ is convergent with respect to the indexed valuation if there exists an element $\underline{x} \in F((z^{-1}))^m$ (called a *limit of* $\{\underline{x}_k\}_{k\ge0}$) which satisfies: for any $(h, v) \in Z_m \times \mathbb{Z}$ there is a positive integer k_0 such that $\mathrm{Iv}(\underline{x}_k - \underline{x}) \ge (h, v)$ whenever $k \ge k_0$.

One can verify that:

(1) If a sequence $\{\underline{x}_k\}_{k\ge0}$ is convergent, then its limit $\underline{x} \in F((z^{-1}))^m$ is unique. Therefore we can write $\underline{x} = \lim_{k\to\infty} \underline{x}_k$.
(2) $F(z)^m$ is dense in $F((z^{-1}))^m$ in the sense that each element in $F((z^{-1}))^m$ is the limit of a sequence from $F(z)^m$.

3. Optimal rational approximation

DEFINITION 4. Let

$$\frac{\underline{p}(z)}{q(z)} = \left(\frac{p_1(z)}{q(z)}, \ldots, \frac{p_m(z)}{q(z)}\right)^{\tau} \in F[z]^m$$

be an m-tuple of rational fractions, where $q(z)$ is the common denominator of the m components. The indexed valuation $\mathrm{Iv}(\underline{r} - \underline{p}(z)/q(z))$ is called the *precision of approximation* of $\underline{r}$ by $\underline{p}(z)/q(z)$. The tuple $\underline{p}(z)/q(z)$ is called an *optimal rational approximant* to $\underline{r}$ if it satisfies the following two conditions:

- $\mathrm{Iv}\left(\underline{r} - \dfrac{\underline{u}(z)}{v(z)}\right) < \mathrm{Iv}\left(\underline{r} - \dfrac{\underline{p}(z)}{q(z)}\right)\ \forall \dfrac{\underline{u}(z)}{v(z)} \in F(z)^m$, $\deg(v(z)) < \deg(q(z))$;
- $\mathrm{Iv}\left(\underline{r} - \dfrac{\underline{u}(z)}{v(z)}\right) \le \mathrm{Iv}\left(\underline{r} - \dfrac{\underline{p}(z)}{q(z)}\right)\ \forall \dfrac{\underline{u}(z)}{v(z)} \in F(z)^m$, $\deg(v(z)) = \deg(q(z))$.

For a non-zero element $r = \sum_{i=0}^{t} b_{-i} z^i + \sum_{i\ge 1} b_i z^{-i}$ in $F((z^{-1}))$, where $t \ge 0$, define $\lfloor r \rfloor = \sum_{i=0}^{t} b_{-i} z^i$ and $\{r\} = \sum_{i\ge 1} b_i z^{-i}$, which are called the *polynomial part* and the *remaining part* of r, respectively [15].

For $\underline{r} = (\ldots, r_j(z), \ldots)^\tau \in F((z^{-1}))^m$, set $\lfloor \underline{r} \rfloor = (\ldots, \lfloor r_j(z) \rfloor, \ldots)^\tau$ and $\{\underline{r}\} = (\ldots, \{r_j(z)\}, \ldots)^\tau$. It is not difficult to see that $\underline{p}(z)/q(z)$ is an optimal rational approximant to $\{\underline{r}\}$ of precision (h,n) if and only if $\lfloor \underline{r} \rfloor + \underline{p}(z)/q(z)$ is an optimal rational approximant to $\underline{r}$ of the same precision. Therefore, it is enough to consider elements $\underline{r}$ with positive valuation ($v(\underline{r}) > 0$) in studying optimal rational approximation of formal Laurent series.

4. Multidimensional continued fraction algorithm. We denote by

$$\mathrm{diag}(r_1, \ldots, r_m), \quad r_j \in F((z^{-1})), \tag{3}$$

the diagonal matrix of order m with the jth diagonal element equal to r_j.

m-Continued Fraction Algorithm (m-CFA, for short). Given $\underline{r} \in F((z^{-1}))^m$ with $\underline{r} \ne \underline{0}$ and $v(\underline{r}) > 0$, initially set $\underline{a}_0 = \underline{0}$, $\Delta_{-1} = I_m = \mathrm{diag}(\ldots, z^{-c_{0,j}}, \ldots)$, $c_{0,j} = 0$ for $1 \le j \le m$, and $\alpha_0 = \underline{r}$. For any integer $k \ge 1$, suppose $\Delta_{k-2} = \mathrm{diag}(\ldots, z^{-c_{k-1,j}}, \ldots), c_{i,j} \in \mathbb{Z}$, and $\underline{0} \ne \alpha_{k-1} = (\ldots, \alpha_{k-1,j}, \ldots)^\tau \in F((z^{-1}))^m$ have been obtained. Then the computations for the kth round are defined by the following steps:

(1) Set $(h_k, c_k) = \mathrm{Iv}(\Delta_{k-2}\alpha_{k-1})$.
(2) Set $\Delta_{k-1} = \mathrm{diag}(\ldots, z^{-c_{k,j}}, \ldots)$, which is an $m \times m$ diagonal matrix, where $c_{k,j} = c_{k-1,j}$ if $j \ne h_k$, and $c_{k,h_k} = c_k$.
(3) Set $\varrho_k = (\ldots, \varrho_{k,j}, \ldots)^\tau \in F((z^{-1}))^m$, where $\varrho_{k,j} = \alpha_{k-1,j}/\alpha_{k-1,h_k}$ if $j \ne h_k$, and $\varrho_{k,h_k} = 1/\alpha_{k-1,h_k}$.
(4) Set $\alpha_k = \{\varrho_k\}$ and $\underline{a}_k = \lfloor \varrho_k \rfloor$. If $\alpha_k = \underline{0}$, then set $\mu = k$, and the algorithm terminates.

Define $\mu = \infty$ if the above procedure never terminates.

By letting m-CFA act on $\underline{r}$, we get an expansion of the form

$$C(\underline{r}) = [\underline{0}, h_1, \underline{a}_1, \ldots, h_k, \underline{a}_k, \ldots], \qquad 1 \le k \le \mu.$$

We call $C(\underline{r})$ the *multi-continued fraction expansion* of $\underline{r}$, and μ the *length* of $C(\underline{r})$.

In what follows we keep the notation $C(\underline{r})$ and all the notations appearing in the process of generating $C(\underline{r})$, and define

$$\underline{a}_k = (a_{k,1}, \ldots, a_{k,j}, \ldots, a_{k,m}). \tag{4}$$

For the case $\mu < \infty$, we see that $\alpha_\mu = \underline{0}$, and it is convenient to set

$$(h_{\mu+1}, c_{\mu+1}) = \mathrm{Iv}(\Delta_\mu \alpha_\mu) = (1, \infty). \tag{5}$$

Theorem 5. *$\alpha_{k-1,h_k} \ne 0$ for $1 \le k \le \mu$. As a consequence, the m-CFA is well defined, and $0 \ne a_{k-1,h_k} \in F[z]$, $\deg(a_{k-1,h_k}) \ge 1$ for $1 \le k \le \mu$.*

Proof. From $(h_k, c_k) = \mathrm{Iv}(\Delta_{k-2}\alpha_{k-1})$, we see that $c_{k-1,h_k} + v(\alpha_{k-1,h_k}) = c_k$, thus, $v(\alpha_{k-1,h_k}) = -c_{k-1,h_k} + c_k \in \mathbb{Z}$, hence $\alpha_{k-1,h_k} \neq 0$.

REMARK. When $m = 1$, the m-CFA is exactly the classical continued fraction algorithm [14] for formal power series. In fact, when $m = 1$, we have $h_k = 1$ for all k, hence both step (1) and step (2) at each round are unnecessary. Now, the 1-CFA is as follows (we write $\underline{r} = r$, $\underline{a}_k = a_k$): Initially, set $a_0 = 0$, $\alpha_0 = r$. For any integer $k \geq 1$, suppose $[a_0, a_1, \ldots, a_{k-1}]$ and $0 \neq \alpha_{k-1} \in F((z^{-1}))$ have been obtained. Then the computations for the kth round are defined by the following steps:

(1) Set $\varrho_k = 1/\alpha_{k-1}$.
(2) Set $\alpha_k = \{\varrho_k\}$ and $a_k = \lfloor \varrho_k \rfloor$. If $\alpha_k = \underline{0}$, then set $\mu = k$, and the algorithm terminates.

Define $\mu = \infty$ if the above procedure never terminates.

5. Three conditions satisfied by the multi-continued fraction expansion $C(\underline{r})$. Define

$$
\begin{cases}
t_0 = 0, \\
t_k = \deg(a_{k,h_k}(z)), \quad 1 \leq k \leq \mu, \\
v_{0,j} = 0, \\
v_{k,j} = \sum_{h_i=j,\, 1\leq i\leq k} t_i, \quad 1 \leq k \leq \mu,\ 1 \leq j \leq m, \\
v_k = v_{k,h_k}, \quad 1 \leq k \leq \mu, \\
D_k = \mathrm{diag}(z^{-v_{k,1}}, \ldots, z^{-v_{k,m}}), \quad 0 \leq k \leq \mu, \\
t_\mu = \infty, \\
(h_{\mu+1}, v_{\mu+1}) = (1, \infty) \quad \text{if } \mu < \infty.
\end{cases}
\tag{6}
$$

THEOREM 6. *For $1 \leq k \leq \mu$, $C(\underline{r})$ satisfies*:

- Condition 1: $t_k \geq 1$,
- Condition 2: $\mathrm{Iv}(D_k \underline{a}_k) = (h_k, v_{k-1,h_k})$,
- Condition 3: $(h_k, v_{k-1,h_k}) < (h_{k+1}, v_{k+1})$.

Before proving Theorem 6 we make some preparations. In particular, we introduce the concept of a D-matrix.

DEFINITION 7. We call a diagonal matrix over $F((z^{-1}))$ a *D-matrix* if each of its diagonal elements is a power of z.

It is clear that both D_k and Δ_{k-1} are D-matrices.

LEMMA 8. *Let $\underline{0} \neq \varrho \in F((z^{-1}))^m$ and $I(\Delta\varrho) = h$, where Δ is a D-matrix. Then*

$$
\mathrm{Iv}(\Delta\varrho) = \begin{cases}
\mathrm{Iv}(\Delta\lfloor\varrho\rfloor) < \mathrm{Iv}(\Delta\{\varrho\}) & \text{if } \lfloor\varrho_h\rfloor \neq 0, \\
\mathrm{Iv}(\Delta\{\varrho\}) < \mathrm{Iv}(\Delta\lfloor\varrho\rfloor) & \text{if } \lfloor\varrho_h\rfloor = 0,
\end{cases}
$$

where ϱ_h is the hth component of ϱ.

Proof. Set $\Delta = \mathrm{diag}(\ldots, z^{-b_j}, \ldots)$. Then $\mathrm{Iv}(\Delta\varrho) = \mathrm{Iv}(z^{-b_h} z^{-v(\varrho_h)} \underline{e}_h)$. Noting that there are no common monomials in $\Delta\{\varrho\}$ and $\Delta\lfloor\varrho\rfloor$, we see that $\mathrm{Iv}(\Delta\{\varrho\}) \neq \mathrm{Iv}(\Delta\lfloor\varrho\rfloor)$, and then $\mathrm{Iv}(\Delta\varrho) = \min\{\mathrm{Iv}(\Delta\lfloor\varrho\rfloor), \mathrm{Iv}(\Delta\{\varrho\})\}$, which leads to the result by observing that $\mathrm{Iv}(\Delta\varrho) = \mathrm{Iv}(\Delta\lfloor\varrho\rfloor) < \mathrm{Iv}(\Delta\{\varrho\})$ if and only if $z^{-v(\varrho_h)}\underline{e}_h \in \lfloor\varrho\rfloor$, and the latter holds true if and only if $\lfloor\varrho_h\rfloor \neq 0$.

Lemma 9.

(1) $t_k = \deg(a_{k,h_k}) = -v(\varrho_{k,h_k}) = v(\alpha_{k-1,h_k}) > 0$ *for* $1 \le k \le \mu$.
(2) $v_{k,j} = c_{k,j}$ *and* $v_k = c_k$ *for* $0 \le k \le \mu$ *and* $1 \le j \le m$. *As a consequence*, $D_k = \Delta_{k-1}$ *for* $0 \le k \le \mu$.
(3) $\mathrm{Iv}(\Delta_{k-1}\varrho_k) = (h_k, v_{k-1,h_k})$.

Proof. (1) Noting that $\alpha_{k-1,h_k} \neq 0$ and α_{k-1} is the remaining part of ϱ_{k-1}, we see that $0 < v(\alpha_{k-1,h_k}) \neq \infty$. Since $a_{k,h_k} = \lfloor\varrho_{k,h_k}\rfloor = \lfloor\alpha^{-1}_{k-1,h_k}\rfloor$, we obtain

$$\begin{aligned} t_k = \deg(a_{k,h_k}) &= -v(\lfloor\varrho_{k,h_k}\rfloor) = -v(\lfloor\alpha^{-1}_{k-1,h_k}\rfloor) \\ &= -v(\alpha^{-1}_{k-1,h_k}) = v(\alpha_{k-1,h_k}) > 0. \end{aligned}$$

(2) By definition,

$$v_{k,j} = \begin{cases} v_{k-1,j} & \text{if } j \neq h_k, \\ v_k = v_{k-1,h_k} + t_k & \text{if } j = h_k. \end{cases}$$

From $\mathrm{Iv}(\Delta_{k-2}\alpha_{k-1}) = (h_k, c_k)$, we see that $c_k = c_{k-1,h_k} + v(\alpha_{k-1,h_k}) = c_{k-1,h_k} + t_k$, so

$$c_{k,j} = \begin{cases} c_{k-1,j} & \text{if } j \neq h_k, \\ c_k = c_{k-1,h_k} + t_k & \text{if } j = h_k. \end{cases} \tag{7}$$

Therefore, the $v_{k,j}$ satisfy the same recurrence relation as $c_{k,j}$, and they have the same initial values: $v_{0,j} = c_{0,j}$, so $v_{k,j} = c_{k,j}$ and $v_k = v_{k,h_k} = c_{k,h_k} = c_k$.

(3) From (7) we see that

$$\Delta_{k-1} = \begin{pmatrix} I_{h_k-1} & 0 & 0 \\ 0 & z^{-t_k} & 0 \\ 0 & 0 & I_{m-h_k} \end{pmatrix} \Delta_{k-2},$$

and

$$\varrho_k = \begin{pmatrix} I_{h_k-1} & 0 & 0 \\ 0 & \alpha^{-1}_{k-1,h_k} & 0 \\ 0 & 0 & I_{m-h_k} \end{pmatrix} \alpha_{k-1}\alpha^{-1}_{k-1,h_k}.$$

Then

$$\Delta_{k-1}\varrho_k = \begin{pmatrix} I_{h_k-1} & 0 & 0 \\ 0 & z^{-t_k}\alpha^{-1}_{k-1,h_k} & 0 \\ 0 & 0 & I_{m-h_k} \end{pmatrix} \Delta_{k-2}\alpha_{k-1}\alpha^{-1}_{k-1,h_k}.$$

Since $v(z^{-t_k}\alpha_{k-1,h_k}^{-1}) = 0$, we obtain

$$v\left(\begin{pmatrix} I_{h_k-1} & 0 & 0 \\ 0 & z^{-t_k}\alpha_{k-1,h_k}^{-1} & 0 \\ 0 & 0 & I_{m-h_k} \end{pmatrix} \Delta_{k-2}\alpha_{k-1}\right) = v(\Delta_{k-2}\alpha_{k-1}) = (h_k, c_k).$$

Thus

$$\mathrm{Iv}(\Delta_{k-1}\varrho_k) = (h_k, c_k - v(\alpha_{k-1,h_k})) = (h_k, v_{k-1,h_k}).$$

Proof of Theorem 6. From Lemma 9 we see that Condition 1 holds true. Noting that $h_k = I(\Delta_{k-1}\varrho_k)$ and $\lfloor \varrho_{k,h_k} \rfloor \neq 0$ (see Lemma 9), from Lemma 8 we get

$$\mathrm{Iv}(\Delta_{k-1}\varrho_k) = \mathrm{Iv}(\Delta_{k-1}\lfloor \varrho_k \rfloor) < \mathrm{Iv}(\Delta_{k-1}\{\varrho_k\}).$$

Since $\underline{a}_k = \lfloor \varrho_k \rfloor$, $\alpha_k = \{\varrho_k\}$ and $\mathrm{Iv}(\Delta_{k-1}\varrho_k) = (h_k, v_{k-1,h_k})$, we get

$$(h_k, v_{k-1,h_k}) = \mathrm{Iv}(\Delta_{k-1}\underline{a}_k) < \mathrm{Iv}(\Delta_{k-1}\alpha_k) = (h_{k+1}, c_{k+1}) = (h_{k+1}, v_{k+1}),$$

which together with $\Delta_{k-1} = D_k$ tells us that $C(\underline{r})$ satisfies Conditions 2 and 3.

6. m-CFA and optimal rational approximations. In this section we show how $C(\underline{r})$ provides optimal rational approximations to $\underline{r}$ by rational fractions $\left(\frac{\underline{p}_k}{q_k}\right)$, $0 \le k \le \mu$, defined below.

Define iteratively the square matrices B_k of order $m+1$ over $F[z]$:

$$\begin{cases} B_0 = I_{m+1}, \\ B_k = B_{k-1}E_{h_k}A(\underline{a}_k), \quad 1 \le k \le \mu, \end{cases} \tag{8}$$

where

$$\begin{cases} E_h = (\underline{e}_1\, \underline{e}_2 \ldots \underline{e}_{h-1}\, \underline{e}_{m+1}\, \underline{e}_{h+1} \cdots \underline{e}_m\, \underline{e}_h), \\ A(\underline{a}_k) = \begin{pmatrix} I_m & \underline{a}_k \\ 0 & 1 \end{pmatrix}. \end{cases} \tag{9}$$

In other words, E_h is the matrix of order $m+1$ obtained by exchanging the hth and $(m+1)$th columns of the identity matrix I_{m+1}.

Define

$$\begin{pmatrix} \underline{p}_k \\ q_k \end{pmatrix} = B_k(0 \ldots 0\, 1)^\tau, \tag{10}$$

which is the last column of B_k, where $\underline{p}_k(z) \in F[z]^m$ and $q_k(z) \in F[z]$.

REMARK. When $m = 1$, write $\underline{p}_k = p_k \in F[z]$, $\underline{a}_k = a_k \in F[z]$; we claim that p_k and q_k satisfy the following recurrence relation:

$$\begin{cases} p_k = p_{k-2} + a_k p_{k-1}, \\ q_k = q_{k-2} + a_k q_{k-1}, \end{cases} \quad \text{for } k \ge 1, \tag{11}$$

where $(p_{-1}, q_{-1}) = (1, 0)$ and $(p_0, q_0) = (0, 1)$, hence the rational fractions $\binom{p_k}{q_k}$ are exactly the rational approximants provided by the classical continued fraction algorithm [14]. In fact, we can prove (11) and

$$B_k = \begin{pmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{pmatrix}, \qquad k \geq 0, \tag{12}$$

together by induction on k. It is easy to check (12) for $k = 0$. Assume

$$B_{k-1} = \begin{pmatrix} p_{k-2} & p_{k-1} \\ q_{k-2} & q_{k-1} \end{pmatrix};$$

then the first column of B_k is

$$B_k(1\ 0)^\tau = B_{k-1}E_1A(a_k)(1\ 0)^\tau = B_{k-1}E_1(1\ 0)^\tau = B_{k-1}(0\ 1)^\tau = \begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix},$$

hence (12) is true because $\binom{p_k}{q_k}$ is the second column of B_k by definition. Then we have

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = B_k(0\ 1)^\tau = B_{k-1}E_1A(a_k)(0\ 1)^\tau = B_{k-1}\begin{pmatrix} 1 \\ a_k \end{pmatrix} = \begin{pmatrix} p_{k-2} + a_k p_{k-1} \\ q_{k-2} + a_k q_{k-1} \end{pmatrix},$$

hence (11) holds true for k.

Define

$$\begin{cases} d_0 = 0, \\ d_k = \sum_{1 \leq i \leq k} t_i, \\ n_k = d_{k-1} + v_k, \\ d_{\mu+1} = t_{\mu+1} = n_{\mu+1} = \infty \quad \text{if } \mu < \infty. \end{cases} \tag{13}$$

From the fact that $n_k = d_{k-1} + v_k = d_k + v_{k-1,h_k}$ and $n_{k+1} = d_k + v_{k+1}$, we see immediately that Condition 3: $(h_k, v_{k-1,h_k}) < (h_{k+1}, v_{k+1})\ \forall 1 \leq k \leq \mu$, is equivalent to the following condition:

$$(h_k, n_k) < (h_{k+1}, n_{k+1}) \qquad \forall 1 \leq k \leq \mu. \tag{14}$$

THEOREM 10.

(1) $\gcd(q_k(z), \ldots, p_{k,j}(z), \ldots) = 1$ *for all* $0 \leq k \leq \mu$, *where* $p_{k,j}(z)$ *is the* j*th component of* $\underline{p}_k(z)$.

(2) $\deg(q_k(z)) = d_k$ *for all* $0 \leq k \leq \mu$.

THEOREM 11. $\mathrm{Iv}(\underline{r} - \underline{p}_k(z)/q_k(z)) = (h_{k+1}, n_{k+1})$. *As a consequence,*

$$\underline{r} = \begin{cases} \dfrac{\underline{p}_\mu(z)}{q_\mu(z)} & \text{if } \mu < \infty, \\[2ex] \lim\limits_{k \to \infty} \dfrac{\underline{p}_k(z)}{q_k(z)} & \text{if } \mu = \infty. \end{cases} \tag{15}$$

We call the rational fraction $\underline{p}_k(z)/q_k(z)$ $(0 \le k \le \mu)$ the kth *rational approximant* of $C(\underline{r})$, and we say $C(\underline{r})$ converges to $\underline{r}$ in the sense that (15) holds. The following theorem shows that $C(\underline{r})$ provides optimal rational approximations to $\underline{r}$.

THEOREM 12. *Assume* $q(z) \in F[z]$, $d_k \le \deg(q(z)) < d_{k+1}$ *and* $\underline{p}(z) \in F[z]^m$ *for some* $0 \le k \le \mu$. *Then*

$$\mathrm{Iv}\left(\underline{r} - \frac{\underline{p}(z)}{q(z)}\right) \le \mathrm{Iv}\left(\underline{r} - \frac{\underline{p}_k(z)}{q_k(z)}\right) = (h_{k+1}, n_{k+1}).$$

As a consequence, no $\underline{p}(z)/q(z)$ *with* $\deg(q(z)) < d_{k+1}$ *approximates* $\underline{r}$ *better than* $\underline{p}_k(z)/q_k(z)$. *In particular*:

(1) *Each* $\underline{p}_k(z)/q_k(z)$, $0 \le k \le \mu$, *is an optimal rational approximant to* $\underline{r}$.

(2) *If* $\underline{p}(z)/q(z)$ *is an optimal rational approximant to* $\underline{r}$, *then* $\deg(q(z)) = d_k$ *for some* k, $0 \le k \le \mu$.

7. Proof of the theorems

7.1. *Proof of Theorem 10.* First we express $q_k(z)$ explicitly. To do this, for $0 \le k \le \mu$ we denote by P_{k-1} the $m \times m$ submatrix of B_k which is made up of the first m columns and the first m rows, and by Q_{k-1} the $1 \times m$ submatrix of B_k made up of the first m columns and the last row; moreover, denote by $\underline{P}_{k-1,j}$ $(\in F[z]^m)$ the jth column of P_{k-1}, and by $Q_{k-1,j}$ $(\in F[z])$ the jth component of Q_{k-1} for $1 \le j \le m$.

LEMMA 13. *For* $1 \le k \le \mu$, *we have*:

$$(1) \qquad B_{k-1}E_{h_k} = \begin{pmatrix} P_{k-1} & P_{k-2,h_k} \\ Q_{k-1} & Q_{k-2,h_k} \end{pmatrix}.$$

$$(2) \qquad \begin{pmatrix} P_{k-1,j} \\ Q_{k-1,j} \end{pmatrix} = \begin{cases} \begin{pmatrix} P_{k-2,j} \\ Q_{k-2,j} \end{pmatrix} & \text{if } j \ne h_k, \\ \begin{pmatrix} \underline{p}_{k-1} \\ q_{k-1} \end{pmatrix} & \text{if } j = h_k. \end{cases}$$

Proof. (1) We have

$$B_{k-1}E_{h_k}\begin{pmatrix} I_m \\ 0 \end{pmatrix} = B_{k-1}E_{h_k}A(\underline{a}_k)\begin{pmatrix} I_m \\ 0 \end{pmatrix} = B_k\begin{pmatrix} I_m \\ 0 \end{pmatrix} = \begin{pmatrix} P_{k-1} \\ Q_{k-1} \end{pmatrix},$$

and

$$B_{k-1}E_{h_k}\begin{pmatrix} \underline{0} \\ 1 \end{pmatrix} = B_{k-1}\underline{e}_{h_k} = \begin{pmatrix} P_{k-2,h_k} \\ Q_{k-2,h_k} \end{pmatrix}.$$

Hence, we get (1).

(2) We have

$$\begin{pmatrix} P_{k-1,j} \\ Q_{k-1,j} \end{pmatrix} = B_{k-1}E_{h_k}\underline{e}_j = \begin{cases} B_{k-1}\underline{e}_j = \begin{pmatrix} P_{k-2,j} \\ Q_{k-2,j} \end{pmatrix} & \text{if } j \neq h_k, \\ B_{k-1}\underline{e}_{m+1} = \begin{pmatrix} \underline{p}_{k-1} \\ q_{k-1} \end{pmatrix} & \text{if } j = h_k. \end{cases}$$

Now, for $k \geq 1$, q_k can be expressed explicitly as

$$\begin{aligned} (16)\qquad & q_k(z) \\ &= (0_{1\times m}\ 1)B_k\begin{pmatrix} \underline{0} \\ 1 \end{pmatrix} = (0_{1\times m}\ 1)B_{k-1}E_{h_k}\begin{pmatrix} I_m & \underline{a}_k \\ \underline{0} & 1 \end{pmatrix}\begin{pmatrix} \underline{0} \\ 1 \end{pmatrix} \\ &= (0_{1\times m}\ 1)\begin{pmatrix} P_{k-1} & P_{k-2,h_k} \\ Q_{k-1} & Q_{k-2,h_k} \end{pmatrix}\begin{pmatrix} \underline{a}_k \\ 1 \end{pmatrix} = Q_{k-1}\underline{a}_k(z) + Q_{k-2,h_k} \\ &= q_{k-1}(z)a_{k,h_k}(z) + \sum_{j\neq h_k,\, Q_{k-1,j}\neq 0,\, 1\leq j\leq m} Q_{k-1,j}a_{k,j}(z) + Q_{k-2,h_k}. \end{aligned}$$

To evaluate the degree of $a_{k,j}(z)$ and to show how $Q_{k-1,j}$ depends on some $q_i(z)$ $(0 \leq i \leq k-1)$, we define a function $l(k,j)$, which is associated to $C(\underline{r})$ and defined on the set $[1,\mu]\times Z_m$ ($[1,\mu] = \{k \in \mathbb{Z} \mid 1 \leq k \leq \mu\}$), in the following way: $l(k,j) = k_0$ if there exists an integer k_0 such that $1 \leq k_0 \leq k$, $h_{k_0} = j$ and $h_i \neq j$ for all $k_0 < i \leq k$; and $l(k,j) = 0$ otherwise. It is clear that

$$(17)\qquad \begin{cases} l(k,h_k) = k, \\ l(k,j) < k \quad \text{if } j \neq h_k, \\ h_{l(k,j)} = j, \\ v_{k,j} = v_{l(k,j)}. \end{cases}$$

LEMMA 14. *For $1 \leq k \leq \mu$, we have*

$$(1)\qquad \begin{pmatrix} P_{k-1,j} \\ Q_{k-1,j} \end{pmatrix} = \begin{cases} \begin{pmatrix} \underline{p}_{l(k,j)-1} \\ q_{l(k,j)-1} \end{pmatrix} & \textit{if } l(k,j) \geq 1, \\ \begin{pmatrix} \underline{e}_j \\ 0 \end{pmatrix} & \textit{if } l(k,j) = 0. \end{cases}$$

As a consequence, $l(k,j) \geq 1$ *if* $Q_{k-1,j} \neq 0$.

(2) *For $j \neq h_k$,*

$$\begin{cases} \deg(a_{k,j}(z)) < d_k - d_{l(k,j)-1} & \textit{if } l(k,j) \geq 1, \\ a_{k,j}(z) \in F & \textit{if } l(k,j) = 0. \end{cases}$$

Proof. (1) For $j = h_k$, we have seen

$$\begin{pmatrix} P_{k-1,h_k} \\ Q_{k-1,h_k} \end{pmatrix} = \begin{pmatrix} \underline{p}_{k-1} \\ q_{k-1} \end{pmatrix} = \begin{pmatrix} \underline{p}_{l(k,h_k)-1} \\ q_{l(k,h_k)-1} \end{pmatrix}.$$

For $j \neq h_k$ we have

$$\begin{pmatrix} P_{k-1,j} \\ Q_{k-1,j} \end{pmatrix} = \begin{pmatrix} P_{k-2,j} \\ Q_{k-2,j} \end{pmatrix} = \dots$$
$$= \begin{cases} \begin{pmatrix} P_{l(k,j)-1,j} \\ Q_{l(k,j)-1,j} \end{pmatrix} = \begin{pmatrix} P_{l(k,j)-1,h_{l(k,j)}} \\ Q_{l(k,j)-1,h_{l(k,j)}} \end{pmatrix} = \begin{pmatrix} \underline{p}_{l(k,j)-1} \\ q_{l(k,j)-1} \end{pmatrix} & \text{if } l(k,j) > 0, \\ \begin{pmatrix} P_{0-1,j} \\ Q_{0-1,j} \end{pmatrix} = \begin{pmatrix} P_{-1,j} \\ Q_{-1,j} \end{pmatrix} = \begin{pmatrix} \underline{e}_j \\ 0 \end{pmatrix} & \text{if } l(k,j) = 0. \end{cases}$$

(2) From $D_k\underline{a}_k = \sum_{1\le j\le m} z^{-v_{k,j}} a_{k,j}\underline{e}_j$ and $\mathrm{Iv}(D_k\underline{a}_k) = (h_k, v_{k-1,h_k})$ and the assumption $j \neq h_k$, we see that

$$(j, v_{k,j} - \deg(a_{k,j})) = \mathrm{Iv}(z^{-v_{k,j}} a_{k,j}\underline{e}_j) > \mathrm{Iv}(D_k\underline{a}_k) = (h_k, v_{k-1,h_k}), \tag{18}$$

and then

$$v_{k,j} - \deg(a_{k,j}) \ge v_{k-1,h_k}. \tag{19}$$

If $l(k,j) > 0$, from (18) we get

$$\begin{aligned} (j, d_k + v_{k,j} - \deg(a_{k,j})) &> (h_k, d_k + v_{k-1,h_k}) = (h_k, n_k) \\ &> (h_{l(k,j)}, n_{l(k,j)}) = (j, n_{l(k,j)}), \end{aligned}$$

so

$$d_k + v_{k,j} - \deg(a_{k,j}) > n_{l(k,j)} = d_{l(k,j)-1} + v_{l(k,j)} = d_{l(k,j)-1} + v_{k,j},$$

hence $\deg(a_{k,j}) < d_k - d_{l(k,j)-1}$. If $l(k,j) = 0$, then $v_{k,j} = 0$, and from (19) we have $\deg(a_{k,j}) \le v_{k,j} - v_{k-1,h_k} = -v_{k-1,h_k} \le 0$, hence $\deg(a_{k,j}) \le 0$, i.e., $a_{k,j}(z) \in F$.

Now we turn to the proof of Theorem 10.

(1) By definition, B_k is a matrix over $F[z]$ and $\det(B_k) = 1$, which leads to assertion (1).

(2) We argue by induction on k. For $k = 0$, we have $q_0(z) = 1$, hence $\deg(q_0) = 0 = d_0$. Assume $\deg(q_i) = d_i$ for $i < k$ and $k \ge 1$. From (16) and Lemma 14 we see that

$$q_k(z) = q_{k-1}(z)a_{k,h_k}(z) + \sum_{j\neq h_k,\, l(k,j)\ge 1,\, 1\le j\le m} q_{l(k,j)-1,j}\, a_{k,j}(z) + Q_{k-2,h_k}.$$

The required result $\deg(q_k(z)) = d_k$ follows by observing the following facts:

- $\deg(q_{k-1}(z)a_{k,h_k}(z)) = d_{k-1} + t_k = d_k$ (induction assumption).

- For $j \neq h_k$ and $l(k,j) \geq 1$, we see that (by induction assumption)
$$\deg(q_{l(k,j)-1}a_{k,j}(z)) < d_{l(k,j)-1} + d_k - d_{l(k,j)-1} = d_k.$$
- If $Q_{k-2,h_k} \neq 0$, then $l(k-1,h_k) \geq 1$, and hence $Q_{k-2,h_k} = q_{l(k-1,h_k)-1}$. So, $\deg(Q_{k-2,h_k}) = \deg(q_{l(k-1,h_k)-1}) = d_{l(k-1,h_k)-1} < d_k$.

7.2. *Proof of Theorem 11.* For $0 \leq k \leq \mu$ we define

$$\underline{r}_k = \underline{r}q_k - \underline{p}_k, \tag{20}$$
$$-R_{k-1} = \underline{r}Q_{k-1} - P_{k-1}. \tag{21}$$

We call $\underline{r}_k$ the kth *remainder vector*, and R_{k-1} the $(k-1)$th *remainder matrix.*

Theorem 11 is an easy consequence of the following

PROPOSITION 15. *For $1 \leq k \leq \mu$, we have*

(1) $R_{k-1}\varrho_k = -R_{k-2,h_k}$ *and* $\underline{r}_k = R_{k-1}\alpha_k$.
(2) $\mathrm{Iv}(R_{k-1}\alpha_k) = \mathrm{Iv}(D_k\alpha_k)$.
(3) $\mathrm{Iv}(\underline{r}_k) = (h_{k+1}, v_{k+1})$.

Proposition 15 will be proved later, now we prove Theorem 11 based on it. From item (3) of Proposition 15 we get immediately

$$\mathrm{Iv}\left(\underline{r} - \frac{\underline{p}_k(z)}{q_k(z)}\right) = \mathrm{Iv}\left(\frac{\underline{r}_k}{q_k(z)}\right) = (h_{k+1}, v_{k+1} + d_k) = (h_{k+1}, n_{k+1}).$$

To prove Proposition 15, we denote by $R_{k-1,j}$ the jth column of R_{k-1}. It is clear that

$$(-I_m,\, \underline{r})B_k = (-R_{k-1},\, \underline{r}_k), \tag{22}$$
$$(-I_m,\, \underline{r})\begin{pmatrix} P_{k-1,j} \\ Q_{k-1,j} \end{pmatrix} = -R_{k-1,j}. \tag{23}$$

LEMMA 16. *For $1 \leq k \leq \mu$, we have*

(1) $\quad (-I_m,\, \underline{r})B_{k-1}E_{h_k} = (-R_{k-1},\, -R_{k-2,h_k})$.

(2) $\quad -R_{k-1,j} = \begin{cases} -R_{k-2,j} & \text{if } j \neq h_k, \\ \underline{r}_{k-1} & \text{if } j = h_k. \end{cases}$

Proof. (1) We have

$$(-I_m,\, \underline{r})B_{k-1}E_{h_k} = (-I_m,\, \underline{r})\begin{pmatrix} P_{k-1} & P_{k-2,h_k} \\ Q_{k-1} & Q_{k-1,h_k} \end{pmatrix} = (-R_{k-1},\, -R_{k-2,h_k}).$$

(2) By (23),

$$-R_{k-1,j} = (-I_m,\, \underline{r})\begin{pmatrix} P_{k-1,j} \\ Q_{k-1,j} \end{pmatrix}$$
$$= \begin{cases} (-I_m,\, \underline{r})\begin{pmatrix} P_{k-2,j} \\ Q_{k-2,j} \end{pmatrix} = -R_{k-2,j} & \text{if } j \neq h_k, \\ (-I_m,\, \underline{r})\begin{pmatrix} \underline{p}_{k-1} \\ q_{k-1} \end{pmatrix} = \underline{r}_{k-1} & \text{if } j = h_k. \end{cases}$$

Proof of Proposition 15(1). We argue by induction on k. It is easy to check $\underline{r}_0 = R_{-1}\alpha_0$. Now assume $\underline{r}_{k-1} = R_{k-2}\alpha_{k-1}$. We have

$$-R_{k-1}\varrho_k - R_{k-2,h_k} = (-R_{k-1},\ -R_{k-2,h_k})\begin{pmatrix} \varrho_k \\ 1 \end{pmatrix}$$
$$= (-I_m,\, \underline{r})B_{k-1}E_{h_k}\begin{pmatrix} \varrho_k \\ 1 \end{pmatrix} = (-I_m,\ \underline{r})B_{k-1}E_{h_k}E_{h_k}\begin{pmatrix} \alpha_{k-1} \\ 1 \end{pmatrix}\alpha_{k-1,h_k}^{-1}$$
$$= (-R_{k-2},\ \underline{r}_{k-1})\begin{pmatrix} \alpha_{k-1} \\ 1 \end{pmatrix}\alpha_{k-1,h_k}^{-1} = (-R_{k-2}\alpha_{k-1} + \underline{r}_{k-1})\alpha_{k-1,h_k}^{-1} = \underline{0},$$

thus $R_{k-1}\varrho_k = -R_{k-2,h_k}$. Then

$$\underline{r}_k = (-I_m,\ \underline{r})B_k\begin{pmatrix} \underline{0} \\ 1 \end{pmatrix} = (-I_m,\ \underline{r})B_{k-1}E_{h_k}A(\underline{a}_k)\begin{pmatrix} \underline{0} \\ 1 \end{pmatrix}$$
$$= (-R_{k-1},\ -R_{k-2,h_k})\begin{pmatrix} \underline{a}_k \\ 1 \end{pmatrix} = (-R_{k-1},\ -R_{k-2,h_k})\begin{pmatrix} \varrho_k - \alpha_k \\ 1 \end{pmatrix}$$
$$= (-R_{k-1},\ -R_{k-2,h_k})\begin{pmatrix} -\alpha_k \\ 0 \end{pmatrix} = R_{k-1}\alpha_k.$$

To prove $\mathrm{Iv}(R_{k-1}\alpha_k) = \mathrm{Iv}(D_k\alpha_k)$, we need to know the relation between R_{k-1} and D_k. For this purpose we introduce two concepts: base matrix and D-component of a base matrix.

Definition 17. We call a square matrix R of order m over $F((z^{-1}))$ a *base matrix* if $R(j) \neq \underline{0}$ and $I(R(j)) = j$ for each $1 \leq j \leq m$, where $R(j)$ denotes the jth column of R. For a base matrix R, the D-matrix $\Delta = \mathrm{diag}(z^{-v_1}, \ldots, z^{-v_m})$ is called the *D-component* of R if $v_j = v(R(j))$ for each j.

Lemma 18. *Let R be a base matrix, and Δ the D-component of R. Then R is invertible, and $\mathrm{Iv}(R\underline{r}) = \mathrm{Iv}(\Delta\underline{r})$ for all $\underline{r} \in F((z^{-1}))^m$.*

Proof. Let $L = R\Delta^{-1}$. It is clear that $\mathrm{Iv}(L_j) = (j, 0)$ for all $1 \le j \le m$, where L_j denotes the jth column of L. It is enough to prove that L is invertible, and $\mathrm{Iv}(L\underline{r}) = \mathrm{Iv}(\underline{r})$ for all $\underline{r} \in F((z^{-1}))^m$, since then $R = L\Delta$ is invertible, and $\mathrm{Iv}(R\underline{r}) = \mathrm{Iv}(L\Delta\underline{r}) = \mathrm{Iv}(\Delta\underline{r})$.

It is clear that $v(\det(L)) = 0$, so $\det(L) \neq 0$, hence L is invertible. Let $\mathrm{Iv}(\underline{r}) = (h, v)$, $L\underline{r} = (r'_1, \ldots, r'_m)^\tau$, $\underline{r} = (r_1, \ldots, r_m)^\tau$, $L = (s_{i,j})$. Then $r'_i = \sum_j s_{i,j} r_j$. Note that $v(s_{i,j}) > 0$ for $j > i$, $v(s_{i,i}) = 0$, and $v(s_{i,j}) \ge 0$ for $j < i$; $v(r_j) > v$ for $j < h$, $v(r_h) = v$, and $v(r_j) \ge v$ for $j > h$. It is easy to check that $v(r'_i) > v$ for $i < h$, $v(r'_h) = v$, $v(r'_i) \ge v$ for $i > h$, based on Theorem 3. Hence, $\mathrm{Iv}(L\underline{r}) = (h, v) = \mathrm{Iv}(\underline{r})$.

LEMMA 19. *$\mathrm{Iv}(R_{k-1,j}) = (j, v_{k,j})$ for $0 \le k \le \mu$. In particular, R_{k-1} is a base matrix, and D_k is the D-component of R_{k-1} for $0 \le k \le \mu$.*

Proof. We reason by induction on k. When $k = 0$, we have $R_{-1} = I_m$, so $R_{-1,j} = \underline{e}_j$, hence $\mathrm{Iv}(R_{-1,j}) = \mathrm{Iv}(\underline{e}_j) = (j, 0) = (j, v_{0,j})$. Now assume $\mathrm{Iv}(R_{i-1,j}) = (j, v_{i,j})$ for $0 \le i < k$ and $1 \le j \le m$. In particular, we assume $\mathrm{Iv}(R_{k-2,j}) = (j, v_{k-1,j})$, hence R_{k-2} is a base matrix, and D_{k-1} $(= \Delta_{k-2})$ is the D-component of R_{k-2}. If $j \neq h_k$, we have seen that $R_{k-1,j} = R_{k-2,j}$, so $\mathrm{Iv}(R_{k-1,j}) = \mathrm{Iv}(R_{k-2,j}) = (j, v_{k-1,j}) = (j, v_{k,j})$. Since $R_{k-1,h_k} = -\underline{r}_{k-1} = -R_{k-2}\alpha_{k-1}$, we conclude that

$$\begin{aligned}\mathrm{Iv}(R_{k-1,h_k}) &= \mathrm{Iv}(R_{k-2}\alpha_{k-1}) = \mathrm{Iv}(\Delta_{k-2}\alpha_{k-1}) \\ &= (h_k, c_k) = (h_k, c_{k,h_k}) = (h_k, v_{k,h_k}).\end{aligned}$$

Proof of Proposition 15(2), (3). From Lemmas 18 and 19 we see immediately that $\mathrm{Iv}(R_{k-1}\alpha_k) = \mathrm{Iv}(D_k\alpha_k)$, which leads to item (2). From (1) and (2) we get

$$\begin{aligned}\mathrm{Iv}(\underline{r}_k) &= \mathrm{Iv}(R_{k-1}\alpha_k) = \mathrm{Iv}(D_k\alpha_k) = \mathrm{Iv}(D_k\alpha_k) \\ &= \mathrm{Iv}(\Delta_{k-1}\alpha_k) = (h_{k+1}, c_{k+1}) = (h_{k+1}, v_{k+1}),\end{aligned}$$

which is (3).

7.3. *Proof of Theorem 12.* The proof of Theorem 12 is based on the following lemma.

LEMMA 20. *Assume $0 \neq b_i(z) \in F[z]$, $\deg(b_i(z)) < t_{i+1}$, $0 \le i \le \mu$. Then*

(1) $\mathrm{Iv}(\{\underline{r}q_i(z)b_i(z)\}) = (h_{i+1}, v_{i+1} - \deg(b_i(z)))$.
(2) $\mathrm{Iv}(\{\underline{r}q_i(z)b_i(z)\}) \neq \mathrm{Iv}(\{\underline{r}q_j(z)b_j(z)\}), \forall 0 \le j \neq i \le \mu$ *and* $b_i(z)b_j(z) \neq 0$.

Proof. (1) Since

$$\mathrm{Iv}(\underline{r}_i b_i(z)) = (h_{i+1}, v_{i+1} - \deg(b_i(z))) > (h_{i+1}, v_{i+1} - t_{i+1}) \ge (h_{i+1}, 0),$$

we obtain $\{\underline{r}_i b_i(z)\} = \underline{r}_i b_i(z)$. Then

$$\{\underline{r}q_i(z)b_i(z)\} = \{(\underline{r}q_i(z) - \underline{p}_i)b_i(z)\} = \{\underline{r}_i b_i(z)\} = \underline{r}_i b_i(z).$$

So, $\mathrm{Iv}(\{\underline{r}q_i(z)b_i(z)\}) = \mathrm{Iv}(\underline{r}_i b_i(z)) = (h_{i+1}, v_{i+1} - \deg(b_i(z)))$.

(2) If $h_{i+1} \neq h_{j+1}$, then (2) is an easy consequence of (1). If $h_{i+1} = h_{j+1}$, we may assume $j < i$. From (1) we have

$$\begin{aligned} v(\{\underline{r}q_i(z)b_i(z)\}) &= v_{i+1} - \deg(b_i(z)) > v_{i+1} - t_{i+1} = v_{i,h_{i+1}} \geq v_{j+1,h_{i+1}} \\ &= v_{j+1,h_{j+1}} = v_{j+1} \geq v_{j+1} - \deg(b_j(z)) = v(\{\underline{r}q_j(z)b_j(z)\}), \end{aligned}$$

which concludes the proof.

We can now prove Theorem 12. Set $d = \deg(q(z))$ and $(h, v) = \mathrm{Iv}(\{\underline{r}q(z)\})$. Since

$$\mathrm{Iv}(\underline{r}q(z) - \underline{p}(z)) \leq \mathrm{Iv}(\{\underline{r}q(z)\}) \quad \text{and} \quad \underline{r} - \frac{\underline{p}(z)}{q(z)} = \frac{\underline{r}q(z) - \underline{p}(z)}{q(z)},$$

we get $\mathrm{Iv}(\underline{r} - \underline{p}(z)/q(z)) \leq (h, v + d)$. It is enough to prove

$$(h, v) \leq (h_{k+1}, n_{k+1} - d),$$

since then we have $\mathrm{Iv}(\underline{r} - \underline{p}(z)/q(z)) \leq (h, v + d) \leq (h_{k+1}, n_{k+1})$. With the assumption $d_k \leq d < d_{k+1}$ we can write $q(z) = \sum_{0 \leq i \leq k} b_i(z)q_i(z)$ for some $b_i(z) \in F[z]$ such that $\deg(b_i(z)) < \deg(q_{i+1}(z)) - \deg(q_i(z)) = t_{i+1}$ for each $i \geq 0$ and $b_i(z) \neq 0$ (note that $q_0(z) = 1$) and $\deg(b_k(z)) = d - d_k \geq 0$. It is clear that $\{\underline{r}q(z)\} = \sum_{0 \leq i \leq k} \{\underline{r}q_i(z)b_i(z)\}$. From Lemmas 20 and 3 we have

$$\begin{aligned} (h, v) &= \mathrm{Iv}(\{\underline{r}q(z)\}) = \min\{\mathrm{Iv}(\{\underline{r}q_i(z)b_i(z)\}) \mid b_i(z) \neq 0, 0 \leq i \leq k\} \\ &\leq \mathrm{Iv}(\{\underline{r}q_k(z)b_k(z)\}) = (h_{k+1}, v_{k+1} - \deg(b_k(z))) \\ &= (h_{k+1}, v_{k+1} + d_k - d) = (h_{k+1}, n_{k+1} - d). \end{aligned}$$

8. Remark. We have focused on m-CFA in this paper. We showed that the m-CFA produces a multi-continued fraction expansion $C(\underline{r})$ of $\underline{r}$ for any multiple Laurent series $\underline{r}$, which provides optimal rational approximations to $\underline{r}$.

For further study, consider an arbitrary data of the expansion form

$$C = [\underline{0}, h_1, \underline{a}_1, \ldots, h_k, \underline{a}_k, \ldots], \quad 1 \leq h_k \leq m, \underline{a}_k \in F[z]^m, 1 \leq k \leq \mu, \tag{24}$$

which satisfies the three conditions formulated for $C(\underline{r})$ in Section 5. We call such a C a multi-continued fraction. From the definition we see that multi-continued fractions are not necessarily identical to $C(\underline{r})$ for some $\underline{r}$. The problem arises whether multi-continued fractions have similar properties to those of $C(\underline{r})$, to be specific, whether any multi-continued fraction C converges to an element $\underline{r}$ in $F((z^{-1}))^m$ and provides optimal rational approximations to $\underline{r}$, and whether one can construct an algorithm which produces such C. The answers to these problems are affirmative, and we will call the expected algorithm the multi-universal continued fraction algorithm (m-UCFA, for short). We will discuss these problems in another paper.

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