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On the mean value of $L(m,\chi)L(n,\overline{\chi})$ at positive integers $m,n\geq 1$

by

HUANING LIU and WENPENG ZHANG (Xi'an)

1. Introduction. Let χ be a Dirichlet character modulo $q\geq 2$, and $L(s,\chi)$ be the Dirichlet L-function corresponding to χ. S. Louboutin [3] and the second author [7] proved that

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |L(1,\chi)|^2=\frac{\pi^2}{12}\cdot\frac{\phi^2(q)}{q^2}\bigg(q\prod_{p\mid q}\bigg(1+\frac{1}{p}\bigg)-3\bigg),$$

where $\phi(q)$ is the Euler function. In the case that $q=p$ is prime, this formula had been proved by H. Walum [6]. Moreover, S. Louboutin [4] studied the mean value of $|L(1,\chi)|^2$ for odd primitive Dirichlet characters. M. Katsurada and K. Matsumoto [2] gave some asymptotic formulae for $\sum_{\chi \bmod q,\, \chi\neq\chi_0}|L(1,\chi)|^2$, where χ_0 is the principal character modulo q.

Furthermore, S. Louboutin [5] proved the following:

PROPOSITION 1.1. *Let $q>2$ and $k,l\geq 1$ denote integers. Set*

$$\phi_l(q)=\prod_{p\mid q}\bigg(1-\frac{1}{p^l}\bigg)\quad \textit{and}\quad \phi(q)=q\phi_1(q).$$

Then for any $k\geq 1$ there exists a polynomial $R_k(X)=\sum_{l=0}^{2k}r_{k,l}X^l$ of degree $2k$ with rational coefficients such that for all $q>2$ we have

$$\frac{2}{\phi(q)}\sum_{\chi(-1)=(-1)^k}|L(k,\chi)|^2=\frac{\pi^{2k}}{2((k-1)!)^2}\sum_{l=1}^{2k}r_{k,l}\phi_l(q)q^{l-2k}.$$

However, he did not determine the coefficients $r_{k,l}$.

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The main purpose of this paper is to study the mean value of the product $L(m,\chi)L(n,\overline{\chi})$ at positive integers $m,n\ge 1$, and give an interesting exact formula, by using the generalized Dedekind sums, Bernoulli polynomials and Bernoulli numbers:

THEOREM 1.1. *Let $q\ge 2$ and $m,n\ge 1$ be positive integers with $m\equiv n \bmod 2$. Set $\varepsilon_{m,n}=1$ if $m\equiv n\equiv 1 \bmod 2$ and $\varepsilon_{m,n}=0$ if $m\equiv n\equiv 0 \bmod 2$. Then*

$$\frac{2}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^m}} L(m,\chi)L(n,\overline{\chi})$$
$$=\frac{(-1)^{(m-n)/2}(2\pi)^{m+n}}{2m!n!}\Bigg(\sum_{l=0}^{m+n} r_{m,n,l}\phi_l(q)q^{l-m-n}-\frac{\varepsilon_{m,n}}{q}B_mB_n\phi_{m+n-1}(q)\Bigg),$$

where

$$r_{m,n,l}=B_{m+n-l}\sum_{\substack{a=0\\ a+b\ge m+n-l}}^{m}\sum_{b=0}^{n} B_{m-a}B_{n-b}\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1},$$

B_m is the Bernoulli number, and $\binom{m}{a}=\frac{m!}{a!(m-a)!}$.

2. Proof of Theorem 1.1. We define the generalized Dedekind sums by

$$s(m,n,q)=\sum_{j=1}^{q-1}B_m\left(\frac{j}{q}\right)B_n\left(\frac{j}{q}\right),$$

where $B_n(x)=\sum_{i=0}^{n}\binom{n}{i}B_ix^{n-i}$ is the nth Bernoulli polynomial. First we establish the connection between $s(m,n,q)$ and the Dirichlet L-functions.

LEMMA 2.1. *For integers $q\ge 2$ and $m,n>0$ with $m\equiv n \bmod 2$, we have*

$$\frac{(2\pi i)^{m+n}q^{m+n-1}}{4m!n!}\left[s(m,n,q)+\frac{m!n!}{(2\pi i)^{m+n}}\left(\sum_{r=1}^{\infty}\frac{1+(-1)^m}{r^m}\right)\left(\sum_{s=1}^{\infty}\frac{1+(-1)^n}{s^n}\right)\right]$$
$$=\sum_{d|q}\frac{d^{m+n}}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=(-1)^m=(-1)^n}}\overline{\chi}(-1)L(m,\chi)L(n,\overline{\chi}).$$

Proof. From Theorem 12.19 of [1] we know that

$$B_n(x)=-\frac{n!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\ne 0}}^{\infty}\frac{e(rx)}{r^n}\quad \text{if } 0<x\le 1,$$

where $e(y) = e^{2\pi i y}$. Then we have

$$\frac{(2\pi i)^{m+n}}{m!n!}\, s(m,n,q) = \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{1}{r^m s^n} \sum_{j=1}^{q-1} e\left(\frac{j(r+s)}{q}\right)$$

$$= q \sum_{\substack{r=-\infty \\ r\neq 0 \\ r+s\equiv 0 \bmod q}}^{\infty} \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{1}{r^m s^n} - \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{1}{r^m s^n}$$

$$= q \sum_{d|q} \sum_{\substack{r=-\infty \\ r\neq 0 \\ r+s\equiv 0 \bmod q \\ \gcd(r,q)=q/d}}^{\infty} \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{1}{r^m s^n} - \left(\sum_{r=1}^{\infty} \frac{1+(-1)^m}{r^m}\right)\left(\sum_{s=1}^{\infty} \frac{1+(-1)^n}{s^n}\right)$$

$$= q \sum_{d|q} \sum_{\substack{r=-\infty \\ r\neq 0 \\ r+s\equiv 0 \bmod d \\ \gcd(r,d)=1}}^{\infty} \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{1}{\left(r\cdot\frac{q}{d}\right)^m \left(s\cdot\frac{q}{d}\right)^n} - \left(\sum_{r=1}^{\infty} \frac{1+(-1)^m}{r^m}\right)\left(\sum_{s=1}^{\infty} \frac{1+(-1)^n}{s^n}\right)$$

$$= \frac{1}{q^{m+n-1}} \sum_{d|q} \frac{d^{m+n}}{\phi(d)} \sum_{\chi \bmod d} \left(\sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} \frac{\chi(r)}{r^m}\right)\left(\sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{\overline{\chi}(-s)}{s^n}\right)$$

$$- \left(\sum_{r=1}^{\infty} \frac{1+(-1)^m}{r^m}\right)\left(\sum_{s=1}^{\infty} \frac{1+(-1)^n}{s^n}\right)$$

$$= \frac{4}{q^{m+n-1}} \sum_{d|q} \frac{d^{m+n}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=(-1)^m=(-1)^n}} \overline{\chi}(-1) L(m,\chi) L(n,\overline{\chi})$$

$$- \left(\sum_{r=1}^{\infty} \frac{1+(-1)^m}{r^m}\right)\left(\sum_{s=1}^{\infty} \frac{1+(-1)^n}{s^n}\right).$$

This proves Lemma 2.1.

Remark. From Lemma 2.1 we know that $s(m,n,q)=0$ if $m \not\equiv n \bmod 2$.

Now we express $s(m,n,q)$ in terms of Bernoulli numbers as follows:

Lemma 2.2. *For integers $q \geq 2$ and $m,n>0$, we have*

$$s(m,n,q) = \sum_{c=0}^{m+n} B_c q^{1-c} \sum_{\substack{a=0 \\ a+b\geq c}}^{m} \sum_{b=0}^{n} B_{m-a} B_{n-b} \frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{c}}{a+b+1} - B_m B_n.$$

Proof. From the properties of Bernoulli polynomials and Bernoulli numbers (see Chapter 12 of [1]) we get

$$
\begin{aligned}
s(m,n,q) &= \sum_{j=1}^{q-1} B_m\left(\frac{j}{q}\right) B_n\left(\frac{j}{q}\right)\\
&= \sum_{j=1}^{q-1}\left[\sum_{a=0}^{m}\binom{m}{a}B_{m-a}j^a q^{-a}\right]\left[\sum_{b=0}^{n}\binom{n}{b}B_{n-b}j^b q^{-b}\right]\\
&= \sum_{a=0}^{m}\sum_{b=0}^{n}\binom{m}{a}\binom{n}{b}B_{m-a}B_{n-b}q^{-a-b}\Big(\sum_{j=1}^{q-1} j^{a+b}\Big)\\
&= \sum_{\substack{a=0\\ a+b>0}}^{m}\sum_{b=0}^{n}\binom{m}{a}\binom{n}{b}B_{m-a}B_{n-b}q^{-a-b}\Big(\sum_{j=1}^{q-1} j^{a+b}\Big) + (q-1)B_mB_n\\
&= \sum_{\substack{a=0\\ a+b>0}}^{m}\sum_{b=0}^{n}\binom{m}{a}\binom{n}{b}B_{m-a}B_{n-b}q^{-a-b}\Big(\frac{1}{a+b+1}\sum_{c=0}^{a+b}\binom{a+b+1}{c}B_c q^{a+b+1-c}\Big)\\
&\qquad\qquad + (q-1)B_mB_n\\
&= \sum_{a=0}^{m}\sum_{b=0}^{n}B_{m-a}B_{n-b}\,\frac{\binom{m}{a}\binom{n}{b}}{a+b+1}\sum_{c=0}^{a+b}\binom{a+b+1}{c}B_c q^{1-c} - B_mB_n\\
&= \sum_{c=0}^{m+n}B_c q^{1-c}\sum_{\substack{a=0\\ a+b\ge c}}^{m}\sum_{b=0}^{n}B_{m-a}B_{n-b}\,\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{c}}{a+b+1} - B_mB_n.
\end{aligned}
$$

This completes the proof of Lemma 2.2.

Now we prove Theorem 1.1. By Lemma 2.1 and the Möbius transformation

$$
G(q) = \sum_{d|q} F(d) \ \Leftrightarrow\ F(q) = \sum_{d|q}\mu\left(\frac{q}{d}\right)G(d)
$$

we get

$$
\begin{aligned}
\frac{q^{m+n}}{\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^m=(-1)^n}} &\overline{\chi}(-1)L(m,\chi)L(n,\overline{\chi})\\
&= \frac{(2\pi i)^{m+n}}{4m!n!}\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{m+n-1}s(m,n,d)\\
&\quad + \frac{1}{4}\Big(\sum_{r=1}^{\infty}\frac{1+(-1)^m}{r^m}\Big)\Big(\sum_{s=1}^{\infty}\frac{1+(-1)^n}{s^n}\Big)\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{m+n-1}.
\end{aligned}
$$

Using Lemma 2.2 we have

$$\begin{aligned}
&\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{m+n-1}s(m,n,d)\\
&\quad=\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{m+n-1}\\
&\qquad\times\left[\sum_{c=0}^{m+n}B_c d^{1-c}\sum_{\substack{a=0\\a+b\geq c}}^{m}\sum_{b=0}^{n}B_{m-a}B_{n-b}\,\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{c}}{a+b+1}-B_mB_n\right]\\
&\quad=\sum_{d|q}\mu\left(\frac{q}{d}\right)\sum_{c=0}^{m+n}B_c d^{m+n-c}\sum_{\substack{a=0\\a+b\geq c}}^{m}\sum_{b=0}^{n}B_{m-a}B_{n-b}\,\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{c}}{a+b+1}\\
&\qquad-B_mB_n\sum_{d|q}\mu\left(\frac{q}{d}\right)d^{m+n-1}.
\end{aligned}$$

Noting that

$$\sum_{d|q}\mu\left(\frac{q}{d}\right)d^c=q^c\prod_{p|q}\left(1-\frac{1}{p^c}\right)=q^c\phi_c(q),$$

we infer that

$$\begin{aligned}
&\frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^m=(-1)^n}}\overline{\chi}(-1)L(m,\chi)L(n,\overline{\chi})\\
&\quad=\frac{(2\pi i)^{m+n}}{4m!n!}\sum_{c=0}^{m+n}B_c q^{-c}\phi_{m+n-c}(q)\sum_{\substack{a=0\\a+b\geq c}}^{m}\sum_{b=0}^{n}B_{m-a}B_{n-b}\,\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{c}}{a+b+1}\\
&\qquad-\frac{(2\pi i)^{m+n}}{4m!n!q}B_mB_n\phi_{m+n-1}(q)\\
&\qquad+\frac{1}{4q}\left(\sum_{r=1}^{\infty}\frac{1+(-1)^m}{r^m}\right)\left(\sum_{s=1}^{\infty}\frac{1+(-1)^n}{s^n}\right)\phi_{m+n-1}(q).
\end{aligned}$$

Now setting $m\equiv n \bmod 2$, and noting that $(-1)^m(1+(-1)^m)=1+(-1)^m$, $i^{m+n}(-1)^m=(-1)^{(m-n)/2}$, and $2\zeta(2k)=(-1)^{k+1}\frac{(2\pi)^{2k}}{(2k)!}B_{2k}$ for any positive integer k, we immediately get

$$\begin{aligned}
&\frac{2}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^m}}L(m,\chi)L(n,\overline{\chi})=\frac{(-1)^{(m-n)/2}(2\pi)^{m+n}}{2m!n!}\\
&\qquad\times\sum_{l=0}^{m+n}\phi_l(q)q^{l-m-n}\left[B_{m+n-l}\sum_{\substack{a=0\\a+b\geq m+n-l}}^{m}\sum_{b=0}^{n}B_{m-a}B_{n-b}\,\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1}\right]
\end{aligned}$$

$$-\frac{(-1)^{(m-n)/2}(2\pi)^{m+n}}{2m!n!q}B_mB_n\phi_{m+n-1}(q)$$
$$+\frac{1}{2q}\bigg(\sum_{r=1}^{\infty}\frac{1+(-1)^m}{r^m}\bigg)\bigg(\sum_{s=1}^{\infty}\frac{1+(-1)^n}{s^n}\bigg)\phi_{m+n-1}(q)$$
$$=\frac{(-1)^{(m-n)/2}(2\pi)^{m+n}}{2m!n!}\bigg(\sum_{l=0}^{m+n}r_{m,n,l}\phi_l(q)q^{l-m-n}-\frac{\varepsilon_{m,n}}{q}B_mB_n\phi_{m+n-1}(q)\bigg).$$

This proves Theorem 1.1.

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Department of Mathematics
Northwest University
Xi'an, Shaanxi, P.R. China
E-mail: hnliu@nwu.edu.cn
wpzhang@nwu.edu.cn

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